Proc. Indian Acad. Sci. (Math. Sci.), Vol. 94, Nos 2 & 3, December 1985, pp. 61–69.

On Extended Hermite-Fezér interpolation based on the zeros of Laguerre polynomials

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Abstract. Szász [2] proved the existence of an interpolatory polynomial which interpolates a given function at given data with some extended points. He also gave its explicit representation. We study the uniform convergence of the same, interpolating at the zeros of Laguerre polynomials with one point (as origin) extension.

Keywords. Interpolation; polynomial approximation; uniform norm; Laguerre polynomials.

1. Introduction

Let there be two disjoint sets of points $\{x_{kn}\}$ $(k = 1, 2, \ldots, n)$ and $\{\xi_{im}\}$, $(i = 1, 2 \ldots, m)$ in the real interval $[A, B]$. $C[A, B]$ is a linear space of continuous functions defined over $[A, B]$. Then it is known [2] that there exists a uniquely determined polynomial $S_N(x)$ of degree $N \leqslant 2n+m-1$. $S_N(x) \in P(x)$ where $P(x)$ is a linear subspace of $C[A, B]$ consisting of all polynomials. $S_N(x)$ satisfies the following conditions:

$$S_N(x_k) = f(x_k),\ S'_N(x_k) = d_k\ (k = 1, \ldots n,\ x_k \equiv x_{kn},\ d_k \equiv d_{kn}), \tag{1}$$

$$S_N(\xi_i) = f(\xi_i),\ (\xi_{im} \equiv \xi_i,\ i = 1, 2 \ldots m). \tag{2}$$

d_k $(k = 1, \ldots n)$ are real numbers. Also $S_N(x)$ is given by

$$S_N(x) = \sum_{k=1}^{n} f(x_k)\frac{\Omega_m(x)}{\Omega_m(x_k)}[1 + c_k(x - x_k)]\, l_k^2(x)$$

$$+ \sum_{i=1}^{n} f(\xi_i)\frac{\omega_n^2(x)}{\omega_n^2(\xi_i)} L_i(x) + \sum_{k=1}^{n} d_k \frac{\Omega_m(x)}{\Omega_m(x_k)}(x - x_k) l_k^2(x)$$

$$\stackrel{\text{def}}{=} \sum_{k=1}^{n} f(x_k) h_k(x) + \sum_{i=1}^{m} f(\xi_i) t_i(x) + \sum_{k=1}^{n} d_k H_k(x), \tag{3}$$

where

$$\omega_n(x) = C_n(x - x_1) \ldots (x - x_n),\ C_n \neq 0; \tag{4}$$

$$\Omega_m(x) = K_m(x - \xi_1) \ldots (x - \xi_m),\ K_m \neq 0; \tag{5}$$

$$c_k = \sum_{i=1}^{n} \frac{1}{\xi_i - x_k} - \frac{\omega''_n(x_k)}{\omega'_n(x_k)},\ (k = 1, 2, \ldots n); \tag{6}$$

$$l_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)}; \tag{7}$$

$$L_i(x) = \Omega_m(x)[\Omega'(\xi_i)(x-\xi_i)]^{-1}. \tag{8}$$

$S_N(x) \equiv S_N(f, d, x)$, $(N \leqslant 2n+m-1)$ called extended Hermite-Fezér interpolation polynomial, has interpolatory property at the roots of $\omega_n(x)$ and $\Omega_m(x)$. Replacing $\omega_n(x)$ by Laguerre polynomials $L_n^{(\alpha)}(x)$ and denoting by x_{kn} ($\equiv x_{kn}^{(\alpha)}$, $k = 1, \ldots, n$) the kth root of $L_n^{(\alpha)}(x)$, from (3) we get for $m = 1$, $\xi_1 = 0$; $d_k = 0$, $k = 1, \ldots, n$;

$$S_N(x) \overset{\text{def}}{=} A_n^{(\alpha)}(f, x) = \left[\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}\right]^2 f(0) + \sum_{k=1}^{n} f(x_k)\frac{x}{x_k} v_k^{(\alpha)}(x)\left\{\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_k)(x-x_k)}\right\}^2, \tag{9}$$

where

$$v_k^{(\alpha)}(x) = 1 - \left\{\frac{1}{x_k} + \frac{L_n^{(\alpha)\prime\prime}(x_k)}{L_n^{(\alpha)\prime}(x_k)}\right\}(x-x_k). \tag{10}$$

$A_n^{(\alpha)}(f, x)$ is called A-quasi-Hermite-Fezér interpolation polynomial of degree $\leqslant 2n$. From differential equation

$$xy'' + (\alpha+1-x)y' + ny = 0,\ y \equiv L_n^{(\alpha)}(x) \text{ (see [3])}; \tag{11}$$

We have

$$v_k^{(\alpha)}(x) = 1 - \frac{(x_k-\alpha)(x-x_k)}{x_k}. \tag{12}$$

2. Results

Let $C(m) = \{f(x)\colon f \text{ is continuous in } [0, \infty), f(x) = O(x^m) \text{ as } x \to \infty;$ $m \geqslant 0 \text{ is an integer.}\}$

Consider the Hermite-Fezér interpolation polynomial

$$H_n^{(\alpha)}(f, x) = \sum_{k=1}^{n} f(x_k)v(x)l_k^2(x),\ \alpha > -1; \tag{13}$$

where

$$v\ (x) = \frac{(x-x_k)(\alpha-x_k)+x}{x_k},\ (k = 1, 2 \ldots, n).$$

Then the first convergence theorem by Szegö ([3] theorem 14.7) is

$$\lim_{n\to\infty} \|f(x) - H_n^{(\alpha)}(f, x)\|_I = 0 \tag{14}$$

$\forall f \in C(s)$ and $I \subset (0, \infty)$ for $\alpha \geqslant 0$, or, $I \subset [0, \infty)$ for $-1 < \alpha < 0$. Furthermore there exists a function in $C(m)$ such that $\{H_n^{(\alpha)}(f, x)\}$ diverges for $\alpha \geqslant 0$ at $x = 0$. As for the rate of the convergence the following result is due to Vertesi [4]

$$\|f(x) - H_n^{(\alpha)}(f, x)\|_I = \begin{cases} O(\omega(f, n^{-1-\alpha})); & -1 < \alpha < -\frac{1}{2} \\ O\left(\omega\left(f, \frac{\log n}{\sqrt{n}}\right)\right); & \alpha \geqslant -\frac{1}{2}, \end{cases} \tag{15}$$

$\forall f \in C(m)$ where $I \subset (0, \infty)$ and $\omega(f, h)$ is modulus of continuity in $[0, \Delta) \supset I$. Again $\forall f \in C(s)$ the following result is due to Szabados [1]

$$\|f(x) - H_n^{(\alpha)}(f, x)\|_{[0,\Delta]} = \begin{cases} O\left(\omega\left(f, \frac{\log n}{\sqrt{n}}\right)\right); -1 < \alpha \leqslant -\frac{1}{2} \\ O(\omega(f, n^{\alpha} \log n)); \ -\frac{1}{2} \leqslant \alpha < 0. \end{cases} \tag{16}$$

Our aim is to construct such interpolatory polynomial which converges to $f(x)$ in $[0, \Delta]$, $\Delta < \infty$ for $\alpha > -1$. However we conclude the following important result

$$\|f(x) - A_n^{(\alpha)}(f, x)\|_{[0,\Delta]} = \begin{cases} O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha - 3}\right), \ -\frac{1}{2} \leqslant \alpha < 1; \\ O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{-2}\right), \ -1 < \alpha < -\frac{1}{2}; \end{cases} \tag{17}$$

where $\omega_\Delta(h)$ is the modulus of continuity on $[0, \Delta + \varepsilon]$ such that $\omega_\Delta(f, h) = O[\omega_\Delta(h)]$ Thus we prove the following:

THEOREM 1.

$$f \in C(m) \Rightarrow (17), \text{ holds.} \tag{18}$$

For the lower estimation, we have

THEOREM 2. *Let* $\lim_{t \to 0} \omega(t) t^{-1} = \infty$ *then* $\exists f \in C(m)$:

$$\|f(x) - A_n^{(\alpha)}(f, x)\|_{[0,\Delta]} > \begin{cases} \sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha - 3}, \ -\frac{1}{2} \leqslant \alpha < 1; \\ \sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{-2}, \ -1 < \alpha < -\frac{1}{2}, \end{cases} \tag{19}$$

The following results shall be used in the sequel for $\alpha > -1$.

$$2\sqrt{x_{kn}^{(\alpha)}} = \frac{1}{\sqrt{n}}[k\pi + O(1)], \ (0 < x_{kn}^{(\alpha)} \leqslant \Omega, \ n = 1, 2 \ldots). \tag{20}$$

$$|L_n^{(\alpha)\prime}(x_k)| \sim k^{-\alpha - 3/2} n^{\alpha + 1}, \ (0 < x_k \equiv x_{kn}^{(\alpha)} \leqslant \Omega, \ n = 1, 2 \ldots). \tag{21}$$

$$|L_n^{(\alpha)}(x)| = \begin{cases} x^{-\alpha/2 - 1/4} O(n^{\alpha/2 - 1/4}), \ cn^{-1} \leqslant x \leqslant \Omega & (22) \\ O(n^{\alpha}), \ 0 \leqslant x \leqslant cn^{-1} & (23) \end{cases}$$

$$\sum_{k=1}^{n} x_k^{m-1} [L_n^{(\alpha)\prime}(x_k)]^{-2} = \frac{\Gamma(n+1)\Gamma(m+\alpha+3)}{\Gamma(n+\alpha+1)} \sim n^{-\alpha},$$

$(m = 1, 2 \ldots, 2n - 3)$.

$$\frac{d}{dx}\{L_n^{(\alpha)}(x)\} = -L_{n-1}^{\alpha+1}(x). \tag{24}$$

References for (20) to (24) is [3]. We also use the following elementary results [4]. Let x_j be the nearest root to x, then for $k \neq j$, $k = j \pm i$, $0 \leqslant x$, $x_k \leqslant \Omega$ we have

$$|x - x_k| = O(1)\frac{ij + i^2}{n} \tag{25}$$

$$(x_{k+1} - x_k) \sim \begin{cases} 1/n, \; k = O(1) \\ 1/\sqrt{n}, \; \varepsilon \leqslant x_k, \; x_{k+1} \leqslant \Omega. \end{cases} \tag{26}$$

$$|f(x) - f(x_j)| = \begin{cases} O(\omega_\Delta(1/n)), \; x = 0 \\ O(1)\,\omega_\Delta(1/\sqrt{n}), \; \delta \leqslant x, \; x_j \leqslant \Delta + \varepsilon \end{cases} \tag{27}$$

$$0 < \xi(\alpha) \leqslant v_k^{(\alpha)}(x) \leqslant M(\alpha) < \infty; \tag{28}$$

for $|x - x_k| \leqslant \xi(\alpha) \leqslant \varepsilon, \alpha < 1; \varepsilon > 0$ is arbitrary and fixed. Also we use without reference that $j \sim 1$ for $0 \leqslant x \leqslant c/n; j \sim \sqrt{n}$ and $k \sim \sqrt{n}$ for $0 < \delta \leqslant x, x_k \leqslant \Omega$ so that we have $i \sim k$ in $0 \leqslant x \leqslant \Delta + \varepsilon$ and $x_k \leqslant \Delta + \varepsilon$.

From (22), we have

$$\sqrt{x}\, L_n^{(\alpha+1)}(x) \sim \sqrt{n}\, L_n^{(\alpha)}(x) \text{ for } x \in J, \tag{29}$$

where

$$J = [c/n, \Delta + \varepsilon] \backslash \left\{ \bigcup_{k=1}^{n} [x_{k,n}^{(\alpha+1)} - c_1/n, x_{kn}^{(\alpha+1)} + c_1/n] \right.$$
$$\left. \bigcup \left\{ \bigcup_{k=1}^{n} [x_{kn}^{(\alpha)} - c_2/n, x_{kn}^{(\alpha)} + c_2/n] \right\} \right\}. \tag{30}$$

Moreover (29) holds in J^* where

$$J^* = [0, c/n] \backslash \{0\}. \tag{31}$$

3. Proofs

3.1 *Proof of theorem* 1.

From $A_n^{(\alpha+1)}(1, x) \equiv 1$ we have

$$|f(x) - A_n^{(\alpha+1)}(f, x)| \leqslant |f(x) - f(0)| \left[\frac{L_n^{(\alpha+1)}(x)}{L_n^{(\alpha+1)}(0)} \right]^2$$
$$+ \sum_{k=1}^{n} |f(x) - f(x_k)|\,|v_k^{(\alpha+1)}(x)| \left[\frac{\sqrt{x}\, L_n^{(\alpha+1)}(x)}{\sqrt{x_{kn}^{(\alpha+1)}}\, L_n^{(\alpha+1)\prime}(x_{kn}^{(\alpha+1)})(x - x_k)} \right]^2,$$
$$(x_k \equiv x_{kn}^{(\alpha+1)}) = A_1 + A_2 \text{ (say)}. \tag{32}$$

$$A_1 = |f(x) - f(0)| \left[\frac{L_n^{(\alpha+1)}(x)}{L_n^{(\alpha+1)}(0)} \right]^2$$
$$= \begin{cases} \omega(f, x), \; 0 \leqslant x \leqslant c/n \\ \omega(f, x)\, O[x^{-\alpha/2 - 3/4}\, O(n^{-\alpha/2 - 3/4})]^2, \; c/n \leqslant x \leqslant \Omega. \end{cases} \tag{33}$$
$$= \begin{cases} O(\omega_\Delta(1/n)); \; 0 \leqslant x \leqslant c/n \\ O(\omega_\Delta(1/\sqrt{n})); \; c/n < x \leqslant \Delta + \varepsilon \end{cases}$$

$$A_2 = \sum_{k=1}^{n} |f(x)-f(x_k)|\,|v_k^{(\alpha+1)}(x)|\left[\frac{\sqrt{x}\,L_n^{(\alpha+1)}(x)}{\sqrt{x_k}\,L_n^{(\alpha+1)\prime}(x_k)(x-x_k)}\right]^2;$$

$(x_k = x_{kn}^{(\alpha+1)})$.

but for $x\in J$ and $x_{kn}^{(\alpha+1)} \leqslant \Delta+\varepsilon$ we have

$$A_2 = O\left\{\sum_{k=1}^{n} \omega(f, |x-x_{kn}^{(\alpha+1)}|)\,|v_k^{(\alpha+1)}(x)|\left[\frac{L_n^{(\alpha)}(x)}{(x-x_{kn}^{(\alpha+1)})\,L_n^{(\alpha)\prime}(x_{kn}^{(\alpha)})}\right]^2\right\}. \quad (34)$$

Let $x\notin J$ then we can choose $x^*\in J$ such that $|x-x^*| \leqslant c/n$. Again we have

$$A_2 = O\left(\sum_{k=1}^{n} \omega(f, |x^*-x_{kn}^{(\alpha)}|)\,|v_k^{(\alpha+1)}(x)|\left[\frac{L_n^{(\alpha)}(x^*)}{(x^*-x_{kn}^{(\alpha)})\,L_n^{(\alpha)\prime}(x_{kn}^{(\alpha)})}\right]^2\right).$$

We write $A_i = O\left(\omega\left(f, \dfrac{ij+i^2}{n}\right)\right)$, so we have

$$A_2 \equiv \sum = \sum_{k=j} + \sum_{\substack{|x-x_k|\leqslant\xi(\alpha)\\ k\neq j}} + \sum_{\substack{|x-x_k|>\xi(\alpha)\\ x_k\leqslant\Delta+\varepsilon}} + \sum_{\substack{|x-x_k|>\xi(\alpha)\\ x_k>\Delta+\varepsilon}} = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}$$

(a) Let $c/n \leqslant x \leqslant \Delta+\varepsilon$ so that by

$$\frac{L_n^{(\alpha)}(x)-L_n^{(\alpha)}(x_j)}{x-x_j} = -L_n^{(\alpha)\prime}(x^*);\ x^*\in[x, x_j]$$

$$\mathrm{I} = O(\omega(f, |x-x_j|)) = O(\omega_\Delta(1/n)), \quad (35)$$

Now $|x-x_k| \leqslant \xi(\alpha) \Rightarrow k \sim \sqrt{n}$ and by (20) $|x-x_k| \sim k^2/n$ and (21), (22) and (28) yield

$$\begin{aligned}
\mathrm{II} &= O\left(\sum_{i=1}^{[\sqrt{n}]} A_i k^{2\alpha+3} n^{-2\alpha-2}\frac{n^2}{k^4} n^{\alpha-1/2} x^{-\alpha-1/2}\right)\\
&= O(x^{-\alpha-1/2} n^{-\alpha-1/2}) \sum_{i=1}^{[\sqrt{n}]} A_i i^{2\alpha-1},\ x\in[c/n, \Delta+\varepsilon],\ k\sim i,\\
&= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-1}\right);\ -1/2 \leqslant \alpha < 1,\\
&= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{-2}\right);\ -1 < \alpha < -1/2.
\end{aligned} \quad (36)$$

$$\begin{aligned}
\mathrm{III} &= O\left(\sum_{i=1}^{[\sqrt{n}]} A_i |v_k^{(\alpha)}(x)| k^{2\alpha+3} n^{-2\alpha-2} n^{\alpha-1/2} x^{-\alpha-1/2} \xi^{-2}\right);\\
&= O\left(\sum_{i=1}^{[\sqrt{n}]} A_i k^{2\alpha+3} n^{-2}\frac{n}{k^2}\right); \quad \left(v_k^{(\alpha)}(x) = O(x_k^{-1})\right.\\
&= \left.O\left(\frac{n}{k^2}\right) \text{ for } x_k \leqslant \Delta+\varepsilon\right); \quad O\left(\sum_{i=1}^{[\sqrt{n}]} A_i k^{2\alpha+1} n^{-1}\right);
\end{aligned}$$

$$= O\left\{\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) k^{2\alpha-1} \frac{k^2}{n}\right\}, (k \leqslant \sqrt{n});$$

$$= O\left\{\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) k^{2\alpha-1}\right\} = O\left\{\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-1}\right\}, (k \sim i). \tag{37}$$

We have $f(x) = O(1)$ and $f(x_k) = O(x_k^m)$, so that

$$\mathrm{IV} = \sum_{k=1}^{n} O(x_k^m) \frac{|x_k - (x_k - \alpha - 1)(x - x_k)|}{x_k} \frac{x}{x_k} \left[\frac{L_n^{(\alpha+1)}(x)}{L_n^{(\alpha+1)\prime}(x_k)}\right]^2 \xi^{-2},$$

$$= \sum_{k=1}^{n} O(x_k^m) x_k^{-2} x (x^{-\alpha/2-1/2-1/4} n^{\alpha/2+1/2-1/4})^2 [L_n^{(\alpha+1)\prime}(x_k)]^{-2},$$

$$= x^{-\alpha-1/2} O(n^{\alpha+1/2}) \sum_{k=1}^{n} x_k^{m-2} [L_n^{(\alpha+1)\prime}(x_k)]^{-2},$$

$$= O(nx^{-1})^{\alpha+1/2} \sum_{k=1}^{n} x_k^{m-1} [L_k^{(\alpha+1)\prime}(x_k)]^{-2};$$

$$\left(x_k > \Delta + \varepsilon \Rightarrow x_k^{-1} < \frac{1}{\Delta+\varepsilon}\right),$$

$$= \begin{cases} O(n^\alpha), x \in [c/n, \Delta+\varepsilon], \alpha \geqslant -1/2, \\ O(n^{-1/2}), x \in [c/n, \Delta+\varepsilon], \alpha < -1/2, \end{cases}$$

$$= \begin{cases} O\left(\sum_{[c\sqrt{n}]}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-1}\right); \alpha \geqslant -1/2, 0 < c < 1, \\ O\left(\sum_{[c\sqrt{n}]}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{-2}\right); \alpha < -1/2, 0 < c < 1. \end{cases}$$

Thus by (32) and I, II, III and IV we have for $c/n \leqslant x \leqslant \Delta + \varepsilon$,

$$|A_n^{(\alpha)}(f, x) - f(x)| = O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-3}\right), -1/2 \leqslant \alpha < 1 \tag{38}$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{-2}\right), -1 < \alpha < -1/2;$$

by substituting α in place of $\alpha + 1$.

(b) Let $0 \leqslant x \leqslant c/n$ we have

$$|f(0) - A_n^{(\alpha+1)}(f, 0)| = 0,$$

and

$$\mathrm{I} = O(1)\omega_\Delta(1/n).$$

$$\mathrm{II} = O\left(\sum_{i=1}^{[\sqrt{n}]} \omega(f, |x - x_k|) |v_k^{(\alpha)}(x)| k^{2\alpha+3} n^{-2\alpha-2} \frac{n^2}{k^4} n^{2\alpha}\right).$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-1}\right), \text{(by (28) and } i \sim k).$$

$$\mathrm{III} = O\left(\sum_{i=1}^{[\sqrt{n}]} \omega(f, |x - x_k| \ v_k^{(\alpha)}(x) | k^{2\alpha+3} n^{-2\alpha-2} \xi^{-2} n^{2\alpha}\right),$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) x_k^{-1} k^{2\alpha+3} n^{-2}\right),$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) k^{2\alpha+1} n^{-1}\right),$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) k^{2\alpha-1} (n^{-1} k^2)\right),$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-1}\right).$$

$$\mathrm{IV} = O(n^{2\alpha}) x \sum_{k=1}^{n} x_k^{m-1} [L_n^{(\alpha+1)\prime}(x_k)]^{-2}$$

$$= O(n^{\alpha-1}) x = O(n^{\alpha-2})$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-1}\right), \ \alpha > -1,$$

So in $[0, c/n]$ we too get

$$|f(x) - A_n^{(\alpha)}(f, x)| = O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-3}\right).$$

Thus in the sense of uniform norm,

$$\| A_n^{(\alpha)}(f, x) - f(x) \|_{[0,\Delta]} = \begin{cases} O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-3}\right), & -1/2 \leqslant \alpha < 1 \\ O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}(i^{-2}\right) & -1 < \alpha < -1/2. \text{ Q.E.D.} \end{cases}$$

4. We use theorem 4.1 of Vertesi [5] for the proof of theorem 2, which is as follows:

4.1. *Theorem* 4.1 *of Vertesi.*

If for the sequence of linear operators $T_n(f, x)$ and the functions $g_n(x)$, $(n = 1, 2, \ldots)$ we have

(a) $g_n(x) \in C(\omega)$,

(b) $T_n(g_n, z_n) \geqslant c_1 \lambda_n(z_n)$ for certain $\{z_n\} C[-1, 1]$, (definition of $\lambda_n(z_n)$).

(c) $\bar{f}(x) \stackrel{\text{def}}{=} Q \sum_{i=1}^{\infty} e_{n_i} g_{n_i}(x) \in C(\omega)$ for certain $\{n_i\}$ and $\{e_{n_i}\}$, $(0 < e_n \leqslant 1, \ e_{n+1} \leqslant e_n)$;

(d) $c_2 \lambda_{n_k}(z_{n_k}) > \sum_{i=k+1}^{\infty} e_{n_i} | T_{n_k}(g_{n_i}; z_{n_k})| + \sum_{i=k}^{\infty} e_{n_i} |g_{n_i}(z_{n_k})|$, $(k = 1, 2 \ldots)$;

with $c_2 < c_1$ then there exists an $f \in C(\omega)$ such that

$$T_n(f, z_n) - f(z_n) > e_n \lambda_n(z_n), \; (n = n_1, n_2 \ldots).$$

4.2 *Proof of theorem* 2.

Let

$$c_\Delta(w) = \{g_n(x): g_n(x) \text{ are continuous for } 0 \leqslant x \leqslant \Delta \text{ and } g_n(x) \equiv 0 \text{ for } x > \Delta,$$
$$\omega(g_n, t) = O(\omega(t)); \; \lim_{t \to 0} \omega(t) t^{-1} = \infty\}.$$

We select a subset of $C_\Delta(\omega)$ as follows

$$g_n(z_n) = g_n(0) = 0, \; 0 \leqslant z_n \leqslant x_1, \; (x_{1n}^{(\alpha)} = x_1),$$
$$g_n(z_n) = \omega(k/\sqrt{n}) \text{ sign } v_k^{(\alpha+1)}(x); \; x_1 \leqslant z_n \leqslant x_k \leqslant \Delta,$$
$$g_n(z_n) = g_n(x_k) = 0, \; x_k \geqslant \Delta.$$

Thus

$$T_n(g_n, z_n) \equiv A_n^{(\alpha+1)}(g_n, z_n) = \sum_{k=0}^{n} g_n(x_k) a_{kn}^{(\alpha+1)}(z_n),$$

where

$$a_{on}^{(\alpha+1)}(z_n) = \left[\frac{L_n^{(\alpha+1)}(z_n)}{L_n^{(\alpha+1)}(0)}\right]^2, \quad a_{kn}^{(\alpha+1)}(z_n)$$
$$= v_k^{(\alpha+1)}(z_n)\left[\frac{\sqrt{z_n}\, L_n^{(\alpha+1)}(z_n)}{\sqrt{x_k}\, L_n^{(\alpha+1)\prime}(x_k)(z_n - x_k)}\right]^2$$

so that

$$A_n^{(\alpha+1)}(g_n, z_n) = \sum_{k=0}^{n} g_n(x_k) a_{kn}^{(\alpha+1)}(z_n),$$
$$\geqslant c_4 \sum_{x_k \leqslant \Delta} \omega(k/\sqrt{n})\left[\frac{\sqrt{z_n}\; L_n^{(\alpha+1)}(z_n)}{\sqrt{x_k}\; L_n^{(\alpha+1)\prime}(x_k)(z_n - x_k)}\right]^2$$
$$\stackrel{\text{def}}{=} \lambda_n(z_n) \sim \left[c_s \sum_{|x - x_k| < \varepsilon} \omega(k/\sqrt{n}) \frac{l_k^2(z_n)}{x_k} + c_6 \sum_{\substack{|x - x_k| \geqslant \varepsilon \\ x_k \leqslant \Delta}} \omega(k/\sqrt{n}) \frac{l_k^2(z_n)}{x_k^2}\right];$$
$$\left(l_k(z_n) = \frac{L_n^{(\alpha)}(z_n)}{L_n^{(\alpha)\prime}(x_k)(z_n - x_k)}\right);$$
$$\sim \left[c_7 \sum_{i=1}^{[\sqrt{n}]} \omega(k/\sqrt{n}) i^{2\alpha - 1} + n^{-1} c_8 \sum_{i=1}^{[\sqrt{n}]} \omega(k/\sqrt{n}) i^{2\alpha - 1}\right]$$
$$\sim c_9 \sum_{i=1}^{[\sqrt{n}]} \omega(i/\sqrt{n}) i^{2\alpha - 1}; \; (-\tfrac{1}{2} \leqslant \alpha < 1).$$

We, therefore, conclude that for certain $f \in C(m)$, $-1/2 \leqslant \alpha < 1$, $\{z_n\} c [0, \Delta]$,

$$\| A_n^{(\alpha+1)}(f, z_n) - f(z_n) \|_{[0, \Delta]} > \sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha - 1}$$

or

$$||A_n^{(\alpha)}(f, z_{n_1}) - f(z_{n_1})||_{[0,\Delta]} > \sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-3}.$$

Similar arguments hold for other inequality. Q.E.D.

Acknowledgements

The author is grateful to Prof. P. Vertesi (Budapest, Hungary), for helpful suggestions. Financial support from UGC, New Delhi is gratefully acknowledged.

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