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Degenerated nonlinear hyperbolic equation with discontinuous coefficients

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Abstract. The partial differential equations with discontinuous coefficients have been extended in various directions by a number of authors [2], [3], [11], [13], [14], [16]. This paper deals with a mixed problem for a degenerated nonlinear hyperbolic equation with discontinuous coefficients.

Keywords. Partial differential equation; discontinuous coefficients; nonlinear hyperbolic equation.

1. Statement of the problem and notations

Let $Q = D \times [0, T]$ where D is a bounded domain in R^n with a smooth boundary Γ, Γ_1 separates D into two domains D_1 and D_2, $\gamma = \Gamma_1 \times [0, T]$, $S = \Gamma[0, T]$, $Q_1 = D_1[0, T]$ and $Q_2 = D_2[0, T]$.

Consider the hyperbolic equation

$$f(t)[u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j}] + \sum_{i=1}^{n} b_i(x, t)u_{x_i} = F(x, t, u) \tag{1}$$

with initial conditions

$$u(x, 0) = u_t(x, 0) = 0, \tag{2}$$

boundary condition

$$u(x, t)|_S = 0, \tag{3}$$

and jumping conditions

$$u(x, t)|_{\gamma+0} = u(x, t)|_{\gamma-0},$$

$$K_1(x)\frac{\mathrm{d}u}{\mathrm{d}N}\bigg|_{\gamma+0} - K_2(x)\frac{\mathrm{d}u}{\mathrm{d}N}\bigg|_{\gamma-0} = k_3 u, \tag{4}$$

where $x = (x_1, \ldots, x_n)$, k_3 is a positive constant,

$$K(x) = \begin{cases} K_1(x), & x \in D_1 \\ K_2(x), & x \in D_2 \end{cases},$$

$$\frac{d}{dN} = \sum_{i,j=1}^{n} a_{ij} \cos(\nu, x_i) \frac{\partial}{\partial x_j},$$

and ν is the exterior unit normal of D_1.

For studying the problem (1)–(4) we have to define the following spaces; $W_2^1(Q) = W_2^1(Q_1) + W_2^1(Q_1)$ is a Sobolev space (see [1]); $E(Q)$ is a set of functions defined in Q and satisfy

$$\iint_Q \frac{\Phi^2(t,x)}{f(t)} < \infty.$$

We will refrain from writing the symbols dx and dxdt.

DEFINITION. The function $u(x,t)$ is called the solution of (1)–(4), if it satisfies

$$\iint_Q [K(x)u_{tt}\Phi + \sum_{i,j=1}^{n} a_{ij} u_{x_i} \Phi_{x_j} K(x) + \sum_{i,j=1}^{n} a_{ij} u_{x_i} K_{x_j} \Phi$$
$$+ \frac{1}{f(t)} \sum_{i=1}^{n} b_i u_{x_i} X\Phi - \frac{1}{f(t)} K(X)\Phi F(x,t,u)]$$
$$+ \int_0 \int_{\Gamma_1}^{T} k_3 u\Phi = 0, \tag{5}$$

and (2), (3) in L^2.

2. Assumption and results

THEOREM. Suppose that

(i) $f(t)$, $t > 0$ is continuous, differentiable and for small t satisfies

$$M_1 t^{\alpha} \leqslant f(t) \leqslant M_2 t^{\alpha}; \quad M_3 t^{\alpha-1} \leqslant f'(t) \leqslant M_4 t^{\alpha-1},$$

where $0 \leqslant \alpha < 1$ M_l, $l = 1, 2, \ldots$ are positive constants.

(ii) $\Sigma_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geqslant \lambda \Sigma_{i=1}^{n} \xi_i^2, \lambda > 0$ and $\partial b_i/\partial t$ is continuous in Q_1, Q_2.

(iii) $F(x,t,u)$ is measurable with respect to x, continuous with respect to t and has bounded derivatives F_t, F_u, in Q_1, Q_2 and $F(x,t,u) \leqslant p(x,t)u + q(x,t)$ where $p, q = 0(t^{\alpha+3/2})$, $F_t = 0(t^{\alpha+1/2})$, then there exists a unique weak solution to problem (1)–(4).

Proof. Let $v_r(x)$ be a linear independent system in $W_2^1(D)$, satisfy condition (4) and compact in $W_2^1(D)$. We seek the solution of problem (1)–(4) in the form (see [9]):

$$u_m(x,t) = \sum_{r=1}^{m} c_r(t) v_r(x).$$

Substitute $u(x, t)$ with $u_m(x, t)$ in (1), multiply by $v_s(x)$ and integrate over D; the result is the following system of ordinary differential equations:

$$\begin{aligned} & f(t) \sum_{r=1}^{m} c_r''(t) \int_D K v_r v_s + f(t) \sum_{r=1}^{m} c_r(t) \int_D \sum_{i,j=1}^{n} a_{ij}(v_r)_{x_i}(v_s)_{x_j} K \\ & + f(t) \sum_{r=1}^{m} c_r(t) \int_{\Gamma_1} k_3 v_r v_s + \sum_{r=1}^{m} c_r(t) \int_D \sum_{i=1}^{n} b_i (v_r)_{x_i} v_s K \\ & + f(t) \sum_{r=1}^{n} c_r(t) \cdot \int_D \sum_{i,j=1}^{n} a_{ij}(v_r)_{x_i} K_{x_j} \Phi \\ & \qquad = \int_D KF\left(x, t \sum_{r=1}^{m} c_r v_r\right) v_s, \end{aligned} \tag{6}$$

$s = (1, \ldots, m)$ with initial conditions,

$$c_r(0) = c_r'(0) = 0. \tag{7}$$

The existence of the solution to the problem (6)–(7) was proved (see [6], [12]).

Now, we will prove that $u_m(x, t)$ converges to $u(x, t)$ which is the weak solution to problem (1)–(4). To do so, we will find some important *a priori* estimates. This is usually where the hard work comes in. Many interesting estimates are discussed in [5]–[8] and [15]. Let $\eta_\varepsilon(x, t)$ be an infinitely differentiable function which tends to zero in $Q - Q_\varepsilon$, $\eta_\varepsilon = 1$ in $Q_{2\varepsilon}$ and $|\eta_\varepsilon| \leqslant 1$ in $Q_{2\varepsilon} - Q_\varepsilon$, where $\varepsilon > 0$. Substitute F with F_ε, where $F_\varepsilon = \eta_\varepsilon F$ in system (6) to get the following system:

$$\begin{aligned} & f(t) \sum_{r=1}^{m} c_r'' \int_D K v_r v_s + f(t) \sum_{r=1}^{m} c_r \int_D \sum_{ij=1}^{n} a_{ij}(v_r)_{x_i}(v_s)_{x_j} K \\ & + f(t) \sum_{r=1}^{m} c_r \int_{\Gamma_1} k_3 v_r v_s + f(t) \sum_{r=1}^{m} c_r \int_D \sum_{ij=1}^{n} a_{ij}(v_r)_{x_i}(K)_{x_j} v_s \\ & + \sum_{r=1}^{m} c_r \int_D \sum_{i=1}^{n} b_i (v_r)_{x_i} v_s K = \int_D K F_\varepsilon\left(x, t, \sum_{r=1}^{m} c_r v_r\right) v_s. \end{aligned} \tag{8}$$

For convenience, we will adopt the notation of u_m to be understood as $u_{m,\varepsilon}$. Multiply (8) by $c_{s,\varepsilon}'\, 1/f(t)t^{\beta + 2\alpha}$, $0 < \beta < 1 - \alpha$; sum over s from 1 to m; integrate with respect to t from 0 to $T_1 < T$; to finally get:

$$\begin{aligned} & \int_0^{T_1} \int_D \frac{1}{t^{\beta + 2\alpha}} K(u_m)_{tt}(u_m)_t + \int_0^{T_1} \int_D \frac{1}{t^{\beta + 2\alpha}} K \sum_{ij=1}^{n} a_{ij}(u_m)_{x_i}(u_m)_{x_j} \\ & + \int_0^{T_1} \int_{\Gamma_1} \frac{1}{t^{\beta + 2\alpha}} k_3 u_m (u_m)_t + \int_0^{T_1} \int_D \frac{1}{f(t) t^{\beta + 2\alpha}} K \sum_{i=1}^{n} b_i (u_m)_{x_i}(u_m)_t \\ & = \int_0^{T_1} \int_D \frac{1}{f(t) t^{\beta + 2\alpha}} K(u_m)_t F_\varepsilon. \end{aligned} \tag{9}$$

Integrate by parts to get:

$$\frac{1}{2}\int_D \frac{1}{t^{\beta+2\alpha}} K(u_m)_t^2\bigg|_{t=T_1} + \frac{1}{2}\int_D \frac{1}{t^{\beta+2\alpha}} K \sum_{ij=1}^{n} a_{ij}(u_m)_{x_i}(u_m)_{x_j}\bigg|_{t=T_1}$$

$$+\frac{1}{2}\int_{\Gamma_1} \frac{1}{t^{\beta+2\alpha}} k_3 u_m^2\bigg|_{t=T_1} + \frac{B+2\alpha}{2}\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha+1}} K(u_m)_t^2$$

$$+\frac{\beta+2\alpha}{2}\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha+1}} K \sum_{ij=1}^{n} a_{ij}(u_m)_{x_i}(u_m)_{x_j}$$

$$+\frac{\beta+2\alpha}{2}\int_0^{T_1}\int_{\Gamma_1} \frac{1}{t^{\beta+2\alpha}} k_3 u_m^2$$

$$= \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} KF_3(u_m)_t - \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} \sum_{i=1}^{n} b_i K(u_m)_{x_i}(u_m)_t$$

$$-\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}} \sum_{ij=1}^{n} a_{ij}(u_m)_{x_i} K_{x_j}(u_m)_t. \tag{10}$$

Estimate the first term on the right hand side using the inequality $|ab| \leqslant 1/2\,a^2 + 1/2b^2$ and the theorem conditions to obtain:

$$\left|\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}f(t)} KF_\varepsilon(u_m)_t\right|$$

$$\leqslant \left|\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}f(t)} Pu_m(u_m)_t\right| + \left|\int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} (u_m)_t q\right|$$

$$\leqslant \frac{1}{2}\int_0^{T_1} \frac{1}{f(t)t^{\beta+2\alpha}} p^2 u_m^2 + \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} (u_m)_t^2$$

$$+\frac{1}{2}\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}} q^2. \tag{11}$$

From the above, it is easy to arrive at

$$\int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} p^2 u_m^2 \leqslant C\frac{T_1^5}{4(2+\beta+2\alpha)}\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha+1}} (u_m)_t^2, \tag{12}$$

where C is a constant independent of m, ε. Similarly, we can estimate the remaining terms in (10), and therefore the inequality (10) can be written as:

$$\int_0^{T_1}\int_D \frac{A}{t^{1+\beta+2\alpha}} (u_m)_t^2 + \int_0^{T_1}\int_D \frac{B}{t^{1+\beta+2\alpha}} \sum_{i=1}^{n} (u_m)_{x_i}^2 + \int_0^{T}\int_{\Gamma_1} \frac{1}{t^{1+\beta+2\alpha}} u^2$$

$$\leqslant \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} q^2, \tag{13}$$

where

$$A = \frac{\beta+2\alpha}{2} - \frac{\text{cnt}}{2\phi} - \frac{\text{cnt}}{2} - \frac{t}{\phi} - \frac{CT_1^5}{8(2+\beta+2\alpha)},$$

and $$B = \frac{\beta + 2\alpha}{2}\lambda - \frac{Ct}{2\phi} - \frac{Ct}{2},$$

choose T_1 sufficiently small such that A and B remain positive. From the integral on the right side in (13), together with the condition on $q(x, t)$ we find (see [4]):

$$\int_0^{T_1}\int_D \frac{1}{t^{1+\beta+2\alpha}}(u_m)_t^2 < C,$$

$$\int_0^{T_1}\int_D \frac{1}{t^{1+\beta+2\alpha}}\sum_{i=1}^{n}(u_m)_{x_i}^2 < C,$$

$$\int_0^{T_1}\int_D \frac{1}{t^{1+\beta+2\alpha}}u_m^2 < C. \tag{14}$$

Multiply (8) by $1/f(t)$; differentiate with respect to t and multiply the result by $C''_{s,\varepsilon}$ $1/t^{\beta}$; sum over s from 1 to m and integrate with respect to t from 0 to $T_1 < T$ to get:

$$\begin{aligned}
&\int_0^{T_1}\int_D \frac{1}{t^{\beta}}K(u_m)_{tt}(u_m)_t + \int_0^{T_1}\int_D \frac{1}{t^{\beta}}K\sum_{ij=1}^{n}(u_m)_{x_i}(u_m)_{x_j} \\
&+ \int_0^{T_1}\int_{\Gamma_1} \frac{1}{t^{\beta}}k_3(u_m)_t(u_m)_{tt} + \int_0^{T_1}\int_D \frac{1}{t^{\beta}f(t)}K\sum_{i=1}^{n}(b_i)_t(u_m)_{x_i}(u_m)_t \\
&+ \int_0^{T_1}\int_D \frac{1}{t^{\beta}f(t)}K\sum_{i=1}^{n}b_iu_m(u_m)_{tt} - \int_0^{T_1}\int_D \frac{f'(t)}{t^{\beta}f^2(t)}K\sum_{i=1}^{n}b_i(u_m)_{x_i}(u_m)_{tt} \\
&+ \int_0^{T_1}\int_D \frac{1}{t^{\beta}}\sum_{ij=1}^{n}a_{ij}(K)_{x_j}(u_m)_{tx_i}(u_m)_{tt} \\
&= \int_0^{T_1}\int_D \frac{1}{t^{\beta}f(t)}(F_\varepsilon)_tK(u_m)_{tt} + \int_0^{T_1}\int_D \frac{1}{t^{\beta}f(t)}(F_\varepsilon)_{u_m}(u_m)_t(u_m)_{tt} \\
&- \int_0^{T_1}\int_D \frac{f'(t)}{t^{\beta}f^2(t)}KF_\varepsilon(u_m)_{tt}.
\end{aligned} \tag{15}$$

From (15), the fact that $|F_\varepsilon| \leq |F|$ and the theorem conditions, we get the next *a priori* estimates:

$$\int_0^{T_1}\int_D \frac{1}{t^{1+\beta}}(u_m)_{tt}^2 < C,$$

$$\int_0^{T_1}\int_D \frac{1}{t^{1+\beta}}(u_m)_{tx_i}^2 < C,$$

$$\int_0^{T_1}\int_D \frac{1}{t^{1+\beta}}(u_m)_t^2 < C. \tag{16}$$

Using the *a priori* estimates (14) and (16) we can choose subsequences $(u_{mk})_t$, $(u_{mk})_{x_i}$, $(u_{mk})_{tt}$, $(u_{mk})_{tx_i}$ converges to u_t, u_{xi}, u_{tt}, u_{tx_i} in L_2. Regarding the embedding theorem (see [1]) u_m converges strongly to u.

To prove the uniqueness, assume u and $\bar{u}$ are two different solutions and $w = u - \bar{u}$. Using the *a priori* estimates in (14) and (16), we clearly get $w = 0$, i.e., $u = \bar{u}$. Thus, the existence and uniqueness have been proved.

REMARK. Similarly, we can prove the existence theorem for problems (1)–(4), in case of $F(x, t, u) = a(x, t)|u|$ where $\sigma \leqslant n + 2/n - 1$ in place of condition 3 (see [10]).

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