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Minimum degree of a graph and the existence of k-factors

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Abstract. Let G be a graph and k a positive integer such that (i) $k|V(G)|$ is even; (ii) $\delta(G) \geqslant \frac{1}{2}[|V(G)|]$, and (iii) $|V(G)| \geqslant 4k-5$. Then G has a k-factor.

Keywords. Minimum degree; graph; k-factors.

All graphs considered are simple and finite. We refer the reader to [1] for standard graph theoretic terms not defined in this paper.

Let k be a positive integer. Then we say that a graph G has a k-factor if it has a k-regular spanning subgraph. If $X, Y \subseteq V(G)$, then $e(X, Y)$ denotes the number of edges having one end-vertex in X and the other in Y. Tutte [4] proved the following theorem.

Tutte's k-factor theorem: A graph G has a k-factor if and only if

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G\backslash D}(x)) \leqslant k|D|$$

for all $D, S \subseteq V(G)$, $D \cap S = \varnothing$, where $q_G(D, S; k)$, denotes the number of components C of $(G\backslash D)\backslash S$ such that $k|V(C)| + e(C, S)$ is odd.

In the same paper he also showed that

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G\backslash D}(x)) - k|D| \equiv k|V(G)| \pmod 2. \tag{1}$$

The following theorem of Dirac [2] is well known.

THEOREM 1. Let G be a graph such that

(i) $|V(G)| \geqslant 3$

(ii) $\delta(G) \geqslant \frac{|V(G)|}{2}$

Then G has a Hamilton cycle.

We will state now a result due to Nash-Williams [3] which considerably sharpens Dirac's theorem for large graphs.

Theorem 2. Let G be a graph such that $\delta(G) \geqslant \frac{1}{2}[|V(G)|]$. Then there exists a set of $[5(|V(G)| + \theta_{|V(G)|} + 10)/224]$ edge-disjoint Hamiltonian cycles of G, where $\theta_{|V(G)|}$ denotes 0 if $|V(G)|$ is even and 1 if $|V(G)|$ is odd.

Theorem 2 has the following corollary.

Corollary 1. Let G be a graph. If $\delta(G) \geqslant \frac{1}{2}[|V(G)|]$ and $|V(G)|$ is even, then G has $[10(|V(G)| + 10)/224]$ edge-disjoint 1-factors.

Proof. It follows immediately from theorem 2 if we consider a Hamilton cycle as a union of two edge-disjoint 1-factors.

The main theorem of this paper is as follows.

Theorem 3. Let G be a graph and k a positive integer such that

(i) $k|V(G)|$ is even

(ii) $\delta(G) \geqslant \dfrac{|V(G)|}{2}$, and

(iii) $|V(G)| \geqslant 4k - 5$.

Then G has a k-factor.

Lemma 1. Let G be a graph and k a positive integer where $k|V(G)|$ is even. Suppose that there exists a pair of disjoint subsets D and S of $V(G)$ such that

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G\backslash D}(x)) > k|D|.$$

If $|D \cup S|$ is maximal with respect to the above inequality, then
(i) $|N_{G\backslash D}(u)| \geqslant k + 1$ and $|N(u) \cap S| \leqslant k - 1$ for every vertex u of $(G\backslash D)\backslash S$.
(ii) $|V(C)| \geqslant 3$ for every component C of $(G\backslash D)\backslash S$.

Proof. Let C be a component of $(G\backslash D)\backslash S$ and let u be a vertex of C. Suppose that $|N_{G\backslash D}(u)| \leqslant k$. Define $S' = S \cup \{u\}$.
Then since

$$q_G(D, S'; k) \geqslant q_G(D, S; k) - 1, \sum_{x \in S'} (k - d_{G\backslash D}(x)) \geqslant \sum_{x \in S} (k - d_{G\backslash D}(x))$$

and by using (1)

$$q_G(D, S'; k) + \sum_{x \in S'} (k - d_{G\backslash D}(x)) > k|D|$$

which contradicts the maximality of $|D \cup S|$. Thus $|N_{G\backslash D}(u)| \geqslant k + 1$.

Now suppose that $|N(u) \cap S| \geqslant k$. Define $D' = D \cup \{u\}$. Then

$$q_G(D', S; k) \geqslant q_G(D, S; k) - 1$$

and

$$\sum_{x \in S} (k - d_{G \setminus D'}(x)) \geqslant \sum_{x \in S} (k - d_{G \setminus D}(x)) + k.$$

Thus

$$q_G(D', S; k) + \sum_{x \in S} (k - d_{G \setminus D'}(x)) > k\,|D'|$$

which contradicts again the maximality of $|D \cup S|$. Thus $|N(u) \cap S| \leqslant k-1$. (ii) It follows immediately from (i) since $|N_{G \setminus D}(u)| - |N(u) \cap S| = |N(u) \cap V(C)|$ and thus $|N(u) \cap V(C)| \geqslant 2$.

LEMMA 2. If G is Hamiltonian, then for every non-empty proper subset X of $V(G)$,

$$\omega(G \setminus X) \leqslant |X|.$$

Proof. Let C be a Hamilton cycle of G. Then, for every non-empty proper subset X of $V(G)$

$$\omega(C \setminus X) \leqslant |X|.$$

But $C \setminus X$ is a spanning subgraph of $G \setminus X$ and so

$$\omega(G \setminus X) \leqslant \omega(C \setminus X).$$

Thus the lemma follows.

Proof of Theorem 3.
Suppose that G does not have a k-factor. Then by Tutte's k-factor theorem there exists $D, S \subseteq V(G)$, $D \cap S = \varnothing$ such that

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G \setminus D}(x)) > k\,|D|. \tag{2}$$

If $|D \cup S|$ is maximal with respect to (2), then by Lemma 1 (ii), for every component C of $(G \setminus D) \setminus S$,

$$|V(C)| \geqslant 3. \tag{3}$$

Let $|V(G)| = n$, $\delta = \min_{x \in S} \{d_{G \setminus D}(x)\}$, $W = (G \setminus D) \setminus S$.

Then since $k\,|V(G)|$ is even, using (1), (2) becomes

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G \setminus D}(x)) \geqslant k\,|D| + 2.$$

Thus

$$\omega((G \setminus D) \setminus S) + (k - \delta)\,|S| \geqslant k\,|D| + 2. \tag{4}$$

If $S = \varnothing$,

$$\omega(G \setminus D) \geqslant k\,|D| + 2. \tag{5}$$

But by theorem 1, G is Hamiltonian. So by lemma 2, $\omega(G\setminus X) \leqslant |X|$ for every non-empty subset X of $V(G)$, which contradicts (5). Thus $S \neq \varnothing$. Now if $\delta \geqslant k+1$,

$$\omega((G\setminus D)\setminus S) \geqslant k|D| + |S| + 2. \tag{6}$$

Since G is Hamiltonian and by using again lemma 2, we get a contradiction. Hence $0 \leqslant \delta \leqslant k$.

Suppose that $0 \leqslant \delta \leqslant k-1$. Then from (4), we have

$$|S| \geqslant \frac{k|D| + 2 - \omega((G\setminus D)\setminus S)}{(k-\delta)}$$

and since $V(G) = D \cup S \cup V(W)$,

$$|S| \geqslant \frac{k(n - |S| - |V(W)|) + 2 - \omega((G\setminus D)\setminus S)}{(k-\delta)}.$$

Thus

$$|S| \geqslant \frac{kn + 2 - (k+1)|V(W)|}{(2k-\delta)}. \tag{7}$$

Let u be an element of S such that $d_{G\setminus D}(u) = \delta$. Then since

$$|D| + \delta \geqslant d(u) \geqslant \frac{n}{2}, \tag{8}$$

$$n - |S| - |V(W)| + \delta \geqslant \frac{n}{2}$$

hence

$$\delta + \frac{n}{2} - |V(W)| \geqslant |S|. \tag{9}$$

From (7) and (9)

$$\frac{n}{2} + \delta - |V(W)| \geqslant \frac{(kn+2)}{(2k-\delta)} - \frac{(k+1)}{(2k-\delta)}|V(W)|,$$

and since

$$\frac{(k+1)}{(2k-\delta)} \leqslant 1 \text{ for } \delta \leqslant k-1,\ \frac{n}{2} + \delta \geqslant \frac{kn+2}{2k-\delta}.$$

Thus

$$4\delta k - n\delta - 2\delta^2 \geqslant 4. \tag{10}$$

Clearly from (10), we have that $\delta \neq 0$, so

$$4k - 2\delta - \frac{4}{\delta} \geqslant n. \tag{11}$$

But $4k - 2\delta - \dfrac{4}{\delta} \leqslant 4k - 6$ when $\delta \geqslant 1$. This contradicts the hypothesis that $n \geqslant 4k - 5$.

Now suppose that $\delta = k$. Then (4) becomes

$$\omega((G\backslash D)\backslash S) \geqslant k\,|D| + 2. \tag{12}$$

By (8),

$$|D| \geqslant \frac{n}{2} - k \text{ so } \omega((G\backslash D)\backslash S) \geqslant \frac{kn}{2} - k^2 + 2, \text{and by (3)}, |V(W)| \geqslant \frac{3kn}{2} - 3k^2 + 6.$$

Since $V(G) = D \cup S \cup V(W)$ and $S \neq \varnothing$

$$n = |D| + |S| + |V(W)| \geqslant \frac{3kn}{2} - 3k^2 + 6 + \frac{n}{2} - k + 1.$$

Thus

$$\frac{6k^2 + 2k - 14}{3k - 1} \geqslant n.$$

But

$$4k - 5 > \frac{6k^2 + 2k - 14}{3k - 1} \geqslant n. \tag{13}$$

So (13) contradicts the hypothesis that $|V(G)| \geqslant 4k - 5$. Hence G has a k-factor.

The condition that $\delta(G) \geqslant \frac{1}{2}[|V(G)|]$ is clearly necessary because the complete bipartite graph $K_{m,m-2}$ does not possess a k-factor and $\delta(K_{m,m-2}) = m - 2 = \frac{1}{2}[|V(G)|] - 1$. The condition that $|V(G)| \geqslant 4k - 5$ is also necessary. We will describe a graph G which does not have a k-factor although $|V(G)| = 4k - 6$ and $\delta(G) = \frac{1}{2}[|V(G)|]$. We form G by joining every vertex of the complete graph $K_{2(k-2)}$ to all the vertices of $(k-1)$ copies of K_2. Let $X = \{u_1, v_1, u_2, v_2, \ldots, u_{k-1}, v_{k-1}\}$ where u_i, v_i denotes the two vertices of each K_2 and $Y = V(K_{2(k-2)})$.

Then if $D = Y$ and $S = X$

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G\backslash D}(x)) > k\,|D|$$

since $q_G(D, S; k) = 0$, $\sum_{x \in S} (k - d_{G\backslash D}(x)) = 2(k-1)^2$ and $k\,|D| = 2k(k-2)$. Thus G does not have a k-factor. Moreover $|V(G)| = |X| + |Y| = 2(k-1) + 2(k-2) = 4k - 6$ and $\delta(G) = 2k - 3$.

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