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A three-layer asymptotic analysis of turbulent channel flow

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Abstract. In the classical theory for large-Reynolds number fully developed channel flow, the solutions obtained by asymptotic-expansion techniques for the outer Karman defect layer and the inner Prandtl wall layer are demonstrated to match through the introduction of an intermediate layer based on a general intermediate limit. From an examination of the results for this general intermediate layer, the distinguished intermediate limit and the corresponding intermediate layer for which the turbulent and laminar contributions to the difference of the Reynolds stress from the wall stress are of the same order of magnitude are identified. The thickness of this distinguished intermediate layer is found to be of the order of the geometric mean of the thicknesses of the outer and inner layers

Keywords. Asymptotic analysis; turbulent channel flour; eddy-viscosity closure.

1. Introduction

The present paper demonstrates that, in terms of the classical theory, for turbulent channel (or pipe) flow, there is a *distinguished* intermediate between the outer defect layer and the inner wall layer. With this demonstration, it follows that this (new) three-layer theory modifies and/or extends the (old) two-layer one.

A review and extension of the asymptotic analysis of turbulent channel flow is presented in §2. The non-dimensional boundary-value problem, without the adoption of closure, is introduced, and the basic formulations of the boundary-value problem in terms of outer-layer and inner-layer variables are developed. The asymptotic representations of the Reynolds stress and the mean velocity for the outer and inner layers, as determined from the solutions of the pertinent boundary-value problem are given, and their behaviors in a region intermediate to the defect and wall layers are, then, considered.

If $R_\tau = u^* h^*/v^*$ is the turbulent Reynolds number, much greater than unity, with h^* the half-depth of the channel, u_τ^* the friction velocity, and v^* the kinematic viscosity coefficient, and if $z = z^*/h^*$ and $\zeta = u_\tau^* z^*/v^* = R_\tau z$ are the outer-layer and inner-layer spatial coordinates, respectively, with z^* the distance from the wall, then λ, the *general* intermediate-layer spatial coordinate, is defined by $\lambda = z/\phi(R_\tau) = \zeta/R_\tau\phi(R_\tau)$, with $\phi(R_\tau) \to 0$, $R_\tau\phi(R_\tau) \to \infty$ as $R_\tau \to \infty$. In this paper, attention is directed to the *particular* (or *distinguished*) intermediate-layer spatial-coordinate case of $\eta = \lambda$, with $\phi(R_\tau) = 1/R_\tau^{1/2} \to 0$ and/or $R_\tau\phi(R_\tau) = R_\tau^{1/2} \to \infty$, such that the defect-layer and the wall-layer variables satisfy $z = \eta/R_\tau^{1/2} \to 0$ and $\zeta = R_\tau^{1/2}\eta \to \infty$, respectively, as $R_\tau \to \infty$, with η fixed. This distinguished intermediate-layer spatial coordinate is the one whose introduction produces an intermediate layer in which the turbulent and laminar contributions to the difference of the Reynolds stress from the wall stress are of the same order of magnitude. If the outer-layer and inner-layer asymptotic solutions for the Reynolds stress and the mean velocity match in a general intermediate layer, then these solutions also match in the distinguished intermediate layer introduced. The solutions for the Reynolds stress and the mean velocity in this distinguished intermediate layer, as determined from the outer-layer and/or inner-layer solutions for these flow quantities, are presented.

The results obtained for this distinguished intermediate layer should be considered for the development of an intermediate-layer distinguished limit. The formulation and solutions for this intermediate-layer distinguished limit are presented. The analyses of the outer-layer and inner-layer distinguished limits, complemented by the analysis of the intermediate-layer distinguished limit, complete the three-layer asymptotic analysis of turbulent channel flow.

This intermediate-layer asymptotic analysis yields quantitative results for the magnitude and location of the maximum of the Reynolds stress that is achieved within this layer. In addition to the behaviour of the Reynolds stress, which, in effect, defines this intermediate layer, attention is directed to the behaviour of the mean velocity in this layer. It is demonstrated here that, to leading order of approximation, this velocity is a constant equal to one-half the value of the channel center-line velocity, and that (not surprisingly) the first correction to this value is of the order of u_τ and is functionally logarithmic. The location (to leading order of approximation) of the "half-velocity point" within this intermediate layer is found in the analysis of §2.

In §3, an eddy-viscosity closure model, namely that of Bush and Fendell [4], is adopted, and the solutions for the outer and inner layers and particularly those for the distinguished intermediate layer for this closure model are determined. The heretofore undetermined constants of the asymptotic analysis of §2 are evaluated in terms of the parameters of this closure model. For the purpose of comparison, the solution for the mixing-length-theory eddy-viscosity closure model of Bush and Fendell [5] for the Reynolds stress in this intermediate layer is presented also.

The existence of an intermediate layer in turbulent shear flows has been considered previously by Long and Chen [7] and by Afzal [2, 3]. Long and Chen proposed a mesolayer (or intermediate-layer) theory, in which the inner-wall-layer and intermediate-layer expansions are developed as developed as a composite expansion that is matched with the outer-defect-layer expansion. They claim that, since the intermediate layer "intrudes" between the outer and inner regions preventing the commonly assumed overlap and the consequent logarithmic profile, their theory replaces the classical two-layer theory. Afzal analyzed the three layers separately and matched the resulting asymptotic solutions in two overlap domains to demonstrate the possibility of two logarithmic regions rather than the one logarithmic region of classical theory.

2. Asymptotic analysis

The non-dimensional boundary-value problem for the steady two-dimensional turbulent flow of a fluid of constant density and viscosity (i.e., ρ^*, ν^* = constant) in a channel is

$$\tau + \frac{1}{R}\frac{du}{dz} = u_\tau^2 (1 - z) \qquad (0 \leqslant z \leqslant 1); \tag{1}$$

$$u \to 1, \frac{du}{dz} \to 0, \tau \to 0 \text{ as } z \to 1, \tag{2a}$$

$$u, \tau \to 0 \text{ as } z \to 0. \tag{2b}$$

The independent variable is $z = z^*/h^*$, the normal coordinate, with h^* the half-depth of the channel ($z = 1$ at the channel centreline, $z = 0$ at the channel wall); the dependent variables are $u = u^*/u_0^* = u(z; R)$, the mean velocity in the axial direction, with u_0^* the mean axial velocity at the centreline, and $\tau = \tau^*/\rho^* u_0^{*2} = (-\overline{u^{*\prime}v^{*\prime}})/u_0^{*2} = \tau(z; R)$, the turbulent shear stress. The parameters are $R = u_0^* h^*/\nu^*$, the Reynolds number, taken to be much greater than unity; and $u_\tau = u_\tau^*/u_0^* = [\nu^* (du^*/dz^*)_{z^*=0}]^{1/2}/u_0^* = u_\tau(R)$, the friction velocity, effectively an eigenvalue (to be determined), less than unity. Specifically, in what follows, it is taken that

$$u_\tau \to 0 \text{ as } R \to \infty, \tag{3a}$$

$$R_\tau = R u_\tau = u_\tau^* h^*/\nu^* \to \infty \text{ as } R \to \infty. \tag{3b}$$

For $R \to \infty$, the classical theory of turbulent channel flow divides the flow into two regions: an outer (or defect) layer and an inner (or wall) layer. The outer layer is the thicker layer, about the centreline and not immediately adjacent to the wall, in which the velocity undergoes only a small perturbation from its centreline value (the Karman velocity defect law), and the laminar stress is small relative to the turbulent stress. The inner layer is the thinner region, adjacent to the wall, in which the velocity is only a small fraction of its centreline value (the Prandtl universal law of the wall), and the turbulent and laminar stress are of comparable magnitude.

Asymptotic analyses of the defect layer and the wall layer follow.

For the defect layer, it is taken that the appropriate spatial variable is z, and the velocity and Reynolds stress variables are

$$u(z; R) = 1 - u_\tau(R_\tau)\, f(z; R_\tau), \tag{4a}$$

$$\tau(z; R) = u_\tau^2(R_\tau) t(z; R_\tau). \tag{4b}$$

The outer limit is defined as z fixed for $R_\tau \to \infty$. Thus, based upon substitution of (4) into (1) and (2a), the boundary-value problem for the defect layer ($0 < z \leqslant 1$) is

$$t - \frac{1}{R_\tau}\frac{df}{dz} = (1 - z); \tag{5}$$

$$f, \frac{df}{dz}, t \to 0 \text{ as } z \to 1. \tag{6}$$

An examination of (5) suggests the following asymptotic expansions for the defect-layer dependent variables:

$$f(z; R_\tau) \cong f_0(z) + \frac{1}{R_\tau} f_1(z) + \frac{1}{R_\tau^2} f_2(z) + \ldots, \tag{7a}$$

$$t(z; R_\tau) \cong t_0(z) + \frac{1}{R_\tau} t_1(z) + \frac{1}{R_\tau^2} t_2(z) + \ldots. \tag{7b}$$

Introduction of these expansions into the defect-layer differential equation produces the sequence

$$t_0 = (1-z), t_1 - \frac{df_0}{dz} = 0, t_2 - \frac{df_2}{dz} = 0, \ldots. \tag{8}$$

Thus, the leading-order approximation for the defect-layer turbulent stress is $t_0(z) = (1-z)$. This approximation cannot satisfy the pertinent boundary condition of (2b), as the formulation of (4) and the expansions of (7) are not valid at the wall.

To supplement the defect layer, the wall layer is introduced. For the wall layer, it is taken that the appropriate spatial variable is $\zeta = u_\tau^* z^*/v^* = R_\tau z$, and the velocity and Reynolds stress variables are

$$u(z; R) = u_\tau(R_\tau) g(\zeta; R_\tau), \tag{9a}$$

$$\tau(z; R) = u_\tau^2(R_\tau) s(\zeta; R_\tau), \tag{9b}$$

with the inner limit defined as ζ fixed for $R_\tau \to \infty$. Based upon substitution of (9) into (1) and (2b), the boundary-value problem for the wall layer $(0 \leqslant \zeta < \infty)$ is

$$s + \frac{dg}{d\zeta} = \left(1 - \frac{1}{R_\tau}\zeta\right); \tag{10}$$

$$g, s \to 0 \text{ as } \zeta \to 0. \tag{11}$$

An examination of (10) suggests the following asymptotic expansions for the wall-layer dependent variables:

$$g(\zeta; R_\tau) \cong g_0(\zeta) + \frac{1}{R_\tau} g_1(\zeta) + \frac{1}{R_\tau^2} g_2(\zeta) + \ldots, \tag{12a}$$

$$s(\zeta; R_\tau) \cong s_0(\zeta) + \frac{1}{R_\tau} s_1(\zeta) + \frac{1}{R_\tau^2} s_2(\zeta) + \ldots. \tag{12b}$$

Introduction of these expansions into the wall-layer differential equation produces the sequence

$$s_0 + \frac{dg_0}{d\zeta} = 1, s_1 + \frac{dg_1}{d\zeta} = -\zeta, s_2 + \frac{dg_2}{d\zeta} = 0, \ldots. \tag{13}$$

It is noted that the leading-order equation of (13) corresponds to constant total stress.

Based on the Karman velocity defect law and the Prandtl law of the wall, through an extension of Millikan's argument (cf., e.g., Afzal [1]), the outer expansion as $z \to 0$ and the inner expansion is $\zeta \to \infty$ for the velocity profile are, respectively,

$$\begin{aligned} u(z; R) &= u(z; R_\tau) \\ &\sim 1 - u_\tau(R_\tau)\Big[(-A_{01}\log z + A_0 + W_0 z + \tfrac{1}{2}C_0 z^2 + \dots) \\ &+ \frac{1}{R_\tau}\left(\frac{J_1}{z} - A_{11}\log z + A_1 + W_1 z + \dots\right) \\ &+ \frac{1}{R_\tau^2}\left(\frac{L_2}{2z^2} + \frac{J_2}{z} - A_{21}\log z + A_2 + \dots\right) + \dots\Big]; \end{aligned} \tag{14a}$$

$$\begin{aligned} u(z; R) &= u(\zeta; R_\tau) \\ &\sim u_\tau(R_\tau)\Big[\left(B_{01}\log\zeta + B_0 + \frac{V_0}{\zeta} + \frac{D_0}{2\zeta^2} + \dots\right) + \frac{1}{R_\tau}\Big(H_1\zeta + B_{11}\log\zeta + B_1 \\ &+ \frac{V_1}{\zeta} + \dots\Big) + \frac{1}{R_\tau^2}\left(\frac{1}{2}E_2\zeta^2 + H_2\zeta + B_{21}\log\zeta + B_2 + \dots\right) + \dots\Big]. \end{aligned} \tag{14b}$$

It can be demonstrated that (14a) and (14b) match in a region between the defect and wall layers through the introduction of an intermediate variable λ, defined by

$$\begin{aligned} &\lambda = z/\phi(R_\tau) = \zeta/R_\tau\phi(R_\tau) \text{ fixed}, \\ &\text{with } \phi(R_\tau) \to 0,\ R_\tau\phi(R_\tau) \to \infty \text{ as } R_\tau \to \infty, \end{aligned} \tag{15a}$$

such that

$$z = \phi(R_\tau)\lambda \to 0, \quad \zeta = R_\tau\phi(R_\tau)\lambda \to \infty. \tag{15b}$$

This matching requires that

$$\begin{aligned} &B_{01} = A_{01} = 1/K_0,\ H_1 = -W_0,\ V_0 = -J_1,\ E_2 = C_0, \\ &B_{11} = A_{11} = 1/K_1,\ D_0 = -L_2,\ H_2 = -W_1,\ V_1 = -J_2, \dots \end{aligned} \tag{16a}$$

and that the following skin friction law be satisfied:

$$\begin{aligned} &u_\tau(R_\tau)\left[\left\{\frac{1}{K_0}\log R_\tau + (A_0 + B_0)\right\} + \frac{1}{R_\tau}\left\{\frac{1}{K_1}\log R_\tau + (A_1 + B_1)\right\} + \dots\right] \\ &= 1: \\ &u_\tau(R_\tau) = \frac{K_0}{\log R_\tau}\left[\left\{1 + \frac{K_0(A_0 + B_0)}{\log R_\tau}\right\} + \frac{K_0/K_1}{R_\tau}\left\{1 + \frac{K_1(A_1 + B_1)}{\log R_\tau}\right\} \right. \\ &\left. + \dots\right]^{-1}. \end{aligned} \tag{16b}$$

The corresponding expressions for the Reynolds stress profile are then determined to be

$$\tau(z;\, R) = \tau(z;\, R_\tau)$$

$$\sim u_\tau^2(R_\tau)\left[(1-z)+\frac{1}{R_\tau}\left(-\frac{A_{01}}{z}+W_0+C_0 z+\ldots\right)+\frac{1}{R_\tau^2}\left(-\frac{J_1}{z^2}-\frac{A_{11}}{z}+\ldots\right)+\frac{1}{R_\tau^3}\left(-\frac{L_2}{z^3}-\frac{J_2}{z^2}+\ldots\right)+\ldots\right]; \tag{17a}$$

$$\tau(z;\, R) = \tau(\zeta;\, R_\tau)$$

$$\sim u_\tau^2(R_\tau)\left[\left(1-\frac{B_{01}}{\zeta}+\frac{V_0}{\zeta^2}+\frac{D_0}{\zeta^3}+\ldots\right)+\frac{1}{R_\tau}\left(-\zeta-H_1-\frac{B_{11}}{\zeta}+\ldots\right)+\frac{1}{R_\tau^2}\left(-E_2\zeta-H_2+\ldots\right)+\ldots\right]. \tag{17b}$$

Subject to (15) and (16), matching again follows.

From (17), it is seen that the leading-order turbulent contribution to the departure of the turbulent stress from $\tau_w(R_\tau) = u_\tau^2(R_\tau)$ is $z = \zeta/R_\tau \to 0$, whereas the leading-order laminar contribution to this departure is $1/K_0\zeta = 1/K_0 R_\tau z \to 0$. These turbulent and laminar contributions are seen to be of the same order of magnitude (Afzal [2]) for

$$\eta = R_\tau^{1/2} z = \zeta/R_\tau^{1/2} \text{ fixed.} \tag{18}$$

The introduction of (18) into (17) yields

$$\tau(z;\, R) = \tau(\eta;\, R_\tau)$$

$$\cong u_\tau^2(R_\tau)\left[1-\frac{1}{R_\tau^{1/2}}\left(\eta+\frac{1}{k_0\eta}\right)+\frac{1}{R_\tau}\left(W_0-\frac{J_1}{\eta^2}\right)+\frac{1}{R_\tau^{3/2}}\left(C_0\eta-\frac{1}{K_1\eta}-\frac{L_2}{\eta^3}\right)+\ldots\right]. \tag{19}$$

Further, introduction of (18) into (14) produces

$$u(z\colon R) = u(\eta;\, R_\tau)$$

$$\cong \frac{1}{2}+u_\tau(R_\tau)\left[\left\{\frac{1}{K_0}\log\eta+\frac{1}{2}(B_0-A_0)\right\}-\frac{1}{R_\tau^{1/2}}\left\{W_0\eta+\frac{J_1}{\eta}\right\}+\frac{1}{R_\tau}\left\{\frac{1}{K_1}\log\eta+\frac{1}{2}(B_1-A_1)-\frac{1}{2}\left(C_0\eta^2+\frac{L_2}{\eta^2}\right)\right\}-\frac{1}{R_\tau^{3/2}}\left\{W_1\eta+\frac{J_2}{\eta}\right\}+\ldots\right]. \tag{20}$$

Thus, $\eta = \lambda$, with $\phi(R_\tau) = 1/R_\tau^{1/2}$, represents a distinguished intermediate variable, whose introduction produces an intermediate layer, in which the turbulent and laminar contributions to the departure of the turbulent stress from u_τ^2 are of the same order of magnitude. These results indicate that a three-layer picture, rather than a two-layer picture, of turbulent channel flow holds. That is, in addition to the defect-layer and wall-layer limits, there exists an intermediate limit. Based on (18)–(20), for the intermediate layer, it is taken that the appropriate spatial variable is

$$\eta = R_\tau^{1/2} z = \zeta / R_\tau^{1/2} = z^* / (h^* \nu^* / u_\tau^*)^{1/2} \text{ fixed;} \tag{21}$$

and the velocity and Reynolds stress variables are

$$u(z;\, R) = \frac{1}{2} + u_\tau(R_\tau) U(\eta;\, R_\tau), \tag{22a}$$

$$\tau(z;\, R) = u_\tau^2(R_\tau)\left[1 - \frac{1}{R_\tau^{1/2}} T(\eta;\, R_\tau)\right]. \tag{22b}$$

The intermediate limit is defined as η fixed for $R_\tau \to \infty$. Substitution of (21) and (22) into (1) produces the following differential equation for the intermediate layer ($0 < \eta < \infty$):

$$T - \frac{dU}{d\eta} = \eta. \tag{23}$$

For this layer, then, all three terms are of equal order, and this intermediate limit is identified as a distinguished limit.

It is taken that the appropriate asymptotic expansions for the intermediate-layer dependent variables are

$$U(\eta;\, R_\tau) \cong U_0(\eta) + \frac{1}{R_\tau^{1/2}} U_1(\eta) + \frac{1}{R_\tau} U_2(\eta) + \ldots, \tag{24a}$$

$$T(\eta;\, R_\tau) \cong T_0(\eta) + \frac{1}{R_\tau^{1/2}} T_1(\eta) + \frac{1}{R_\tau} T_2(\eta) + \ldots. \tag{24b}$$

Introduction of these expansions into the intermediate-layer differential equation produces the sequence

$$T_0 - \frac{dU_0}{d\eta} = \eta,\; T_1 - \frac{dU_1}{d\eta} = 0,\; T_2 - \frac{dU_2}{d\eta} = 0, \ldots. \tag{25}$$

To the orders of approximation considered, the solutions for the velocity and Reynolds stress functions, indicated by (19) and (20), are

$$U_0 = \frac{1}{K_0} \log \eta + \frac{1}{2}(B_0 - A_0),\; T_0 = \left(\eta + \frac{1}{K_0 \eta}\right); \tag{26a}$$

$$U_1 = -\left(W_0 \eta + \frac{J_1}{\eta}\right),\; T_1 = -\left(W_0 - \frac{J_1}{\eta^2}\right); \tag{26b}$$

$$U_2 = \frac{1}{K_2} \log \eta + \frac{1}{2}(B_1 - A_1) - \frac{1}{2}\left(C_0 \eta^2 + \frac{L_2}{\eta^2}\right),$$

$$T_2 = -\left(C_0 \eta - \frac{1}{K_1 \eta} - \frac{L_2}{\eta^3}\right). \tag{26c}$$

It can be shown that these solutions for the intermediate-layer dependent variables match directly to both the defect-layer and wall-layer solutions for these variables.

From (22) and (26), it follows that the velocity and turbulent stress for the intermediate layer are

$$u(z;\ R) = u(\eta;\ R_\tau)$$

$$\cong \frac{1}{2} + u_\tau(R_\tau)\left[\left\{\frac{1}{K_0}\log\eta + \frac{1}{2}(B_0 - A_0)\right\} + \dots\right], \tag{27a}$$

$$\tau(z;\ R) = \tau(\eta;\ R_\tau)$$

$$\cong u_\tau^2(R_\tau)\left[1 - \frac{1}{R_\tau^{1/2}}\left\{\eta + \frac{1}{K_0\eta}\right\} + \dots\right], \tag{27b}$$

where

$$u_\tau(R_\tau) \cong \frac{K_0}{\log R_\tau}\left[\left\{1 - \frac{K_0(B_0 + A_0)}{\log R_\tau}\right\} + \dots\right]. \tag{28}$$

To the order of approximation considered in (27a), it is seen that

$$u \to \tfrac{1}{2} \text{ as } \eta \to \eta_{\frac{1}{2}} = \exp\{-\tfrac{1}{2}K_0(B_0 - A_0)\}. \tag{29}$$

For $K_0 \cong 0.41$ and $B_0 \cong 5.0$, $A_0 \cong 0.65$, $\eta_{1/2} \cong 0.41$. Further, from (27b) it is seen that

$$\tau \to \tau_{\max} \cong u_\tau^2\left[1 - \frac{2}{K_0^{1/2}R_\tau^{1/2}} + \dots\right] \text{ as } \eta \to \eta_{\max} = \frac{1}{K_0^{1/2}}. \tag{30}$$

For $K_0 \cong 0.41, \eta_{\max} \cong 1.56$. A study of the experimental data for pipe and channel flows (Afzal [3]) suggests the values $\eta_{\frac{1}{2}} \cong 0.44$ and $\eta_{\max} \cong 1.85$.

3. Eddy-viscosity closure

With the adoption of the eddy-viscosity closure model, defined by

$$\tau = \frac{\varepsilon}{R}\frac{\mathrm{d}u}{\mathrm{d}z}, \tag{31}$$

where $\varepsilon = \varepsilon^*/\nu^* = \varepsilon(z;\ R)$ is the kinematic eddy viscosity, the nondimensional boundary-value problem of (1) and (2) becomes

$$\frac{(1+\varepsilon)}{R}\frac{\mathrm{d}u}{\mathrm{d}z} = u_\tau^2(1-z); \tag{32}$$

$$u \to 1,\ \frac{\mathrm{d}u}{\mathrm{d}z} \to 0 \quad \text{as } z \to 1, \tag{33a}$$

$$u, \varepsilon \to 0 \quad \text{as } z \to 0. \tag{33b}$$

In order to solve this boundary-value problem, the kinematic eddy viscosity is specified [4] as

$$\varepsilon(z;\ R) = \varepsilon(z;\ R_\tau) = R_\tau M(z) N(\zeta), \tag{34}$$

where $\zeta = R_\tau z$. The functions $M(z)$ and $N(\zeta)$ have the following asymptotic behaviours:

$$M(z) = \kappa z M_0(z) \to \kappa z(1 + \kappa_2 z^2 + \ldots) \to 0 \quad \text{as } z \to 0, \tag{35a}$$

$$M(z) \to M_1 \text{ (algebraically) as } z \to 1; \tag{35b}$$

$$N(\zeta) = N_0(\zeta) \to C\zeta^2 + \ldots \to 0 \quad \text{as } \zeta \to 0, \tag{36a}$$

$$N(\zeta) = N_0(\zeta) \to 1 \text{ (exponentially)} \quad \text{as } \zeta \to \infty. \tag{36b}$$

Here, $\kappa, \kappa_2, \ldots, M_1, C, \ldots$ are constants of order unity.

For the outer layer, based on (4), (31) can be written as

$$t = -\frac{\varepsilon}{R_\tau}\frac{df}{dz}, \tag{37a}$$

where, from (35) and (36),

$$\frac{\varepsilon}{R_\tau} = \frac{\varepsilon(z;\, R_\tau)}{R_\tau} \approx \kappa z\, M_0(z). \tag{37b}$$

In turn, the combination of (37) with (27) and (28) produces

$$t_0 = (1 - z) = -\kappa z\, M_0(z)\frac{df_0}{dz}, \tag{38a}$$

$$t_1 = \frac{df_0}{dz} = -\kappa z\, M_0(z)\frac{df_1}{dz}, \ldots, \tag{38b}$$

such that the boundary-value problems for $f_0(z), f_1(z), \ldots$ are

$$\frac{df_0}{dz} = -\frac{(1 - z)}{[\kappa z M_0(z)]}, \; f_0 \to 0 \quad \text{as } z \to 1; \tag{39a}$$

$$\frac{df_1}{dz} = \frac{(1 - z)}{[\kappa z\, M_0(z)]^2}, \; f_1 \to 0 \quad \text{as } z \to 1; \ldots. \tag{39b}$$

The integration of (39) leads to

$$f_0(z) = \frac{1}{\kappa}[f_{00}(z) - f_{01}(z)]:$$

$$f_{00}(z) = \frac{[z - \log z]}{M_0(z)}, f_{01}(z) = (\kappa/M_1) + \int_z^1 \frac{M_0'(k)}{M_0(k)} f_{00}(k)\, dk: \tag{40a}$$

$$f_1(z) = -\frac{1}{\kappa^2}[f_{10}(z) - f_{11}(z)]:$$

$$f_{10}(z) = \frac{[(1/z) + \log z]}{M_0^2(z)}, f_{11}(z) = \left(\frac{\kappa}{M_1}\right)^2 + 2\int_z^1 \frac{M_0'(k)}{M_0(k)} f_{10}(k)\, dk; \ldots. \tag{40b}$$

Thus, for the eddy-viscosity closure model, the constants of (14a) are $A_{01} = 1/K_0 = 1/\kappa$, $A_0 = -f_{01}(0)/\kappa$, $W_0 = 1/\kappa, \ldots$, $J_1 = -1/\kappa^2$, $A_{11} = 1/K_1 = 1/\kappa^2$, $A_1 = f_{11}(0)/\kappa^2, \ldots$.

Correspondingly, for the inner layer,

$$s = \varepsilon \frac{\mathrm{d}g}{\mathrm{d}\zeta}, \tag{41a}$$

where

$$\varepsilon = \varepsilon(\zeta; R_\tau) \cong \kappa\zeta\left(1 + \frac{1}{R_\tau^2}\kappa_2\zeta^2 + \ldots\right)N_0(\zeta). \tag{41b}$$

Thus, the boundary-value problems for $g_0(\zeta), g_1(\zeta), \ldots$ are

$$\frac{\mathrm{d}g_0}{\mathrm{d}\zeta} = \frac{1}{[1+\kappa\zeta N_0(\zeta)]}, \; g_0 \to 0 \text{ as } \zeta \to 0; \tag{42a}$$

$$\frac{\mathrm{d}g_1}{\mathrm{d}\zeta} = -\frac{\zeta}{[1+\kappa\zeta N_0(\zeta)]}, \; g_1 \to 0 \text{ as } \zeta \to 0; \ldots. \tag{42b}$$

The solutions of these equations are

$$g_0(\zeta) = \frac{1}{\kappa}[g_{00}(\zeta) + g_{01}(\zeta)]:$$

$$g_{00}(\zeta) = \log(1+\kappa\zeta), \; g_{01}(\zeta) = \kappa^2 \int_0^\zeta \frac{v[1-N_0(v)]\,\mathrm{d}v}{(1+\kappa v)[1+\kappa v N_0(v)]}; \tag{43a}$$

$$g_1(\zeta) = -\frac{1}{\kappa^2}[g_{10}(\zeta) + g_{11}(\zeta)]:$$

$$g_{10}(\zeta) = [\kappa\zeta - \log(1+\kappa\zeta)], \; g_{11}(\zeta) = \kappa^3 \int_0^\zeta \frac{v^2(1-N_0(v)]\,\mathrm{d}v}{(1+\kappa v)[1+\kappa v N_0(v)]}; \ldots. \tag{43b}$$

From (43), it is seen that the constants of (14b) are $B_{01} = 1/K_0 = 1/\kappa$, $B_0 = [g_{01}(\infty) + \log\kappa]/\kappa$, $V_0 = 1/\kappa^2, \ldots$, $H_1 = -1/\kappa$, $B_{11} = 1/K_1 = 1/\kappa^2$, $B_1 = -[g_{11}(\infty) - \log\kappa]/\kappa^2, \ldots$.

For the intermediate layer, based on (22), (31) can be written as

$$\left(1 - \frac{1}{R_\tau^{1/2}}T\right) = \frac{\varepsilon}{R_\tau^{1/2}}\frac{\mathrm{d}U}{\mathrm{d}\eta}, \tag{44a}$$

where, from (35) and (36),

$$\frac{\varepsilon}{R_\tau^{1/2}} = \frac{\varepsilon(\eta; R_\tau)}{R_\tau^{1/2}} \cong \kappa\eta\left(1 + \frac{1}{R_\tau}\kappa_2\eta^2 + \ldots\right). \tag{44b}$$

The combination of (44) with (24) and (25) produces

$$dU_0/d\eta = \frac{1}{(\kappa\eta)}, \tag{45a}$$

$$dU_1/d\eta = -1 + \kappa\eta^2/(\kappa\eta)^2, \tag{45b}$$

$$dU_2/d\eta = \frac{1+\kappa\eta^2}{(\kappa\eta)^3} - \frac{\kappa_2\eta^2}{(\kappa\eta)}, \ldots. \tag{45c}$$

Integration of (45) leads to the following solutions for the velocity functions:

$$U_0 = \frac{1}{\kappa}\log\eta + U_{00}, \tag{46a}$$

$$U_1 = -\left(\frac{\eta}{\kappa} - \frac{1}{\kappa^2\eta}\right) + U_{10}, \tag{46b}$$

$$U_2 = \frac{1}{\kappa^2}\log\eta + U_{20} - \frac{1}{2}\left(\frac{\kappa_2\eta^2}{\kappa} + \frac{1}{\kappa^3\eta^2}\right), \ldots. \tag{46c}$$

Here, $U_{00}, U_{10}, U_{20}, \ldots$ are constants of integration. A comparison of (26) and (46) indicates that, for the eddy-viscosity closure model adopted here,

$$A_{01} = B_{01} = 1/K_0 = 1/\kappa,\ \frac{1}{2}(B_0 - A_0) = U_{00} = \frac{1}{2\kappa}[g_{01}(\infty) + f_{01}(0)$$
$$+\log\kappa];$$
$$W_0 = -H_1 = 1/\kappa,\ J_1 = -V_0 = -1/\kappa^2,\ U_{10} = 0;$$
$$A_{11} = B_{11} = 1/K_1 = 1/\kappa^2,\ \frac{1}{2}(B_1 - A_1) = U_{20} = -\frac{1}{2\kappa^2}[g_{11}(\infty) + f_{11}(0)$$
$$-\log\kappa],$$
$$C_0 = E_2 = \kappa_2/\kappa,\ L_2 = -D_0 = 1/\kappa^3; \ldots. \tag{47}$$

From (25) and (45), the Reynolds stress functions are

$$T_0 = 1 + \kappa\eta^2/(\kappa\eta), \tag{48a}$$

$$T_1 = -1 + \kappa\eta^2/(\kappa\eta)^2, \tag{48b}$$

$$T_2 = \frac{1+\kappa\eta^2}{(\kappa\eta)^3} - \frac{\kappa_2\eta^2}{(\kappa\eta)}, \ldots. \tag{48c}$$

Thus, for the eddy-viscosity closure model adopted,

$$S(\eta; R_\tau) = \frac{\tau(\eta; R_\tau)}{u_\tau^3(R_\tau)}$$
$$= 1 - \frac{1}{R_\tau^{1/2}}\frac{1+\kappa\eta^2}{(\kappa\eta)} + \frac{1}{R_\tau}\frac{1+\kappa\eta^2}{(\kappa\eta)^2} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{49}$$

In turn, the first derivative of S with respect to η is

$$\frac{dS}{d\eta} = \frac{1}{R_\tau^{1/2}} \frac{1-\kappa\eta^2}{\kappa\eta^2} - \frac{1}{R_\tau} \frac{2}{\kappa^2\eta^3} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{50}$$

From (49) and (50), it is seen that $\eta \to \eta_{max}(R_\tau)$, $S(\eta; R_\tau) \to S_{max}(R_\tau)$, a maximum, with

$$\eta_{max}(R_\tau) = \frac{1}{\kappa^{1/2}}\left[1 - \frac{1}{\kappa^{1/2} R_\tau^{1/2}} + O\left(\frac{1}{R_\tau}\right)\right], \tag{51a}$$

$$S_{max}(R_\tau) = 1 - \frac{2}{\kappa^{1/2} R_\tau^{1/2}} + \frac{2}{\kappa R_\tau} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{51b}$$

For the purpose of comparison, it is determined that the employment of the mixing-length-theory eddy-viscosity closure model of Bush and Fendell [5] yields the following Reynolds stress in the intermediate layer:

$$S(\eta; R_\tau) = 1 - \frac{1}{R_\tau^{1/2}} \frac{1+\kappa\eta^2}{(\kappa\eta)} + \frac{1}{R_\tau} \frac{1+\kappa\eta^2}{2(\kappa\eta)^2} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{52}$$

In turn, it is found that

$$\eta_{max}(R_\tau) = \frac{1}{\kappa^{1/2}}\left[1 - \frac{1}{2\kappa^{1/2} R_\tau^{1/2}} + O\left(\frac{1}{R_\tau}\right)\right], \tag{53a}$$

$$S_{max}(R_\tau) = 1 - \frac{2}{\kappa^{1/2} R_\tau^{1/2}} + \frac{1}{\kappa R_\tau} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{53b}$$

From the preceding, it is seen that the difference in the Reynolds stress of the zero-equation and mixing-length-theory eddy-viscosity closure models is of $O(1/R_\tau)$; the difference in the location of the maximum Reynolds stress of the two models is of $O(1/R_\tau^{1/2})$.

4. Results and discussion

The asymptotic analysis presented shows that turbulent channel flow consists of three layers: an outer layer, an inner layer, and an intermediate layer. The outer and inner layers correspond to the outer and inner limits of classical theory and have the length scales h^* and v^*/u_τ^*, respectively. The intermediate layer corresponds to the distinguished intermediate limit of the classical theory and has a length scale of $(h^* v^*/u_\tau^*)^{1/2}$. (It is seen that the intermediate-layer length scale is the geometric mean of the outer-layer and inner-layer length scales.) This distinguished intermediate layer is identified as being the one in which the turbulent and laminar contributions to the difference of the Reynolds stress from the wall stress are of the same order of magnitude.

The present analysis quantitatively yields the magnitude and location of the maximum of the Reynolds stress within the intermediate layer, in the process of determining its asymptotic behaviour in this layer. Further, through this three-layer analysis, it is found that the mean velocity achieves a value that is one-half of its centreline value within this intermediate layer, in the process of determining its asymptotic behaviour in the layer.

The intermediate-layer velocity distribution, i.e., $(u-\frac{1}{2})/u_\tau$ as a function of η and R_τ, obtained from the experimental data of Laufer [6] is displayed in figure 1. Also displayed in this figure is the velocity distribution in this layer as determined from §2 of the present analysis,

$$\frac{(u-\frac{1}{2})}{u_\tau}=\left[\frac{1}{K_0}\log\eta+\frac{1}{2}(B_0-A_0)\right]+O(R_\tau^{-1/2}), \tag{54}$$

with $K_0 = 0.41$ and $B_0 = 5.0$, $A_0 = 0.65$, such that $1/K_0 = 2.4$, $\frac{1}{2}(B_0 - A_0) = 2.2$. The figure shows that a substantial logarithmic region with universal slope and intercept does, in fact, exist.

The determination of the friction velocity $u_\tau(R)$ and the turbulent Reynolds number $R_\tau(R)$ is independent of whether a three-layer or a two-layer picture of turbulent channel flow is considered. It is found that

$$u_\tau \cong \frac{K_0}{\log R}\left[1+\frac{\log\log R-\{K_0(A_0+B_0)+\log K_0\}}{\log R}+\ldots\right]\to 0; \tag{54a}$$

$$R_\tau \cong \frac{K_0 R}{\log R}\left[1+\frac{\log\log R-\{K_0(A_0+B_0)+\log K_0\}}{\log R}+\ldots\right]\to \infty \tag{54b}$$

in the limit R, $\log R\to\infty$, which establishes the validity of the asymptotic analyses presented.

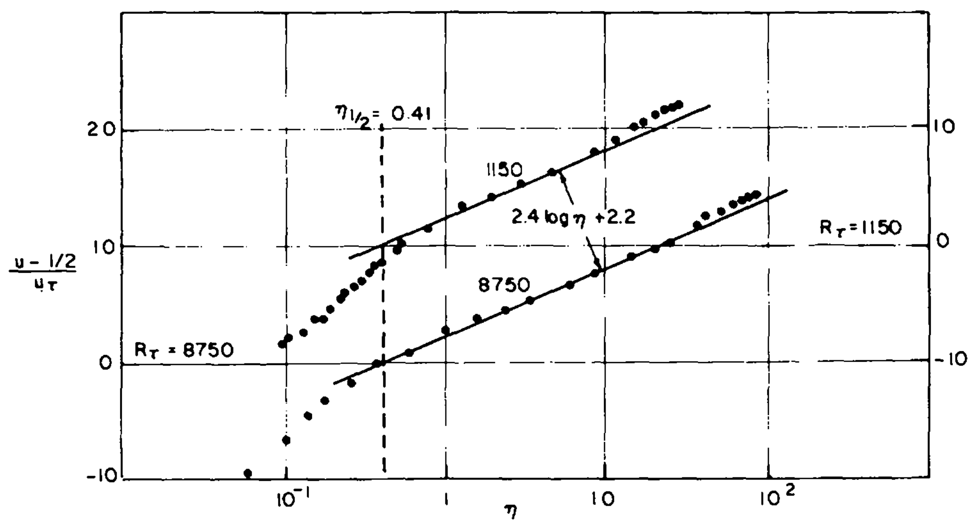

Figure 1. Intermediate-layer distribution as a function of η and R_τ (Laufer [6]).

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