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Schiffer variation of complex structure and coordinates for Teichmüller spaces

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Abstract. Schiffer variation of complex structure on a Riemann surface X_0 is achieved by punching out a parametric disc $\bar{D}$ from X_0 and replacing it by another Jordan domain whose boundary curve is a holomorphic image of $\partial\bar{D}$. This change of structure depends on a complex parameter ε which determines the holomorphic mapping function around $\partial\bar{D}$.

It is very natural to look for conditions under which these ε-parameters provide local coordinates for Teichmüller space $T(X_0)$, (or reduced Teichmüller space $T^{\#}(X_0)$). For compact X_0 this problem was first solved by Patt [8] using a complicated analysis of periods and Ahlfors' [2] τ-coordinates.

Using Gardiner's [6], [7] technique, (independently discovered by the present author), of interpreting Schiffer variation as a quasi conformal deformation of structure, we greatly simplify and generalize Patt's result. Theorems 1 and 2 below take care of all the finite-dimensional Teichmüller spaces. In Theorem 3 we are able to analyse the situation for infinite dimensional $T(X_0)$ also. Variational formulae for the dependence of classical moduli parameters on the ε's follow painlessly.

Keywords. Riemann surfaces; Teichmüller spaces; quasiconformal mappings.

1. Introduction

We are interested in making explicit variations of complex-structure on a Riemann surface X_0 so that the variation parameters provide complex-analytic and real-analytic coordinates (respectively) on the Teichmüller space $T(X_0)$ and reduced Teichmüller space $T^{\#}(X_0)$. Such variations, obtained by changing the complex structure on disjoint discs in X_0, were introduced by Schiffer, see [9].

In two interesting papers Gardiner [6], [7], showed that Schiffer's variation can be achieved by quasiconformal (q.c.) deformation, and that Schiffer's variational formulae are equivalent to q.c. variational formulae involving appropriate Beltrami differentials. The technique is applied in the present article to give a very general and simple solution to the coordinatisation problem for moduli space mentioned at the beginning.

Instead of using periods and Ahlfors' τ-coordinates as in Patt [8], we use Bers coordinates for our analysis. We prove that if Schiffer variations are carried out independently in d suitably-chosen disjoint discs on X_0, with arbitrarily specified boundaries and/or almost-arbitrarily specified centres, then the ε-parameters provide local complex-analytic coordinates for $T(X_0)$ around X_0. See Theorem 1. Here d is the (finite) complex dimension of $T(X_0)$.

Even when X_0 is not of finite conformal type, but the reduced space $T^{\#}(X_0)$ is a d-dimensional real-analytic manifold, we can use the real parts of the ε's as local real-analytic coordinates for $T^{\#}(X_0)$, (Theorem 2).

In §5 we have a theorem for infinite dimensional Teichmüller spaces using a countable family of discs for variation of structure on X_0. That such an analysis is possible testifies again to the power of interpreting Schiffer variation as q.c. deformation.

2. Preliminaries

Let X_0 be an arbitrary Riemann surface and t a (holomorphic) local parameter around a point $p \in X_0$. Without loss of generality we assume that $t(p) = 0$ and that the image of t contains a disc of radius greater than one around 0. We call the open domain $D = t^{-1}(\Delta)$ a parametric unit disc on X_0 with centre p, (where Δ is the open unit disc in $\mathbf{C}$).

We denote the boundary of D by $\partial D = \beta = \{x \in X_0 : |t(x)| = 1\}$. Note that, owing to the profusion of conformal Riemann mappings, the Jordan curve β on X_0 can be chosen with a great degree of arbitrariness.

A new Riemann surface, X_ε^*, will be defined by making the following 'Schiffer variation' of complex structure on the disc D. Indeed,

$$t^*(t) = t + \frac{\varepsilon}{t}, \quad \varepsilon \in \mathbf{C} \tag{1}$$

is a holomorphic function in an annular neighbourhood of β and maps β to a Jordan curve β^* in the t^*-plane for small ε. The Jordan domain with boundary β^* is denoted D^*; D^* is of course a bounded simply-connected region of the t^*-plane.

X_ε^* is obtained now by removing D from X_0 and filling in the hole with $\bar{D}^*$ (bar denotes closure)—the boundary identification being given by (1). So x on β is identified with $t^*(t(x))$ on β^*. On $\bar{D}^*$ we use t^* as a holomorphic coordinate, and on $X_\varepsilon^* - \bar{D}^* = X_0 - \bar{D}$ we use the original coordinates from X_0. Note that on $\partial\bar{D}^* = \beta^* \subset X_\varepsilon^*$ we may use either t or t^* as holomorphic coordinates. Clearly X_ε^* becomes a well-defined Riemann surface topologically equivalent (but in general *not* conformally equivalent) to X_0. Obviously, if X_0 is topologically marked (by a choice of generators for $\pi_1(X_0)$) so is X_ε^*.

From now on let $X_0 = U/G$, G a torsion-free Fuchsian group operating on the upper half-lane, U, or on the unit disc Δ, (whichever is convenient). We recall briefly relevant points regarding the Teichmüller space $T(X_0) = T(G)$ and reduced Teichmüller space $T^{\#}(X_0) = T^{\#}(G)$.

For this purpose let κ denote the holomorphic cotangent bundle of X_0. A Beltrami differential μ on X_0 is a L^∞ section of the bundle $\bar{\kappa} \otimes \kappa^{-1}$ over X_0, so it is represented in local parameters on X_0 by

$$\mu = \mu(z)\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}, \; \|\mu\|_\infty < \infty.$$

We call the complex Banach space of Beltrami differentials $L^\infty(X_0) = L^\infty(G) = L^\infty(\bar\kappa \otimes \kappa^{-1})$. The open unit ball in $L^\infty(X_0)$ is denoted $M(X_0) = M(G)$ and is called the Banach manifold of proper Beltrami differentials.

Any $\mu \in M(X_0)$ defines a 'Riemannian metric' $\lambda|dz + \mu d\bar z|$, whose conformal class gives X_0 a conformal (= complex) structure. Indeed, local homeomorphic solutions of the Beltrami equation $\bar\partial w = \mu \cdot \partial w$, with the coefficient μ, provide holomorphic local coordinates for the new complex structure. X_0 with this complex structure is denoted X_μ.

Now, if $\varphi: X_0 \to Y$ is a q.c. homeomorphism onto another Riemann surface Y, then the complex dilatation of φ, denoted $(\mu(\varphi)(= \bar\partial\varphi/\partial\varphi)$, forms a proper Beltrami differential on X_0. Indeed φ becomes biholomorphic from $X_{\mu(\varphi)}$ to Y.

We define $\mu, \nu \in M(X_0)$ to be equivalent $(\sim)$ if there is a biholomorphism between X_μ and X_ν homotopic to the identity where throughout the homotopy the ideal boundary of X_0 remains pointwise fixed. We define μ and ν to be weakly equivalent $(\tilde{\#})$ if the condition for this homotopy on the ideal boundary is dropped. We set

$$T(X_0) = M(X_0)/\sim \text{ and } T^\#(X_0) = M(X_0)/\tilde{\#}.$$

Both spaces parametrize marked Riemann surfaces which are quasiconformally homeomorphic to X_0. We denote the natural projections from $M(X_0)$ to $T(X_0)$ and $T^\#(X_0)$ by Φ and $\Phi^\#$ respectively. $T(X_0)$ itself of course projects onto the (usually smaller) space $T^\#(X_0)$.

If X_0 is of finite type (g, k), (i.e. a compact genus g surface with k deleted (or distinguished) points), then $T(X_0) \equiv T^\#(X_0)$ inherits a (unique) complex structure of a $(3g - 3 + k)$-dimensional complex mainfold making Φ a holomorphic submersion. If X_0 is not of finite type but $G = \pi_1(X_0)$ is finitely generated, then the Schottky double $\hat X_0$ of X_0 is of finite type (g', k'), and $T^\#(X_0)$ embeds ('by doubling') as a real analytic manifold of real dimension $(3g' - 3 + k')$ in $T(\hat X_0)$—the latter being a complex manifold of the same number of complex dimensions. These are the only situations where $T(X_0)$ or $T^\#(X_0)$ are finite-dimensional (Earle [4]).

Let $Q(X_0)$ denote the integrable holomorphic quadratic differentials on X_0, i.e. the holomorphic sections ψ of $\kappa \otimes \kappa$ over X_0 such that the L^1-norm is finite:

$$\|\psi\| = \iint_{X_0} |\psi(z)| dx dy < \infty. \tag{2}$$

Of course $Q(X_0) \subset L^1(\kappa \otimes \kappa)$, and this latter Banach space has the usual duality-pairing with $L^\infty(\bar\kappa \otimes \kappa^{-1}) = L^\infty(X_0)$ by

$$\langle \psi, \mu \rangle = \iint_{X_0} \psi\mu\, dz \hat{A}\, d\bar z, \quad \psi \in L^1(\kappa \otimes \kappa),\ \mu \in L^\infty(\bar\kappa \otimes \kappa^{-1}). \tag{3}$$

In case X_0 is of type (g, k), $Q(X_0)$ is a complex vector space of dimension equal to the complex dimension of $T(X_0)$, (Riemann-Roch). In any case $T(X_0)$ is well known to be a complex Banach manifold and $T^\#(X_0)$ a real Banach manifold with Φ and $\Phi^\#$ analytic submersions. We require the following classical 'Teichmüller's Lemma' and a variant:

LEMMA 1. The kernel of the differential of Φ at $\mu = 0$ is

$$N(X_0) = Q(X_0)^{\perp} = \{v \in L^{\infty}(X_0): \langle \psi, v \rangle = 0, \text{ for all } \psi \in Q(X_0)\}.$$

Thus the holomorphic tangent space to $T(X_0)$ at X_0 is $Q(X_0)^* = L^{\infty}(X_0)/N(X_0)$.

The embedding of $T^{\#}(X_0)$ in $T(\hat{X}_0)$ is by extending $\mu \in M(X_0)$ to $\mu^{\text{ext}} \in M(\hat{X}_0)$ using the obvious reflection.

LEMMA 2. The kernel of the differential of $\Phi^{\#}$ at $\mu = 0$ is

$$N^{\#}(X_0) = \{v \in L^{\infty}(X_0): \langle \psi, v^{\text{ext}} \rangle = 0, \quad \text{for all } \psi \in Q(\hat{X}_0)\}.$$

The real-analytic tangent space to $T^{\#}(X_0)$ at X_0 is $L^{\infty}(X_0)/N^{\#}(X_0)$. $Q(\hat{X}_0)$ is the (real) direct sum of two copies of $Q^{\#}(X_0)$, where $Q^{\#}(X_0)$ comprises those integrable holomorphic quadratic differentials on X_0 which are real on the ideal boundary of X_0. Clearly, $L^{\infty}(X_0)/N^{\#}(X_0)$ is the real dual space of $Q^{\#}(X_0)$. In fact, μ in $L^{\infty}(X_0)$ acts on $Q^{\#}(X_0)$ as the linear functional

$$l_{\mu}(\psi) = \text{Re}\left\{\iint_{X_0} \psi\mu \, dxdy\right\}.$$

For Lemma 1 see Ahlfors [1], and for Lemma 2 see Earle [4, p. 60].

Suppose $T(X_0)$ is finite dimensional and $\{\mu_1, \ldots, \mu_d\}$ is a C-basis for $L^{\infty}(X_0)/N(X_0)$. Then clearly, by Lemma 1, the map from a neighbourhood of the origin in $\mathbf{C}^d$ to $T(X_0)$ which sends

$$(\tau_1, \ldots, \tau_d) \mapsto \Phi(\tau_1\mu_1 + \ldots + \tau_d\mu_d) \tag{4}$$

is (the inverse of) a holomorphic coordinate system for a neighbourhood of X_0 in $T(X_0)$. The $(\tau_1, \ldots, \tau_d)$ are called 'Bers coordinates'. An analogous statement holds for real-analytic coordinates in a finite-dimensional $T^{\#}(X_0)$ using Lemma 2.

3. Two main theorems

On a marked Riemann surface X_0 we carry out independent Schiffer variations in $n(\geqslant 1)$ disjoint parametric unit discs $D_1, \ldots, D_n$ centred at $p_1, \ldots, p_n$ with parameter t_k in D_k. $\partial D_k = \beta_k$ is mapped to β_k^* as in (1) by:

$$t_k^*(t_k) = t_k + \frac{\varepsilon_k}{t_k}.$$

The new marked Riemann surface is

$$X_\varepsilon^* = X_{\varepsilon_1, \ldots, \varepsilon_n}^* \text{ in } T(X_0).$$

The double $\hat{X}_\varepsilon^*$ of X_ε^* is an element of $T^{\#}(X_0) \subsetneq T(\hat{X}_0)$. Here ε denotes $(\varepsilon_1, \ldots, \varepsilon_n)$.

THEOREM 1. For Schiffer variation on n disjoint discs as above the map S:

$$(\varepsilon_1, \ldots, \varepsilon_n) \overset{S}{\mapsto} X_\varepsilon^* \tag{5}$$

is holomorphic from a neighbourhood of 0 in $\mathbf{C}^n$ into $T(X_0)$.

If $d =$ complex dimension of $T(X_0)$ is finite, then, given any d points $\{p_1, \ldots, p_d\}$ on X_0 it is possible to choose parametric unit discs with centres $\{p'_1, \ldots, p'_d\}$ lying in arbitrarily small neighbourhoods of the original points so that the variation parameters $(\varepsilon_1, \ldots, \varepsilon_d)$ are holomorphic coordinates for $T(X_0)$ around X_0.

Indeed, if we specify d disjoint parametric unit discs on X_0 with boundaries $\{\beta_1, \ldots, \beta_d\}$, it is possible to choose local parameters for these very discs so that the corresponding ε's again provide holomorphic local coordinates on $T(X_0)$.

The variation parameters corresponding to parametric discs centred at any $\{p_1, \ldots, p_d\}$ are local coordinates if and only if any ψ in $Q(X_0)$ that vanishes at each p_k vanishes identically.

Remark. It is noteworthy that the last statement, which is a corollary of the proof, depends only on the points p_k and *not* on the local parameters.

THEOREM 2. For Schiffer variations in any n disjoint discs on X_0 the map $S^\#$:

$$(\varepsilon_1, \ldots, \varepsilon_n) \overset{S^\#}{\mapsto} \hat{X}_\varepsilon^*, \ (\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)), \tag{6}$$

from a neighbourhood of 0 in $\mathbf{C}^n$ into $T^\#(X_0)(\subsetneq T(\hat{X}_0))$ is real-analytic.

If $d =$ real dimension of $T^\#(X_0)$ is finite, then it is possible to choose d disjoint parametric unit discs on X_0 so that the real parts of $(\varepsilon_1, \ldots, \varepsilon_d)$ provide real-analytic local coordinates for a neighbourhood of X_0 in $T^\#(X_0)$.

Once again the centres of the discs can be required to lie in arbitrarily small open regions, and/or the boundaries $\{\beta_1, \ldots, \beta_d\}$ of the variation-discs can be prescribed beforehand.

The real-parts of the variation parameters for the t_k-discs D_k, centered at p_k, $k = 1, \ldots, d$, provide local coordinates if and only if any ψ in $Q^\#(X_0)$ whose local expressions $\psi_k(t_k)\mathrm{d}t_k^2$ satisfy $\mathrm{Re}(\psi_k(0)) = 0$, (each $k = 1, \ldots, d$) identically vanishes. This time the condition depends not only on the centres of the discs but also on the local parameters.

Proof of Theorem 1

We only need to show S holomorphic with respect to each ε_j separately, so we may restrict attention to variation in one disc D with parameter t. As in Gardiner [6], we produce an explicit q.c. homeomorphism $\varphi_\varepsilon: X_0 \to X_\varepsilon^*$. In fact, let

$$t^* = \varphi_\varepsilon(t) = t + \varepsilon\bar{t} \text{ on } |t| \leqslant 1.$$

It is easy to check that φ_ε maps $\bar{D}$ onto $\bar{D}^*$ with the correct boundary identification, and φ_ε is a C^∞ diffeomorphism for $|\varepsilon| < 1$. (Note, φ_ε maps the radius vector to exp $(i\theta)$

proportionally upon the radius vector to $t^*[\exp(i\theta)]$). Thus:

$$\varphi_\varepsilon = \begin{cases} t + \varepsilon\bar{t} \text{ on } \bar{D} \\ \text{Identity on } X_0 - D \end{cases} \tag{7}$$

is clearly a marking-preserving q.c. homeomorphism of X_0 onto X_ε^*.
The complex dilatation of φ_ε is $\mu(\varphi_\varepsilon) \in M(X_0)$ where

$$\mu(\varphi_\varepsilon) = \begin{cases} \varepsilon\dfrac{\mathrm{d}\bar{t}}{\mathrm{d}t} \quad \text{on } D, \\ 0 \text{ on } X_0 - D, \end{cases} \|\mu(\varphi_\varepsilon)\|_\infty = |\varepsilon| < 1. \tag{8}$$

Since $\mu(\varphi_\varepsilon)$ evidently depends holomorphically on ε and

$$S(\varepsilon) = X_{\mu(\varphi_\varepsilon)} = \Phi(\mu(\varphi_\varepsilon))$$

we see that S is holomorphic.

Suppose now that independent variations are carried out in n disjoint parametric discs $D_1, \ldots, D_n$. We see then:

$$X^*_{\varepsilon_1,\ldots,\varepsilon_n} = \Phi(\varepsilon_1\mu_1 + \cdots + \varepsilon_n\mu_n) \tag{9}$$

where,

$$\mu_k = \begin{cases} \mathrm{d}\bar{t}_k/\mathrm{d}t_k \text{ on } D_k, \\ 0 \text{ on } X_0 - D_k \end{cases} \quad k = 1, \ldots, n. \tag{10}$$

Therefore, by definition of the Bers coordinates, $(\varepsilon_1, \ldots, \varepsilon_d)$ will be holomorphic coordinates for $T(X_0)$ precisely when the $\{\mu_1, \ldots, \mu_d\}$ given by (10) form a C-basis for $L^\infty(X_0)/N(X_0) = Q(X_0)^*$.

The special form of our Beltrami differentials in (10) shows that μ_k, as an element of $Q(X_0)^*$, is the linear functional

$$l_k(\psi) = -2i\pi\,\psi_k(0), \tag{11}$$

where $\psi = \psi_k(t_k)\mathrm{d}t_k^2$ in the parametric t_k-disc D_k. This is simply because, by the mean value theorem,

$$\langle \psi, \mu_k \rangle = \iint_{|t_k| \leqslant 1} \psi_k(t_k)\,\mathrm{d}t_k \wedge \mathrm{d}\bar{t}_k = -2i\pi\,\psi_k(0).$$

Suppose we make a change of parameter for D_k from t_k to $\tilde{t}_k$, $\tilde{t}_k$ being centred at a new point q_k within D_k. Of course, the t_k to $\tilde{t}_k$ transformation is a Möbius automorphism of the unit disc that throws 0 to $t_k(q_k)$. The linear functional $\tilde{l}_k$ in $Q(X_0)^*$, corresponding to Schiffer variation with centre q_k and $\tilde{t}_k$-disc D_k, is of course

$$\tilde{l}_k(\psi) = -2i\pi\tilde{\psi}_k(0).$$

Here $\psi = \tilde{\psi}_k(\tilde{t}_k)\mathrm{d}\tilde{t}_k^2$ in the $\tilde{t}_k$ local coordinate. But then the equality $\psi_k(t_k)\mathrm{d}t_k^2 = \tilde{\psi}_k(\tilde{t}_k)\mathrm{d}\tilde{t}_k^2$ shows that

$$\tilde{l}_k(\psi) = a\psi_k(q_k), \tag{12}$$

upto some non-zero constant a. Since non-zero multiples do not affect linear independence conditions, it is enough to find q_k in the given neighbourhoods of p_k such that the corresponding evaluations at q_k are d C-linearly independent functionals on $Q(X_0)$.

This is easy to do as follows.

Claim. For any t_k-disc D_k, and any neighbourhood A_k of the centre of D_k, the linear functionals $l_a(\psi) = \psi_k(a)$, $a \in A_k, \psi \in Q(X_0)$, span $Q(X_0)^*$.

Proof. If $\psi_k \equiv 0$ on A_k then ψ itself is identically zero. Now set

$$S_k = \{l_a : a \in A_k\}, \ k = 1, \ldots, d.$$

These are subsets of $Q(X_0)^*$ such that each one spans all of $Q(X_0)^*$. All we have to do is to choose d linearly independent vectors $\{\sigma_1, \ldots, \sigma_d\}$, with $\sigma_k \in S_k$. But this is always possible because of the following:

Fact from linear algebra. Let $S_1, \ldots, S_n$ be subsets of any vector space V such that each S_k spans a subspace of dimension at least n. Then there is a set $\{\sigma_1, \ldots, \sigma_n\}$ of n linearly independent vectors in V, σ_k being from S_k for each $k = 1, \ldots, n$.

Proof. A trivial induction on n.

We have evidently completed the proof of all assertions in Theorem 1. From the proof it is clear that both the restrictions on the positions of the centres and the fixing of the boundaries may be imposed simultaneously.

Remark. Notice that no choice of discs can make the ε's global coordinates for $T(X_0)$. This is because otherwise our formula (9) would give a global holomorphic cross-section for the projection Φ, and Earle [5] has shown that this is impossible if $d > 1$.

However, since $T(X_0)$ is arc-connected we see by a compactness argument that one can pass from any complex structure to any other by a finite series of successively applied Schiffer variations carried out in suitably chosen sets of d discs.

Proof of Theorem 2

This theorem is interesting precisely when X_0 is not of finite type but its fundamental group is finitely generated.

Clearly, the q.c. map $\varphi_\varepsilon : X_0 \to X_\varepsilon^*$ extends by reflection to a q.c. map $\hat{\varphi}_\varepsilon : \hat{X}_0 \to \hat{X}_\varepsilon^*$, and the Beltrami coefficient $\mu(\hat{\varphi}_\varepsilon) \in M(\hat{X}_0)$ is the extension by reflection of $\mu(\varphi_\varepsilon) \in M(X_0)$. Thus,

$$\hat{X}_\varepsilon^* = \Phi^\# (\mu(\varphi_\varepsilon)) \tag{13}$$

and clearly therefore, the Schiffer map $S^\#$: Neighbourhood of 0 in $\mathbf{C}^n \to T^\#(X_0)$, is real-analytic.

To prove that the real parts of $(\varepsilon_1, \ldots, \varepsilon_d)$ give real-analytic coordinates on $T^\#(X_0)$ around X_0 we are again reduced to showing that for suitable choice of discs

$\{D_1, \ldots, D_d\}$ on X_0 the Beltrami differentials μ_k of (10) form a **R**-basis for $L^\infty(X_0)/N^\#(X_0)$.

As in the proof of Theorem 1, using Lemma 2 now instead of Lemma 1, we identify the μ_k as real linear functionals l_k on $Q^\#(X_0)$, where

$$l_k(\psi) = \mathrm{Re}(\pi\psi_k(0)). \tag{14}$$

where ψ in $Q^\#(X_0)$ has the local expression $\psi_k(t_k)dt_k^2$ in the t_k-disc D_k (with centre p_k). This time a change of local parameter, even preserving the centre, can effect a non-trivial change in the corresponding functional. Indeed, l_k gets replaced by

$$l'_k(\psi) = a\pi \,\mathrm{Re}\,[\exp(i\theta)\psi_k(0)], \text{ some real } \theta, \tag{15}$$

where a is a non-zero real constant. (Again a can be ignored for purposes of **R**-linear independence.) Note that any real θ is achievable by suitable change of parameter.

To prove Theorem 2 it is clearly sufficient to demonstrate the existence of q_k in the given neighbourhoods A_k of p_k, and reals θ_k, such that the linear functionals

$$\tilde{l}_k(\psi) = \mathrm{Re}\,[\exp(i\theta_k)\cdot\psi_k(q_k)],\ k = 1, \ldots, d,$$

form a linearly independent set in $(Q^\#(X_0))^*$.

But, as before, the sets

$$S_k = \{l_{q,\theta} \in (Q^\#(X_0))^* : l_{q,\theta}(\psi) = \mathrm{Re}\,[\exp(i\theta)\psi_k(q)],\ q \in A_k,\ \theta \in \mathbf{R}\} \tag{16}$$

span all of $(Q^\#(X_0))^*$ because $\psi_k \equiv 0$ on A_k again implies $\psi \equiv 0$. So the same 'Fact from linear algebra' used in the previous proof does the needful.

All the assertions are now evident.

A question: Can one choose $[\frac{1}{2}(d+1)]$ discs on X_0 so that using d real *and* imaginary parts of the corresponding complex ε's we get real analytic coordinates for $T^\#(X_0)$?

4. Variational formulae

From our analysis Patt's variational formulae follow painlessly. As usual define the period mappings, $\pi_{ij}: T(X_0) \to \mathbf{C}$, by

$$\pi_{ij}(X_\mu) = \int_{b_j} \omega_i,$$

where $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ is the canonical homology basis on the compact genus $g(\geqslant 2)$ marked Riemann surface $X_\mu \in T(X_0)$, and $(\omega_1, \ldots, \omega_g)$ is the canonical dual basis of holomorphic 1-forms.

Applying the bilinear relations simply for differentials of the first kind, following Ahlfors [2], we can deduce Rauch's variational formula, ((17) below), for π_{ij}, in the tangent direction μ at $X_0 \in T(X_0)$ for any smooth μ. But then, by Teichmüller's Lemma (Lemma 1), the formula (17) must hold for arbitrary bounded measurable Beltrami differentials μ. This is because, by the Ahlfors-Weill section formula, any tangent direction has a *very* smooth (in fact real-analytic) Beltrami differential as representative.

$$\pi_{ij}(X_{\varepsilon\mu}) - \pi_{ij}(X_0) = \varepsilon\left[\iint_{X_0} (\omega_i \otimes \omega_j)\mu\right] + O(\varepsilon^2) \tag{17}$$

i.e.

$$d_{X_0}\pi_{ij}(d_0\Phi(\mu)) = \langle \omega_i \otimes \omega_j, \mu \rangle$$

We would like to understand the change in π_{ij} with Schiffer variation of complex structure. Let $\omega_i = \omega_i(t)\mathrm{d}t$ in the t-disc D, then we know $\mu(\varphi_\varepsilon)$ as in (8), so:

$$\begin{aligned}\pi_{ij}(X_\varepsilon^*) - \pi_{ij}(X_0) &= \varepsilon \iint_D \omega_i(t)\omega_j(t)\mathrm{d}t \wedge \mathrm{d}\bar{t} + O(\varepsilon^2) \\ &= -2i\pi\varepsilon\omega_i(0)\omega_j(0) + O(\varepsilon^2).\end{aligned} \tag{18}$$

This last result was deduced in Patt [8], (his equation (29)), as one of his central results; he uses differentials of the third kind and a complicated analysis. See Gardiner [6, p. 379] for a similar proof of a somewhat different variation for π_{ij}.

5. Schiffer variations in infinite-dimensional moduli spaces

Consider now X_0 such that $T(X_0)$ and/or $T^*(X_0)$ is infinite dimensional. Choose countably many disjoint parametric unit discs $(D_1, D_2, \ldots)$ on X_0 with corresponding Schiffer variations $(\varepsilon_1, \varepsilon_2, \ldots) = \varepsilon$. Clearly, as long as

$$\varepsilon \in l^\infty_\Delta = \text{unit ball in the Banach space } l^\infty \text{ of bounded complex sequences}$$

we get our q.c. map $\varphi_\varepsilon: X_0 \to X_\varepsilon^*$ with $\|\mu(\varphi_\varepsilon)\|_\infty \leqslant \|\varepsilon\|_\infty$. Thus we meaningfully define the Schiffer variation maps

$$\begin{aligned}&S: l^\infty_\Delta \to T(X), \text{ and}\\ &S^*: l^\infty_\Delta \to T^*(X_0) \subsetneq T(\hat{X}_0)\end{aligned} \tag{19}$$

just as before, (S^* by doubling X_ε^*).

Now, $T(X_0)$ is a Banach manifold—an open subset (*via* the Bers embedding) of the complex Banach space $B(G)$, ($X_0 = U/G$, G Fuchsian),

$$\begin{aligned}B(G) = \{\varphi \in \mathrm{Hol}(U) : \|\varphi\| = 4\|\varphi(z)y^2\|_\infty < \infty,\\ \text{and } \varphi \text{ induces a quadratic form on } X_0\}\end{aligned} \tag{20}$$

Also, $B(G)$ is known to be the dual of the separable Banach space

$$\begin{aligned}A(G) = Q(X_0) = \{\psi \in \mathrm{Hol}(U) : \iint_{U/G} |\psi| < \infty, \psi \text{ induces a quadratic}\\ \text{form on } X_0\},\end{aligned} \tag{21}$$

via the usual Weil-Petersson pairing, namely $(\psi, \varphi) = \iint\limits_{U/G} \psi(z)\overline{\varphi(z)}y^2\,\mathrm{d}x\mathrm{d}y$.

Now, from our knowledge of $\mu(\varphi_\varepsilon)$ we can actually calculate the derivative at 0 of S:

$$d_0 S : l^\infty \to Q(X_0)^* \equiv L^\infty(X_0)/Q(X_0)^\perp.$$

Indeed,

$$d_0 S(c_1, c_2, \ldots) = \left(c_1 \frac{\mathrm{d}\bar{t}_1}{\mathrm{d}t_1} + c_2 \frac{\mathrm{d}\bar{t}_2}{\mathrm{d}t_2} + \ldots\right) \bmod Q(X_0)^\perp \tag{22}$$

as is clear since $S(\varepsilon) = \Phi(\mu(\varphi_\varepsilon))$.

Consider the following bounded linear map

$$\theta : Q(X_0) \to l^1 \tag{23}$$

given by integration over the discs D_j:

$$\theta(\psi) = \left(\iint\limits_{D_1} \psi, \iint\limits_{D_2} \psi, \ldots\right) \tag{24}$$

where $\iint\limits_{D_k} \psi$ is of course $\iint\limits_{|t_k| \leq 1} \psi_k(t_k)\mathrm{d}t_k \wedge \mathrm{d}\bar{t}_k = -2i\pi\psi_k(0)$. Obviously, the operator norm $\|\theta\| \leq 1$.

THEOREM 3. The map $d_0 S$ of (22) is precisely the dual of the map θ of (24). Consequently, the Schiffer variation map S provides local holomorphic coordinates to $T(X_0)$ around X_0 if and only if θ is an isomorphism of Banach spaces.

Proof. Let $c = (c_1 c_2, \ldots) \in l^\infty$. Then c determines a Beltrami differential μ_c in $L^\infty(X_0)$ by

$$\mu_c = \begin{cases} c_1 \dfrac{\mathrm{d}\bar{t}}{\mathrm{d}t_1} \text{ on } D_1 \\ \vdots \\ c_k \dfrac{\mathrm{d}\bar{t}_k}{\mathrm{d}t_k} \text{ on } D_k \\ \vdots \\ 0 \text{ elsewhere on } X_0. \end{cases} \tag{25}$$

Clearly $\mu_c \pmod{Q(X_0)^\perp}$ is exactly $d_0 S(c)$.

Then $d_0 S(c)$, as a linear functional on $Q(X_0)$, is

$$\langle \psi, \mu_c \rangle = c_1 \iint\limits_{D_1} \psi + c_2 \iint\limits_{D_2} \psi + \ldots$$

$=$ the pairing of the l^∞-sequence c with the l^1-sequence $\theta(\psi)$.

This establishes the duality. The second statement now follows from the inverse function theorem for Banach spaces. This duality, for *arbitrary* Teichmüller spaces $T(G)$, is proved below.

Notice that in the finite-dimensional case the injectivity of θ (using d discs) was necessary and sufficient for the Schiffer parameters for the discs D_k to provide coordinates. Even for general $T(X_0)$ we see now that θ is injective if and only if each ψ in $Q(X_0)$ that vanishes at all the centres of D_k vanishes identically. This fits with the last assertion of Theorem 1.

Theorem 3 *and the Bers embedding*

The duality of Theorem 3 connects up with the Weil-Petersson pairing and the Bers embedding for *arbitrary* $T(G)$, *G a Fuchsian group with or without torsion. In this case the parametric discs D_j should be chosen within a fundamental domain for G in U.*

This general proof of $\theta^* = d_0 S$ is specially instructive since it hinges on a well-known reproducing formula which is ubiquitous in Teichmüller theory, namely,

$$\frac{12}{\pi}\iint_U \frac{\varphi(\zeta)\eta^2}{(z-\bar{\zeta})^4}\,\mathrm{d}\xi\mathrm{d}\eta = \varphi(z) \tag{26}$$

for any φ in $B(G)$ and any z in U.

Indeed, let Φ: $M(G) \to B(G)$ be Bers' natural projection. Its derivative at 0 is a map from $L^\infty(G)$ onto $B(G)$ given by:

$$d_0\Phi(\mu) = a\iint_U \frac{\overline{\mu(z)}}{(\bar{z}-\zeta)^4}\,\mathrm{d}x\mathrm{d}y \in B(G), \tag{27}$$

(a is a nonzero constant). See Bers [3] for these standard facts. (Since $B(G) = A(G)^*$ is a Banach space of holomorphic functions on U rather than on the lower half-plane the formulae here are (very) slightly modified.)

The tangent vector at X_0 in $T(X_0)$ corresponding to $d_0S(c)$ is then $d_0\Phi(\mu_c)$, where μ_c is the Beltrami differential in (25) lifted to U as a G-invariant $(-1, 1)$ form, (still called μ_c). Thus,

$$d_0S(c) = \varphi \in B(G), \text{ where } \varphi \text{ is } d_0\Phi(\mu_c).$$

Given any ψ in $A(G)$ we are required to show that the Weil-Petersson pairing (ψ, φ) is precisely

$$c_1\iint_{D_1}\psi + c_2\iint_{D_2}\psi + \ldots = \langle\psi, \mu_c\rangle.$$

But notice that

$$\begin{aligned}(\psi,\varphi) &= \iint_{U/G}\psi(\zeta)\overline{\varphi(\zeta)}\eta^2\,\mathrm{d}\xi\mathrm{d}\eta\\ &= \langle\psi, \bar{\varphi}\eta^2\rangle.\end{aligned}$$

So the question is whether $\bar{\varphi}\eta^2$ and μ_c are equivalent linear functionals on $A(G)$. By Teichmüller's Lemma we know that this happens if and only if their difference is in the kernel of the map $d_0\Phi$. Thus we desire to check whether

$$d_0\Phi(\bar{\varphi}\eta^2) = d_0\Phi(\mu_c). \tag{28}$$

But the right side is, by definition, φ itself. The formula (27) for $d_0\Phi$ says therefore that (28) is indeed true (upto a fixed constant) because of the classic reproducing formula (26). We are through.

We conclude by observing that Shields and Williams [10] have proved that $A(1)$ is abstractly isomorphic to l^1. This fact is of course very relevant to the choice of Schiffer variation discs D_j for coordinatisation of universal Teichmüller space, $T(1)$.

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