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Rayleigh wave scattering due to a rigid barrier in a liquid halfspace

K K MANN and P S DESHWAL

Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, India

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Abstract. The paper presents a theoretical formulation for studying scattering of Rayleigh waves due to the presence of rigid barriers in oceanic waters. The Wiener-Hopf technique has been employed to solve the problem. Exact solution has been obtained in terms of Fourier integrals whose evaluation gives the reflected, transmitted and scattered waves. The scattered waves have the behaviour of cylindrical waves originating at the edge of the barrier. Numerical results for the amplitude of the scattered waves have been obtained for small depth of the barrier.

Keywords. Rayleigh wave scattering; rigid barrier; liquid halfspace; Wiener-Hopf technique.

1. Introduction

We propose to study the problem of scattering of Rayleigh waves due to rigid barriers in the surface of oceanic waters. Ursell [8] considered the effect of a vertical barrier of finite depth, fixed in an infinitely deep sea, on normally incident surface waves. Faulkner [4] extended his solution to a three-dimensional case to study diffraction of an obliquely incident surface wave by a vertical barrier of finite depth. Deshwal [1, 2] studied problems of diffraction of compressional waves due to rigid barriers in a liquid halfspace and a liquid layer. The rigid barrier permits no displacement across it. The method of solution is the Wiener-Hopf technique [5].

2. Statement of the problem

The rigid barrier is held in the free surface of sea water, taken to be a homogenous isotropic liquid halfspace. The halfspace lies to the side of positive z-axis pointing vertically downwards with x-axis in the free surface. The barrier is taken along the z-axis and is of small depth H. A time harmonic two-dimensional Rayleigh wave is incident at the barrier in figure 1.

The two-dimensional wave equation is

$$\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \bar{\phi}}{\partial t^2} + \frac{\varepsilon}{c^2}\frac{\partial \bar{\phi}}{\partial t}, \tag{1}$$

c is the velocity of Rayleigh waves and ε is small positive damping constant. Let the

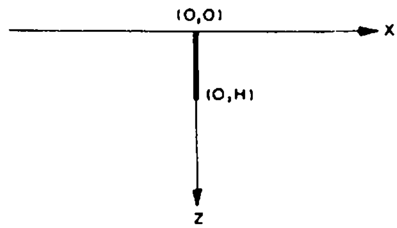

Figure 1.

potential for a time-harmonic wave be

$$\bar{\phi}(x, z, t) = \phi(x, z) \exp(-i\omega t). \tag{2}$$

Then equation (1) reduces to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + k^2 \phi = 0 \tag{3}$$

where

$$k^2 = (\omega^2 + i\varepsilon\omega)/c^2, \quad k = k_1 + ik_2. \tag{4}$$

The imaginary part of k is small and positive. Let the incident and total potentials be

$$\phi_i(x, z) = A_0 \exp(-i\alpha_0 x) \exp(-\gamma_0 z), \tag{5}$$

$$\phi_t(x, z) = \phi_i(x, z) + \phi(x, z), \tag{6}$$

α_0 is the root of the Rayleigh equation and $\gamma_0 = (\alpha_0^2 - k^2)^{1/2}$. The displacement components (u, w) at any point (x, z) are given by

$$u = \partial\phi_t/\partial x, \quad w = \partial\phi_t/\partial z \tag{7}$$

3. Boundary conditions

The conditions on the boundaries are

(i) $\phi(x, z)$ is bounded when $z \to \infty$, (8)

(ii) $\phi_t(x, z) = 0$, on $z = 0$, for all x, (9)

(iii) $u = \partial\phi_t/\partial x = 0$, on $x = 0$, $0 \leqslant z \leqslant H$. (10)

We assume that for given z

$$\left.\begin{aligned} |\phi(x, z)| &\sim D_1 \exp(-k_2 x) \text{ as } x \to \infty \\ &\sim D_2 \exp(k_2 x) \text{ as } x \to -\infty. \end{aligned}\right] \tag{11}$$

where D_1 and D_2 are constants, so that we can define the Fourier transforms

$$\bar{\phi}(\alpha, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(x, z) \exp(i\alpha x) \, dx, \tag{12}$$

$$\bar{\phi}_{+}(\alpha, z) = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} \phi(x, z) \exp(i\alpha x) \, dx, \tag{13}$$

$$\bar{\phi}_{-}(\alpha, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} \phi(x, z) \exp(i\alpha x) \, dx. \tag{14}$$

Let us further take $\alpha = \sigma + i\tau$, then (13) gives

$$|\bar{\phi}_{+}(\alpha, z)| \leqslant \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} \phi(x, z) \exp(-\tau x) \, dx. \tag{15}$$

On account of the assumption made for the form of $\phi(x, z)$, $\bar{\phi}(\alpha, z)$ is bounded at $x = \infty$, only when $\tau > -k_2$. Hence $\bar{\phi}_{+}(\alpha, z)$ is regular in the region $\tau > -k_2$ of the complex α-plane. Similarly, it can be shown that $\bar{\phi}_{-}(\alpha, z)$ is regular in the region $\tau < k_2$ of the α-plane. Since

$$\bar{\phi}(\alpha, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(x, z) \exp(i\alpha x) \, dx = \bar{\phi}_{+}(\alpha, z) + \bar{\phi}_{-}(\alpha, z). \tag{16}$$

Therefore

$$|\bar{\phi}| \leqslant |\bar{\phi}_{+}| + |\bar{\phi}_{-}| \tag{17}$$

shows that $\bar{\phi}(\alpha, z)$ is regular in the strip $-k_2 < \tau < k_2$ of the α-plane. Similarly derivatives of $\bar{\phi}(\alpha, z)$ are also regular in the same strip.

4. Solution of the problem

Multiplying (3) by $\exp(i\alpha x)/(2\pi)^{1/2}$ and integrating between $x = -\infty$ and $x = \infty$, noting that $\partial\phi/\partial x$ vanishes at $x = \pm\infty$, we have

$$(d^2\bar{\phi}/dz^2) - \gamma^2\bar{\phi} = 0, \tag{18}$$

where

$$\gamma = \pm(\alpha^2 - k^2)^{1/2}. \tag{19}$$

We choose that sign before the radical which makes the real part of $\gamma \geqslant 0$, for all α. The solution of (18) is

$$\bar{\phi}(\alpha, z) = A(\alpha)\exp(-\gamma z) + B(\alpha)\exp(\gamma z). \tag{20}$$

Applying boundary condition that $\phi(x, z)$ and hence $\bar{\phi}(\alpha, z)$ is bounded when z tends to ∞, we must take in (20)

$$B(\alpha) = 0. \tag{21}$$

Hence

$$\bar{\phi}(\alpha, z) = A(\alpha)\exp(-\gamma z). \tag{22}$$

This can be written as

$$\bar{\Phi}_+(\alpha, z) + \bar{\Phi}_-(\alpha, z) = A(\alpha)\exp(-\gamma z). \tag{23}$$

Putting $z = H$ in (23) and in its derivative with respect to z, and eliminating $A(\alpha)$ between the equations so obtained, we have

$$\bar{\Phi}_+(\alpha) + \bar{\Phi}_-(\alpha) = -\gamma^{-1}[\bar{\Phi}'_+(\alpha) + \bar{\Phi}'_-(\alpha)]. \tag{24}$$

We have used $\bar{\Phi}_+(\alpha)$, $\bar{\Phi}_-(\alpha)$ for $\bar{\Phi}_+(\alpha, H)$ and $\bar{\Phi}_-(\alpha, H)$. Similar notions are used for their derivatives. Changing α to $-\alpha$ in (24), we get

$$\bar{\Phi}_+(-\alpha) + \bar{\Phi}_-(-\alpha) = -\bar{\gamma}^1[\bar{\Phi}'_+(-\alpha) + \bar{\Phi}'_-(-\alpha)]. \tag{25}$$

Adding (24) and (25), we have

$$\bar{\Phi}_+(\alpha) + \bar{\Phi}_-(\alpha) + \bar{\Phi}_+(-\alpha) + \bar{\Phi}_-(-\alpha)$$

$$= -\bar{\gamma}^1[\bar{\Phi}'_+(\alpha) + \bar{\Phi}'_-(\alpha) + \bar{\Phi}'_+(-\alpha) + \bar{\Phi}'_-(-\alpha)]. \tag{26}$$

Again multiplying (3) by $\exp(i\alpha x)/(2\pi)^{1/2}$ and integrating with respect to x, as x varies from 0 to ∞, we get

$$\frac{d^2\bar{\Phi}_+(\alpha, z)}{dz^2} - \gamma^2\bar{\Phi}_+(\alpha, z) \doteq \frac{1}{(2\pi)^{1/2}}\left(\frac{\partial\phi}{\partial x}\right)_0 - \frac{i\alpha}{(2\pi)^{1/2}}(\phi)_0. \tag{27}$$

Changing α to $-\alpha$ in (27) and adding the two equations, we find

$$\frac{d^2}{dz^2}[\bar{\Phi}_+(\alpha, z) + \bar{\Phi}_+(-\alpha, z)] - \gamma^2[\bar{\Phi}_+(\alpha, z) + \bar{\Phi}_+(-\alpha, z)]$$

$$= \frac{2}{(2\pi)^{1/2}}(\partial\phi/\partial x)_0, \tag{28}$$

where $(\partial\phi/\partial x)_0$ means value of $\partial\phi/\partial x$ at $x = 0$. To find the value of $(\partial\phi/\partial x)_0$, we use condition (10) to find out

$$(\partial\phi/\partial x)_0 = i\alpha_0 A_0 \exp(-\gamma_0 z). \tag{29}$$

Using this value in (28) we have

$$\frac{d^2}{dz^2}[\bar{\Phi}_+(\alpha, z) + \bar{\Phi}_+(-\alpha, z)] - \gamma^2[\bar{\Phi}_+(\alpha, z) + \bar{\Phi}_+(-\alpha, z)]$$

$$= \frac{2i\alpha_0 A_0 \exp(-\gamma_0 z)}{(2\pi)^{1/2}}. \tag{30}$$

The complete solution of (30) is given by

$$\bar{\Phi}_+(\alpha, z) + \phi(-\alpha, z) = A_1(\alpha)\exp(\gamma z) + A_2(\alpha)\exp(-\gamma z)$$

$$-\frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0 A_0 \exp(-\gamma_0 z)}{(\alpha^2 - \alpha_0^2)}. \tag{31}$$

To eliminate the arbitrary constants $A_2(\alpha)$ and $A_1(\alpha)$, we use (9) to find

$$(\phi(x, z))_{z=0} = -A_0 \exp(-i\alpha_0 x). \tag{32}$$

Using (32) in (13), we get

$$\bar{\phi}_+(\alpha, 0) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty (\phi)_{z=0} \exp(i\alpha x)\,dx = \frac{-A_0}{(2\pi)^{1/2}} \left| \frac{\exp[i(\alpha-\alpha_0)x]}{i(\alpha-\alpha_0)} \right|_0^\infty. \tag{33}$$

The integral in (33) is convergent on the line $\tau = \operatorname{Im}(\alpha_0)$, except at the point $\alpha = \alpha_0$. Hence from (33), we have

$$\bar{\phi}_+(\alpha, 0) = \frac{-iA_0}{(2\pi)^{1/2}(\alpha-\alpha_0)}. \tag{34}$$

Changing α to $-\alpha$ in (34), we have

$$\bar{\phi}_+(-\alpha, 0) = \frac{iA_0}{(2\pi)^{1/2}(\alpha+\alpha_0)}. \tag{35}$$

Adding (34) and (35), we find that

$$\bar{\phi}_+(\alpha, 0) + \bar{\phi}_+(-\alpha, 0) = \frac{-2i\alpha_0 A_0}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}. \tag{36}$$

Again putting $z = 0$ in (31) and using (36), we see that

$$A_1(\alpha) + A_2(\alpha) = 0, \text{ i.e., } A_2(\alpha) = -A_1(\alpha). \tag{37}$$

The solution of (31) is now

$$\bar{\phi}_+(\alpha, z) + \bar{\phi}_+(-\alpha, z) = 2A_1 \sinh \gamma z - \frac{2}{(2\pi)^{1/2}} \frac{i\alpha_0 A_0 \exp(-\gamma_0 z)}{(\alpha^2-\alpha_0^2)}. \tag{38}$$

Putting $z = H$ in (38), we get

$$\bar{\phi}_+(\alpha) + \bar{\phi}_+(-\alpha) = 2A_1 \sinh \gamma H - \frac{2i\alpha_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}. \tag{39}$$

Differentiating (38) with respect to z and putting $z = H$, we have

$$\bar{\phi}'_+(\alpha) + \bar{\phi}'_+(-\alpha) = 2A_1\gamma \cosh \gamma H + \frac{2i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}. \tag{40}$$

Eliminating $A_1(\alpha)$ between (39) and (40), we get

$$\bar{\phi}_+(\alpha) + \bar{\phi}_+(-\alpha) = \bar{\gamma}^1 \tanh \gamma H [\bar{\phi}_+(\alpha) + \bar{\phi}'_+(-\alpha)]$$
$$- \frac{2i\alpha_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)} - \frac{2i\alpha_0\gamma_0 A_0}{(2\pi)^{1/2}\gamma(\alpha^2-\alpha_0^2)} \tanh \gamma H \exp(-\gamma_0 H). \tag{41}$$

Again multiplying (3) by $\exp(i\alpha x)/(2\pi)^{1/2}$ and integrating with respect to x, between the limits $-\infty$ to 0, we have

$$\frac{d^2}{dz^2}[\bar{\Phi}_-(\alpha, z)] - \gamma^2[\bar{\Phi}_-(\alpha, z)] = \frac{-1}{(2\pi)^{1/2}}\left(\frac{\partial \phi}{\partial x}\right)_{x=0} + \frac{i\alpha}{(2\pi)^{1/2}}(\phi)_{x=0}. \quad (42)$$

Replacing α by $-\alpha$ in (42) and adding it to the resulting equation, it is found that

$$\frac{d^2}{dz^2}[\bar{\Phi}_-(\alpha, z) + \bar{\Phi}_-(-\alpha, z)] - \gamma^2[\bar{\Phi}_-(\alpha, z) + \bar{\Phi}_-(-\alpha, z)]$$

$$= \frac{-2}{(2\pi)^{1/2}}\left(\frac{\partial \phi}{\partial x}\right)_{x=0} \quad (43)$$

Using (29) in (42), we get

$$\frac{d^2}{dz^2}[\bar{\Phi}_-(\alpha, z) + \bar{\Phi}_-(-\alpha, z)] - \gamma^2[\bar{\Phi}_-(\alpha, z) + \bar{\Phi}_-(-\alpha, z)]$$

$$= \frac{-2}{(2\pi)^{1/2}} i\alpha_0 A_0 \exp(-\gamma_0 z). \quad (44)$$

The complete solution of (44) is given by

$$\bar{\Phi}_-(\alpha, z) + \bar{\Phi}_-(-\alpha, z) = \beta_1(\alpha)\exp(\gamma z) + \beta_2(\alpha)\exp(-\gamma z) + \frac{2}{(2\pi)^{1/2}} \frac{i\alpha_0 A_0 \exp(-\gamma_0 z)}{\alpha^2 - \alpha_0^2}. \quad (45)$$

To eliminate $\beta_1(\alpha)$ and $\beta_2(\alpha)$, we use (32) in

$$\bar{\Phi}_-(\alpha, 0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} (\phi(x, z))_{z=0} \exp(i\alpha x)\,dx, \quad (46)$$

to get

$$\bar{\Phi}_-(\alpha, 0) = \frac{-A_0}{(2\pi)^{1/2}} \left| \frac{\exp[i(\alpha - \alpha_0)x]}{i(\alpha - \alpha_0)} \right|_{-\infty}^{0}. \quad (47)$$

The integral in (47) is convergent on the line $\tau = \text{Im}(\alpha_0)$, except the point $\alpha = \alpha_0$, and we find

$$\bar{\Phi}_-(\alpha, 0) = \frac{iA_0}{(2\pi)^{1/2}(\alpha - \alpha_0)}. \quad (48)$$

Changing α to $-\alpha$ in (48) and adding the resulting equation to (48), wet get

$$\bar{\Phi}_-(\alpha, 0) + \bar{\Phi}_-(-\alpha, 0) = \frac{2i\alpha_0 A_0}{(2\pi)^{1/2}(\alpha^2 - \alpha_0^2)}. \quad (49)$$

Putting $z = H$ in (45) and using (49), we find

$$\beta_1(\alpha) + \beta_2(\alpha) = 0, \text{ i.e., } \beta_2(\alpha) = -\beta_1(\alpha). \quad (50)$$

Using (45) and (50), we obtain

$$\bar{\Phi}_-(\alpha, z) + \bar{\Phi}_-(-\alpha, z) = 2\beta_1 \sinh \gamma z + \frac{2}{(2\pi)^{1/2}} \frac{i\alpha_0 A_0 \exp(-\gamma_0 z)}{(\alpha^2 - \alpha_0^2)}. \quad (51)$$

Putting $z = H$ in (51) and in its derivative with respect to z, we get the following equations

$$\bar{\Phi}_-(\alpha)+\bar{\Phi}_-(-\alpha) = 2\beta_1 \sinh\gamma H + \frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0 A_0 \exp(-\gamma_0 H)}{(\alpha^2-\alpha_0^2)}, \quad (52)$$

$$\bar{\Phi}'_-(\alpha)+\bar{\Phi}'_-(-\alpha) = 2\beta_1\gamma \cosh\gamma H - \frac{2i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}. \quad (53)$$

Eliminating β_1, between (52) and (53), we have

$$\bar{\Phi}_-(\alpha)+\bar{\Phi}_-(-\alpha) = \bar{\gamma}^1 \tanh\gamma H\,[\bar{\Phi}'_-(\alpha)+\bar{\Phi}'_-(-\alpha)] +$$

$$\frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0 A_0 \exp(-\gamma_0 H)}{(\alpha^2-\alpha_0^2)} + \frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)}{(\alpha^2-\alpha_0^2)\gamma}\tanh\gamma H. \quad (54)$$

Adding (54) and (41), we obtain

$$\bar{\Phi}_+(\alpha)+\bar{\Phi}_+(-\alpha)+\bar{\Phi}_-(\alpha)+\bar{\Phi}_-(-\alpha) = \bar{\gamma}^1 \tanh\gamma H\,[\bar{\Phi}'_+(\alpha)+\bar{\Phi}'_+(-\alpha) + \bar{\Phi}'_-(\alpha)+\bar{\Phi}_-(-\alpha)] \quad (55)$$

From (55) and (26), it can be easily seen that

$$(1+\tanh\gamma H)[\bar{\Phi}_+(\alpha)+\bar{\Phi}_+(-\alpha)+\bar{\Phi}_-(\alpha)+\bar{\Phi}_-(-\alpha)] = 0. \quad (56)$$

Since $(1+\tanh \gamma H) \neq 0$, therefore, we can write (56) as

$$\bar{\Phi}_+(\alpha)+\bar{\Phi}_-(-\alpha) = -\bar{\Phi}_+(-\alpha)-\bar{\Phi}_-(\alpha). \quad (57)$$

This is a functional equation of Wiener-Hopf type defined in the strip $-k_2 < \tau < k_2$. The function $\bar{\phi}_+(\alpha)+\bar{\phi}_-(-\alpha)$ is analytic in the region $\tau > -k_2$ and $-\bar{\phi}_-(\alpha)-\bar{\phi}_+(-\alpha)$ in the region $\tau < k_2$ of the complex α-plane. The two functions have the strip $-k_2 < \tau < k_2$ as common region of analyticity, in which the two functions are identically equal to each other. Therefore, by analytic continuation, the two functions, in their respective regions of analyticity, are the representatives of an entire function $F(\alpha)$. From (13), we see that

$$|\bar{\phi}_+(\alpha, z)| \leqslant \frac{1}{(2\pi)^{1/2}}\int_0^\infty |\phi(x, z)|\exp(-\tau x)\mathrm{d}x. \quad (58)$$

$\bar{\Phi}_+(\alpha, z)$ tends to zero as $|\alpha|$ tends to ∞. Similarly $\bar{\Phi}_+(-\alpha)$, $\bar{\Phi}_-(\alpha)$ and $\bar{\Phi}_-(-\alpha)$ tend to zero as $|\alpha|$ tends to ∞. Hence by Liouville's theorem, $F(\alpha)$ is zero at all points of the α-plane. Equating to zero, each member of (57) to find out

$$\bar{\Phi}_+(\alpha) = -\bar{\Phi}_-(-\alpha), \quad \bar{\Phi}_+(-\alpha) = -\bar{\Phi}_-(\alpha).$$

Similarly

$$\bar{\Phi}'_+(\alpha) = -\bar{\Phi}'_-(-\alpha), \quad \bar{\Phi}'_+(-\alpha) = -\bar{\Phi}'_-(\alpha). \quad (59)$$

Subtracting (24) from (41) and using (59), we get

$$2\bar{\Phi}_+(-\alpha) = \bar{\gamma}^1(1+\tanh\gamma H)\bar{\Phi}'_+(\alpha) + \bar{\gamma}^1(\tanh\gamma H - 1)\bar{\Phi}'_+(-\alpha)$$

$$-\frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0 A_0\exp(-\gamma_0 H)}{(\alpha^2-\alpha_0^2)} - \frac{2i\alpha_0\gamma_0 A_0\tanh\gamma H\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)\gamma}$$

$$= \frac{\exp(\gamma H)}{\gamma\cosh\gamma H}\bar{\Phi}'_+(\alpha) - \frac{\exp(-\gamma H)\bar{\Phi}'_+(-\alpha)}{\gamma\cosh\gamma H}$$

$$-\frac{2i\alpha_0\gamma_0 A_0\exp(-\gamma_0 H)\exp(\gamma H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)\gamma\ \cosh\gamma H}$$

$$+\frac{2i\alpha_0\gamma_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)\gamma} - \frac{2i\alpha_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)},$$

i.e.

$$2\bar{\Phi}_+(-\alpha) = \frac{\bar{\Phi}'_+(\alpha)}{(\alpha^2-k^2)H}\left(\frac{\exp(\gamma H)\cdot\gamma H}{\cosh\gamma H}\right) - \frac{\bar{\Phi}'_+(-\alpha)}{(\alpha^2-k^2)H}\left(\frac{\exp(-\gamma H)\cdot\gamma H}{\cosh\gamma H}\right)$$

$$-\frac{2i\alpha_0\gamma_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-k^2)H(\alpha^2-\alpha_0^2)}\left(\frac{\exp(\gamma H)\cdot\gamma H}{\cosh\gamma H}\right)$$

$$+\frac{2i\alpha_0\gamma_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}\gamma(\alpha^2-\alpha_0^2)} - \frac{2i\alpha_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}. \tag{60}$$

This differential equation is solved by the Wiener-Hopf technique.

Let us now factorize $\cosh\gamma H\exp(-\gamma H)/\gamma H$. We write

$$\exp(-\gamma H) = \exp[-T_+(\alpha) - T_-(\alpha)], \tag{61}$$

where

$$T_+(\alpha) = \gamma H\pi^{-1}\cos^{-1}(\alpha/k) \sim i\alpha H\pi^{-1}\log(2\alpha/k),\ \text{as}\ \alpha\to\infty, \tag{62}$$

and

$$T_-(\alpha) = T_+(-\alpha). \tag{63}$$

The factorization of $\cosh\gamma H\exp(-\gamma H)/\gamma H$ as an infinite product is

$$L(\alpha) = L_+(\alpha)L_-(\alpha) = [\exp(-\gamma H)\cosh\gamma H]/\gamma H$$

$$= \frac{\exp[-T_+(\alpha)-T_-(\alpha)]}{H(\alpha+k)^{1/2}(\alpha-k)^{1/2}}\prod_{n=1}^{\infty}[1-k^2b^2_{n-1/2}+\alpha^2b^2_{n-1/2}], \tag{64}$$

where

$$L_-(\alpha) = \frac{\exp[x(\alpha)-T_-(\alpha)]}{[H(\alpha-k)]^{1/2}}\prod_{n=1}^{\infty}[(1-k^2b^2_{n-1/2})^{1/2}+i\alpha b_{n-1/2}]\cdot$$

$$\exp(-i\alpha b_{n-1/2}), \tag{65}$$

$b_{n-1/2} = H/(n-1/2)\pi$ and $x(\alpha)$ is an arbitrary function to make a suitable behaviour of

$L_-(\alpha)$ as $|\alpha| \to \infty$. The behaviour of $L_-(\alpha)$ as $|\alpha| \to \infty$ is given by

$$L_-(\alpha) \sim \exp \frac{\exp[x(\alpha) + i\alpha H\pi^{-1}\log(-2\alpha/k)]}{[H(\alpha-k)]^{1/2}}$$
$$\prod_{n=1}^{\infty}\left[1+\frac{i\alpha H\pi^{-1}}{n-1/2}\right]\exp\left(\frac{-i\alpha H\pi^{-1}}{n-1/2}\right). \tag{66}$$

The infinite product in (66) is obtained using the result (Noble, [5] ex 1.10, p. 41)

$$\prod_{n=1}^{\infty}\left[1+\frac{\alpha}{n-1/2}\right]\exp\left(\frac{-\alpha}{n-1/2}\right) \sim \exp(-c_1\alpha)2^{-2\alpha}\exp(\alpha+1/2)\alpha^{-\alpha}, \tag{67}$$

where $c_1 = 0.5772$ is Euler's constant. Therefore

$$L_-(\alpha) \sim \exp[x(\alpha) + i\alpha H\pi^{-1}(1 - c_1 + \log(\pi/2kH)) - \alpha H/2]/[\pi(\alpha-k)]^{1/2}, \tag{68}$$

$L_-(\alpha)$ is asymptotic to $1/(\alpha)^{1/2}$ as $|\alpha| \to \infty$ if we choose

$$x(\alpha) = -i\alpha H\pi^{-1}[1 - c_1 + \log(\pi/2kH)] + \alpha H/2. \tag{69}$$

Substituting the value of $\exp(-\gamma H)\cosh\gamma H/\gamma H$ in (60), we have

$$\begin{aligned} 2\bar{\Phi}_+(-\alpha) = {} & \frac{\bar{\Phi}'_+(\alpha)}{(\alpha^2-k^2)HL_+(\alpha)L_-(\alpha)} - \frac{\bar{\Phi}'_+(-\alpha)\exp(-2\gamma H)}{(\alpha^2-k^2)HL_+(\alpha)L_-(\alpha)} \\ & - \frac{2i\alpha_0 A_0\gamma_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}H(\alpha^2-k^2)(\alpha^2-\alpha_0^2)L_+(\alpha)L_-(\alpha)} \\ & + \frac{2i\alpha_0\gamma_0 A_0\exp(-\gamma_0 H)}{(\alpha^2-k^2)^{1/2}(2\pi)^{1/2}(\alpha^2-\alpha_0^2)} \\ & - \frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0 A_0\exp(-\gamma_0 H)}{(\alpha^2-\alpha_0^2)}. \end{aligned} \tag{70}$$

This can be written as

$$\begin{aligned} & 2\bar{\Phi}_+(-\alpha)(\alpha-k)L_-(\alpha) + \frac{\bar{\Phi}'_+(-\alpha)\exp(-2\gamma H)}{H(\alpha+k)L_+(\alpha)} \\ & - \frac{2i\alpha_0\gamma_0 A_0\exp(-\gamma_0 H)(\alpha-k)^{1/2}\cdot L(-\alpha)}{(\alpha+k)^{1/2}\cdot(2\pi)^{1/2}(\alpha^2-\alpha_0^2)} \\ & + \frac{2i\alpha_0 A_0(\alpha-k)L_-(\alpha)\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)} \\ & = \frac{\bar{\Phi}'_+(\alpha)}{H(\alpha+k)L_+(\alpha)} - \frac{2i\alpha_0 A_0\gamma_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}H(\alpha+k)(\alpha^2-\alpha_0^2)L_+(\alpha)}. \end{aligned} \tag{71}$$

The term $\bar{\Phi}'_+(-\alpha)\exp(-2\gamma H)/H(\alpha+k)\ L_+(\alpha)$ in (71) is analytic inside the strip $-k_2 < \tau < k_2$ and we have shown that $L_+(\alpha)$ behaves as $1/(\alpha)^{1/2}$ as $|\alpha|$ tends to ∞, inside the said region, the above mentioned term tends to zero. The term under consideration can be expressed as sum of two functions one of which is analytic in the region $\tau < k_2$ and other is analytic in the region $\tau > -k_2$. Terms in (71) are decomposed as

$$
\begin{aligned}
&2\bar{\Phi}_+(-\alpha)(\alpha-k)L_-(\alpha)+\frac{1}{H(\alpha+k)L_+(\alpha)}[\bar{\phi}'_+(-\alpha)\exp(-2\gamma H)\\
&-\bar{\phi}'_+(-p_n)\exp[-2H(p_n^2-k^2)^{1/2}]\\
&-\frac{2i\alpha_0A_0\gamma_0\exp(-\gamma_0H)}{(2\pi)^{1/2}(\alpha+\alpha_0)(\alpha+k)^{1/2}}\left[\frac{(\alpha-k)^{1/2}\cdot L_-(\alpha)}{\alpha-\alpha_0}-\frac{(p_i-k)^{1/2}\;L_-(p_i)}{p_i-\alpha_0}\right]\\
&+\frac{2i\alpha_0A_0\exp(-\gamma_0H)}{(2\pi)^{1/2}(\alpha+\alpha_0)}\left[\frac{(\alpha-k)L_-(\alpha)}{\alpha-\alpha_0}-\frac{(\alpha_0+k)L_-(-\alpha_0)}{2\alpha_0}\right]\\
&+\frac{i\gamma_0A_0\exp(-\gamma_0H)}{(2\pi)^{1/2}\;H(\alpha-\alpha_0)(\alpha_0+k)L_+(\alpha_0)}\\
&=\frac{\bar{\phi}'_+(\alpha)}{H(\alpha+k)L_+(\alpha)}-\frac{\bar{\phi}'_+(-p_n)\exp(-2H(p_n^2-k^2)^{1/2})}{H(\alpha+k)L_+(\alpha)}\\
&+\frac{2i\alpha_0\gamma_0A_0(p_i-k)^{1/2}L_-(p_i)\exp(-\gamma_0H)}{(2\pi)^{1/2}(\alpha+\alpha_0)(\alpha+k)^{1/2}(p_i-\alpha_0)}\\
&-\frac{iA_0(\alpha_0+k)L_-(-\alpha_0)\exp(-\gamma_0H)}{(2\pi)^{1/2}(\alpha+\alpha_0)}-\frac{2i\alpha_0\gamma_0A_0\exp(-\gamma_0\mathrm{H})}{(2\pi)^{1/2}H(\alpha-\alpha_0)}\\
&\times\left[\frac{1}{(\alpha+k)(\alpha+\alpha_0)L_+(\alpha)}-\frac{1}{2\alpha_0(\alpha_0+k)L_+(\alpha_0)}\right],
\end{aligned}
\tag{72}
$$

where p_n are roots of $(\alpha+k)L_+(\alpha)=0$ and $p_i=-\alpha_0, -k$. The functions on the left and right sides of (72) are analytic in the regions $\tau < k_2$ and $\tau > -k_2$ respectively of the α plane. Hence, by analytic continuation, they represent an entire function $R(\alpha)$. Moreover $L_\pm(\alpha)$ reduce to $1/(\alpha)^{1/2}$ as $|\alpha|$ tends to ∞, each term on the right side tends to zero as $\alpha\to\infty$. Hence, by Liouville's theorem, $R(\alpha)$ is zero at every point of the complex plane. Equating to zero the right side of (72), we get

$$
\begin{aligned}
\bar{\phi}'_+(\alpha)&=\bar{\phi}'_+(-p_n)\exp[-2H(p_n^2-k^2)^{1/2}]\\
&-\frac{2i\alpha_0\gamma_0HA_0(\alpha+k)^{1/2}(p_i-k)^{1/2}\;L_-(p_i)L_+(\alpha)\exp(-\gamma_0H)}{(2\pi)^{1/2}(\alpha+\alpha_0)(p_i-\alpha_0)}\\
&+\frac{iH(\alpha+k)(\alpha_0+k)A_0L_+(\alpha)L_-(-\alpha_0)\exp(-\gamma_0H)}{(2\pi)^{1/2}(\alpha+\alpha_0)}\\
&+\frac{2i\alpha_0\gamma_0A_0\exp(-\gamma_0H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}\\
&-\frac{i\gamma_0(\alpha+k)A_0L_+(\alpha)\exp(-\gamma_0H)}{(2\pi)^{1/2}(\alpha-\alpha_0)(\alpha_0+k)L_+(\alpha_0)}.
\end{aligned}
\tag{73}
$$

Changing α to $-\alpha$ in (73) and subtracting the resulting equation, we find

$$\bar{\phi}'_+(\alpha)-\bar{\phi}'_+(-\alpha)=\bar{\phi}'_+(\alpha)+\bar{\phi}'_-(\alpha)=\bar{\phi}'(\alpha)$$

$$=-\frac{2i\alpha_0\gamma_0 A_0 H(p_i-k)^{1/2}L_-(p_i)\exp(-\gamma_0 H)}{(2\pi)^{1/2}(p_i-\alpha_0)}$$

$$\cdot\left[\frac{(\alpha+k)^{1/2}L_+(\alpha)}{\alpha+\alpha_0}+\frac{(k-\alpha)^{1/2}L_+(-\alpha)}{\alpha-\alpha_0}\right]$$

$$+\frac{iHA_0(\alpha_0+k)L_-(-\alpha_0)\exp(-\gamma_0 H)}{(2\pi)^{1/2}}\left[\frac{(\alpha+k)L_+(\alpha)}{\alpha+\alpha_0}+\frac{(k-\alpha)L_+(-\alpha)}{\alpha-\alpha_0}\right]$$

$$-\frac{i\gamma_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha_0+k)L_+(\alpha_0)}\left[\frac{(\alpha+k)L_+(\alpha)}{\alpha-\alpha_0}+\frac{(k-\alpha)L_+(-\alpha)}{\alpha+\alpha_0}\right]. \tag{74}$$

Differentiating (23) with respect to z and putting $z=H$, we get

$$\bar{\phi}'_+(\alpha)+\bar{\phi}'_-(\alpha)=\bar{\phi}'(\alpha)=-\gamma A(\alpha)\exp(-\gamma H) \tag{75}$$

From here

$$A(\alpha)=-\frac{1}{\gamma}[\bar{\phi}'_+(\alpha)+\bar{\phi}'_-(\alpha)]\exp(\gamma H). \tag{76}$$

Using (76) and (74), we find

$$A(\alpha)=\frac{2i\alpha_0\gamma_0 HA_0(p_i-k)^{1/2}\cdot L_-(p_i)\exp(\gamma-\gamma_0)H}{(2\pi)^{1/2}(p_i-\alpha_0)\gamma}$$

$$\left[(\alpha+k)^{1/2}\cdot\frac{L_+(\alpha)}{\alpha+\alpha_0}+\frac{(k-\alpha)^{1/2}\ L_+(-\alpha)}{\alpha-\alpha_0}\right]$$

$$-\frac{iH A_0(\alpha_0+k)L_-(-\alpha_0)\exp(\gamma-\gamma_0)H}{(2\pi)^{1/2}\gamma}\left[\frac{(\alpha+k)L_+(\alpha)}{\alpha+\alpha_0}+\frac{(k-\alpha)L_+(-\alpha)}{\alpha-\alpha_0}\right]$$

$$+\frac{i\gamma_0 A_0\exp(\gamma-\gamma_0)H}{(2\pi)^{1/2}\gamma(\alpha_0+k)L_+(\alpha_0)}\left[\frac{(\alpha+k)L_+(\alpha)}{\alpha-\alpha_0}+\frac{(k-\alpha)L_+(-\alpha)}{\alpha+\alpha_0}\right]. \tag{77}$$

By the inverse Fourier transforms of (12), we have

$$\phi(x,z)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty+i\tau}^{\infty+i\tau}\bar{\phi}(\alpha,z)\exp(-i\alpha x)\,\mathrm{d}x$$

$$=\frac{1}{(2\pi)^{1/2}}\int_{-\infty+i\tau}^{\infty+i\tau}A(\alpha)\exp(-\gamma z-i\alpha x) \tag{78}$$

where

$\tau=\mathrm{Im}(\alpha_0)$, $\alpha\neq\alpha_0$ and $A(\alpha)$ is given by (77).

5. Evaluation of the integral

To evaluate integral (78) the contour is taken along the line $\tau = \mathrm{Im}(\alpha_0)$ as shown in figure 2 avoiding the point $\alpha = \alpha_0$. $\alpha = -\alpha_0$ is the pole and $\alpha = \pm k$ are the branch points and $\mathrm{Re}(\gamma) \geqslant 0$ for all α. The condition $\mathrm{Re}(\gamma) = 0$, as discussed by Ewing and Press [3] gives the parts of hyperbola to be used as branch cut with branch points at $\alpha = \pm k$. Applying Cauchy's residue theorem to the contour chosen, we get

$$\int_{-\infty + i\tau}^{\infty + i\tau} + \int_{AB} + \int_{\Sigma} + \int_{L_\alpha} + \int_{CD} = 2\pi i \text{ (sum of residues)}. \tag{79}$$

The presence of factor $\exp(-i\alpha x) = \exp(-i\sigma x)\exp(-\tau x)(\alpha = \sigma - i\tau)$, makes the integral along infinite circular arcs AB and CD vanish. Σ is a small circle indenting the point $\alpha = \alpha_0$ and L_α is the branch cut around $\alpha = -k$. The contribution of the pole at $\alpha = -\alpha_0$ is

$$-A_0 \exp(-\gamma_0 z)\cdot \exp(i\alpha_0 x), \quad (p_i = -k). \tag{80}$$

This is the reflected wave in the region $x > 0$. From (79), we obtain

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty + i\tau}^{\infty + i\tau} A(\alpha)\exp(-\gamma z)\cdot \exp(-i\alpha x)\,\mathrm{d}\alpha$$

$$= -A_0 \exp(-\gamma_0 z)\cdot \exp(i\alpha_0 x) - \frac{1}{(2\pi)^{1/2}} \int_{L_\alpha} A(\alpha)\exp(-\gamma z)\cdot \exp(-i\alpha x)\,\mathrm{d}\alpha$$

$$- \frac{1}{(2\pi)^{1/2}} \int_{\Sigma} A(\alpha)\exp(-\gamma z)\cdot \exp(-i\alpha x)\,\mathrm{d}\alpha. \tag{81}$$

Indentation around $\alpha = \alpha_0$ (Roos [6]) contributes

$$\phi_1(x, z) = \frac{A_0}{2} \exp(-i\alpha_0 x)\cdot \exp(-\gamma_0 z), \qquad x > 0, \tag{82}$$

$$= -\frac{A_0}{2} \exp(-i\alpha_0 x)\cdot \exp(-\gamma_0 z), \qquad x < 0. \tag{83}$$

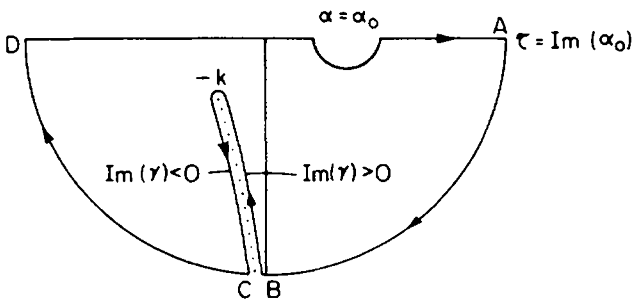

Figure 2.

The term in (82) is the incident wave and in (83) is the wave transmitted to the side $x < 0$.
We now consider

$$I(x, z) = \frac{1}{(2\pi)^{1/2}} \int_{L_a} A(\alpha) \exp(-\gamma z) \cdot \exp(-i\alpha x) \, d\alpha. \tag{84}$$

On L_a $\mathrm{Re}(\gamma) = 0$, and $\mathrm{Im}(\gamma)$ has different values on the two sides of the branch cut. Integrating along two sides of the branch cut, we get

$$\begin{aligned} I(x, z) = -\frac{1}{\pi} \int_{-k}^{-i\infty} \Bigg\{ \Bigg[& \frac{L_-(p_i) 2i\alpha_0 \gamma_0 A_0 H (p_i - k)^{1/2} \, (k - \alpha)^{1/2}}{(p_i - \alpha_0)(\alpha - \alpha_0)} \\ & - \frac{iHA_0(\alpha_0 + k)(k - \alpha) L_-(-\alpha_0)}{(\alpha - \alpha_0)} + \frac{i\gamma_0 A_0 (k - \alpha)}{(\alpha_0 + k) L_+(\alpha_0)(\alpha + \alpha_0)} \Bigg] \\ & \cosh \gamma_1 (z - H) L_+(-\alpha) + \Bigg[\frac{2i\alpha_0 \gamma_0 A_0 (p_i - k)^{1/2} \, (\alpha + k)^{1/2} \, L_-(p_i)}{(p_i - \alpha_0)(\alpha + \alpha_0)} \cosh \gamma_1 z \\ & + \left(\frac{iA_0(\alpha_0 + k)(\alpha + k) L_-(-\alpha_0)}{\alpha + \alpha_0} - \frac{i_0 A_0 (\alpha + k)}{H(\alpha_0 + k)(\alpha - \alpha_0) L_+(\alpha_0)} \right) \sinh \gamma_1 z \Bigg] \\ & \times \frac{\cosh \gamma_1 H}{L_-(\alpha) \gamma_1} \Bigg\} \frac{\exp(-\gamma_0 z) \exp(-i\alpha x)}{\gamma_1} \, d\alpha. \end{aligned} \tag{85}$$

In obtaining (85), we have kept in mind that $L_-(\alpha)$ being analytic in the lower half plane, does not change its value on two sides of the branch cut and it is the factor $(\alpha + k)^{1/2}$ of $\gamma = \pm (\alpha + k)^{1/2} \cdot (\alpha - k)^{1/2}$, which changes sign alongsides of branch cut. We have also used $\gamma = \mathrm{Im}\,\gamma = \gamma_1 > 0$ in the integral (85). The main contribution to the integral comes from the neighbourhood of the branch point $\alpha = -k$. To find it, we put $\alpha = -k - iu$, so that the neighbourhood of $\alpha = -k$, transforms into neighbourhood of $u = 0$. Under the substitution $\alpha = -k - iu$, the branch integral (85) transforms into a line integral as

$$\begin{aligned} I(x, z) = \frac{\exp(ikx - \gamma_0 H)}{-\pi} \int_0^{\infty} \Bigg\{ \Bigg[& \frac{-2\alpha_0 \gamma_0 A_0 H (p_i - k)^{1/2} (2k + iu)^{1/2} L_-(p_i)}{(p_i - \alpha_0)(k + iu + \alpha_0)} \\ & + \frac{HA_0(\alpha_0 + k)(2k + iu) L_-(-\alpha_0)}{k + iu + \alpha_0} - \frac{\gamma_0 A_0 (2k + iu)}{(\alpha_0 + k)(k + iu - \alpha_0) L_+(\alpha_0)} \Bigg] \cdot \\ & \times \cosh \gamma_1 (z - H) \, L_+(k + iu) \\ & + \Bigg[\frac{-2\alpha_0 \gamma_0 A_0 (p_i - k)^{1/2} (-iu)^{1/2} L_-(p_i) \cosh \gamma_1 z}{(k + iu - \alpha_0)(p_i - \alpha_0)} \\ & + \left(\frac{A_0(\alpha_0 + k)(iu) L_-(-\alpha_0)}{k + iu - \alpha_0} - \frac{\gamma_0 A_0 (iu)}{H(\alpha_0 + k)(k + iu + \alpha_0) L_+(\alpha_0)} \right) \sinh \gamma_1 z \Bigg] \\ & \times \frac{\cosh \gamma_1 H}{L_-(-k - iu) \gamma_1} \Bigg\} \frac{\exp(-ux)}{\gamma_1} \, du. \end{aligned} \tag{86}$$

This can be written as

$$I(x,z) = -\frac{\exp(ikx-\gamma_0 H)}{\pi}\int_0^\infty \left\{\left[\frac{G_1(u)}{\gamma_1}\cosh\gamma_1(z-H)\right]\right.$$

$$+\frac{G_2(u)}{\gamma_1}[\cosh\gamma_1(z+H)+\cosh\gamma_1(z-H)]$$

$$\left.+G_3(u)[\sinh\gamma_1(z+H)+\sinh\gamma_1(z-H)]\right\}\exp(-ux)\,du, \tag{87}$$

where

$$G_1(u) = \left[\frac{-2\alpha_0\gamma_0 A_0 H(p_i-k)^{1/2}\cdot(2k+iu)^{1/2}\cdot L_-(p_i)}{(p_i-\alpha_0)(k+iu+\alpha_0)}\right.$$

$$+\frac{HA_0(\alpha_0+k)(2k+iu)L_-(-\alpha_0)}{(k+iu+\alpha_0)}$$

$$\left.-\frac{\gamma_0 A_0(2k+iu)}{(\alpha_0+k)(k+iu-\alpha_0)L_+(\alpha_0)}\right]L_+(k+iu),$$

$$G_2(u) = -\frac{\alpha_0\gamma_0 A_0(p_i-k)^{1/2}\cdot(-iu)^{1/2}\cdot L_-(p_i)}{(p_i-\alpha_0)(k+iu-\alpha_0)L_+(-k-iu)}, \tag{88}$$

$$G_3(u) = \frac{i}{2}\left[\frac{A_0(\alpha_0+k)L_-(-\alpha_0)}{(k+iu-\alpha_0)} - \frac{\gamma_0 A_0}{H(\alpha_0+k)(k+iu+\alpha_0)L_+(\alpha_0)}\right]$$

$$\times\frac{u}{\gamma_1^2 L_-(-k-iu)}.$$

Further

$$\gamma^2 = 2iku - u^2 = 2ik_1 u - 2k_2 u - u^2. \tag{89}$$

Since the real part of the γ is zero; therefore from (89), we have

$$k_1 = 0, \quad \gamma^2 = -2k_2 u, \text{ for all small } u. \tag{90}$$

To evaluate the integral (87) in the neighbourhood of $u = 0$, we expand $G_1(u)$, $G_2(u)$ and $G_3(u)$ around $u = 0$, i.e.

$$G_1(u) = G_1(0) + uG_1'(0) + \frac{u^2}{\underline{|2}}G''(0) + \ldots \tag{91}$$

and similar expressions of $G_2(u)$ and $G_3(u)$. Since u is small, we retain $G_1(0)$, $G_2(0)$ and $G_3(0)$ and neglect the rest in expansion of $G_1(u)$, $G_2(u)$ and $G_3(u)$. The integral (87) results in

$$I(x,z) \sim -\frac{\exp(ikx-\gamma_0 H)}{i\pi}\int_0^\infty\left\{\frac{G_1(0)}{(2k_2u)^{1/2}}\cos(2k_2u)^{1/2}(z-H)\right.$$

$$+\frac{G_2(0)}{(2k_2u)^{1/2}}\cdot[\cos(2k_2u)^{1/2}(z+H)+\cos(2k_2u)^{1/2}(z-H)]$$

$$-G_3(0)\,[\sin(2k_2u)^{1/2}(z+H)+\sin(2k_2u)^{1/2}(z-H)]\Big\}\exp(-ux)\,du. \tag{92}$$

To evaluate the integrals in (92), we use the results (Sneddon, [7] p. 200)

$$\int_0^\infty \sin(u)^{1/2}x\exp(-ktu)\,du=\frac{(\pi)^{1/2}x\exp(-x^2/4kt)}{2(kt)^{3/2}}, \tag{93}$$

$$\int_0^\infty \frac{\cos(u)^{1/2}x}{(u)^{1/2}}\exp(-ktu)\,du=\frac{(\pi)^{1/2}}{(kt)^{1/2}}\exp(-x^2/4kt), \tag{94}$$

We obtain

$$\begin{aligned}I(x,z)\sim -\frac{\exp(ikx-\gamma_0H-i\pi/2)}{(2k_2\pi x)^{1/2}}&\Bigg[[G_1(0)\exp(-k_2(z-H)^2/2x)]\\&+G_2(0)[\exp(-k_2(z+H)^2/2x)+\exp[-k_2(z-H)^2/2x]]\\&-G_3(0)\Bigg[\frac{k_2(z+H)}{x}\exp(-k_2(z+H)^2/2x)\\&+\frac{k_2(z-H)}{x}\exp(-k_2(z-H)^2/2x)\Bigg]\Bigg]\end{aligned} \tag{95}$$

At points at large distances from the barrier, we have

$$-k_2[x^2+(z\pm H)^2]^{1/2}\sim -k_2x-k_2(z\pm H)^2/2x.$$

Then

$$\begin{aligned}I(x,z)=\frac{\exp(-\gamma_0H-i\pi/2)}{(2k_2\pi x)^{1/2}}&\Big\{G_1(0)\exp(-k_2[x^2+(z-H)^2]^{1/2})\\&+G_2(0)[\exp(-k_2[x^2+(z+H)^2]^{1/2})+\exp(-k_2[x^2\\&+(z-H)^2]^{1/2})-\frac{k_2}{x}G_3(0)[(z+H)\exp(-k_2[x^2+(z+H)^2]^{1/2})\\&+(z-H)\exp(-k_2[x^2+(z-H)^2]^{1/2})\Big\}\end{aligned} \tag{96}$$

$$\begin{aligned}=\frac{\exp(-\gamma_0H-i\pi/2)}{(2k_2\pi x)^{1/2}}&\Big\{G_1(0)\exp(-k_2r_1)+G_2(0)[\exp(-k_2r_2)\\&+\exp(-k_2r_1)]-k_2G_3(0)\Bigg[\frac{(z+H)}{x}\exp(-k_2r_2)\\&+\frac{(z-H)}{x}\exp(-k_2r_1)\Bigg]\Big\},\end{aligned} \tag{97}$$

where

$$r_1^2=x^2+(z-H)^2, \tag{98}$$

$$r_2^2=x^2+(z+H)^2, \tag{99}$$

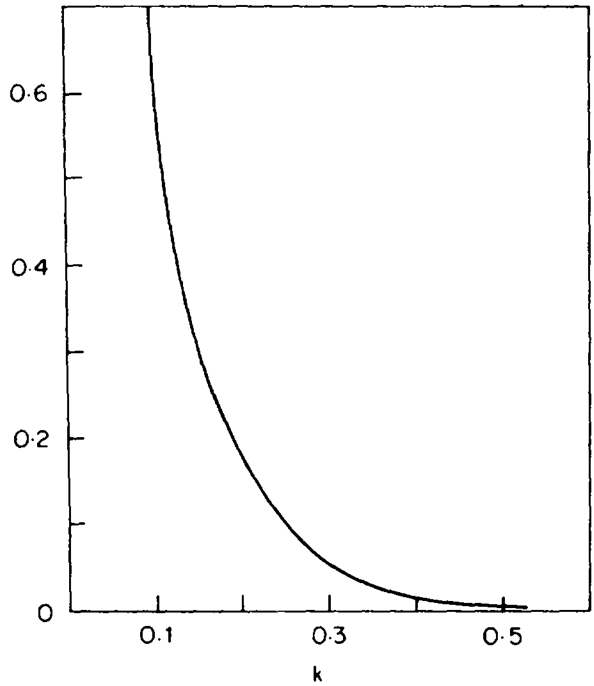

Figure 3.

and

$$G_1(0) = \left[-\frac{2\alpha_0 \gamma_0 A_0 H (p_i - k)^{1/2} (2k)^{1/2} L_-(p_i)}{(p_i - \alpha_0)(k + \alpha_0)} \right.$$

$$\left. + \frac{H A_0 (\alpha_0 + k)(2k) L_-(-\alpha_0)}{k + \alpha_0} - \frac{\gamma_0 A_0 2k}{(\alpha_0 + k)(k - \alpha_0) L_+(\alpha_0)} \right] L_+(k), \quad (100)$$

$$G_2(0) = \frac{\alpha_0 \gamma_0 A_0 (p_i - k)^{1/2} (-i)^{1/2} L_-(p_i)}{(k - \alpha_0)(p_i - \alpha_0) i (2k)^{1/2} L_-(-k)}, \quad (101)$$

$$G_3(0) = \frac{i}{2} \left[\frac{A_0 (\alpha_0 + k) L_-(-\alpha_0)}{(k - \alpha_0)} - \frac{\gamma_0 A_0}{H(\alpha_0 + k)(k + \alpha_0) L_+(\alpha_0)} \right]$$

$$\times \frac{1}{(-2k) L_-(-k)}. \quad (102)$$

6. Results

We now summarize the results obtained in the present paper. The reflected wave in region $x > 0$ is given by (80). Its magnitude is the same as that of the incident wave. The wave transmitted to the region $x < 0$ is given by (83) and has half the amplitude of the incident wave. The scattered waves are obtained in (97). They are of the form $\exp(-k_2 r_1)/(r_1)^{1/2}$ or $\exp(-k_2 r_2)/(r_2)^{1/2}$ where $r_1 = [x^2 + (z-H)^2]^{1/2}$ and $r_2 = [x^2 + (z+H)^2]^{1/2}$. These are cylindrical waves originating at the edge (0, H) of the barrier and at its image (0, $-H$) in the free surface. The graph of the amplitude of the scattered wave in terms of the wave number k is given by figure 3. It has a large value for $k = 0$ and decreases rapidly as k increases. The results are obtained when $H = 0.1$ km at a point $r = 10$ km in the free surface of the oceanic liquid.

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