Proc. Indian Acad. Sci. (Math. Sci.), Vol. 93, Nos 2 & 3, December 1984, pp. 137–145

Principal bundles on the affine line

M S RAGHUNATHAN and A RAMANATHAN

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

Abstract. We prove that any principal bundle on the affine line over a perfect field with a reductive group as structure group comes from the base field by base change.

Keywords. Affine line; reductive algebraic group; principal bundle.

1. Introduction

Let k be an arbitrary field and G a linear algebraic group over k. Let $\mathbf{A}$ denote the affine line Spec $k[T]$, where $k[T]$ is the polynominal ring in one variable over k. We prove the following result in this paper.

THEOREM 1.1. Assume that G is connected and reductive. Let $B \to \mathbf{A}$ be a principal G-bundle on $\mathbf{A}$ such that $B \underset{\text{Spec } k}{\times} \text{Spec } k_s$ is trivial over Spec $k_s[T]$ where k_s is the separable closure of k. Then there is a principal G-bundle B_0 over Spec k such that B is k-isomorphic to the pull back of B_0 by the structure morphism $\mathbf{A} \to \text{Spec } k$.

If k is algebraically closed it follows from a theorem of Steinberg [16] that all G-bundles on $\mathbf{A}$, with G connected, are trivial (cf. [13]). Thus if k is perfect (in particular if the characteristic of k is zero) the assumption of triviality over k_s is always satisfied. On the other hand Knus *et al* [11] have shown that for separably closed non-perfect fields k there exist non-trivial $PGL(n)$-bundles on $\mathbf{A}$. However it seems likely that for simply connected groups G such a phenomenon does not occur.

If the base field k is of positive characteristic the connectedness assumption on G is essential even if k is algebraically closed, for Artin-Schrier extensions of $k(T)$ provide non-trivial finite Galois coverings of $\mathbf{A}$ [12, III §4].

Thus all the assumptions made in the theorem are essential.

The situation regarding non-reductive groups seems rather complicated and we make no efforts to examine it here. The result for many special cases is known: (i) for $GL(n)$ and inner forms of $GL(n)$ it is a reinterpretation of the fact that projective modules over $D[X]$, D a division algebra, are free (ii) for $G = Sp(n)$ it is the same as the classification of non-degenerate alternating forms over $k[X]$ and (iii) for $G = SO(n)$, char $k \neq 2$, it is due to Harder (see [10]). He used the method of extending from $\mathbf{A}$ to the projective line $\mathbf{P}$ which idea we have followed here.

2. Preliminary reductions

Throughout this paper k will denote an arbitrary field, k_s its separable closure and $\bar{k}$ its algebraic closure. For any scheme X over Spec k and a k-algebra R we denote the base change $X \times_k \operatorname{Spec} R$ by X_R and the R-valued points of X, *i.e.* the set of k-morphisms $\operatorname{Spec} R \to X$, by $X(R)$.

2.1 By a (principal) G-bundle over X we mean a scheme $E \to X$ over X together with an action of G (on the right) on E such that the morphism

$$E \times_k G \to E \times_X E$$

given by $(e,g) \to (e,eg)$ where e,g are R-valued points of E,G respectively, is an isomorphism (cf. [12], Chapter III §4 and [14]).

2.2 It is known ([14] or [12] Chapter III §4.2) that for a smooth G any G-bundle is locally trivial in the étale topology, *i.e.* X is covered by étale morphisms $f: U \to X$ such that f^*E is isomorphic to the trivial bundle $U \times G$ on U.

2.3 A section $\sigma: X \to E$ canonically gives rise to a trivialisation $\tilde{\sigma}: X \times G \to E$ defined by $\tilde{\sigma}(x,g) = \sigma(x)g$ where x,g are R-valued points of X,G and $\sigma(x)g$ is the translate of $\sigma(x)$ by g given by the action of G on X.

2.4 If F is a quasi-projective scheme and the smooth group G acts on F we can then form, using étale descent, the associated bundle $E(F) \to X$ with fibre F. Recall, [14], that $E(F)$ is the quotient of $E \times_k F$ by equivalence $(e,f) \sim (eg, g^{-1}f)$ where e,f,g are R-valued points of E,F,G respectively.

2.5 A section $\sigma: X \to E$ also gives a trivialisation $X \times F \to E(F)$ of any associated bundle $E(F)$ defined by sending (x,f) to the equivalence class of $(\sigma(x), f)$ in $E \times F$ where x,e,f are R-valued points X,E,F. We call an isomorphism $X \times F \to E(F)$ an *allowable trivialisation* if it comes from a section of E as above.

2.6 Let π be the Galois group of k_s over k. Then a standard argument ([12], Ex. 2.6 p. 93 and Chapter III §4) shows that the isomorphism classes of G-bundles on the affine line $\mathbf{A} = \operatorname{Spec} k[T]$ which become trivial on $\mathbf{A}_{k_s}$ are in natural one to one correspondence with the Galois cohomology set $H^1(\pi, G(k_s[T]))$, (see [15]). Thus our theorem can be interpreted as saying that the natural map $H^1(\pi, G(k_s)) \to H^1(\pi, G(k_s[T]))$ is a bijection when G is a connected reductive group. (The injectivity of this map is clear since it has a section given by restriction to a k-rational point).

PROPOSITION 2.7. *Let S be a torus over k. Then any S-bundle on $\mathbf{A}$ becomes trivial on $\mathbf{A}_{k_s}$ and is obtained from an S-bundle on* Spec *k by base change.*

Proof. The torus S_{k_s} over k_s splits into a product of, say n copies of, the multiplicative group G_m. Now a G_m-bundle is equivalent to a line bundle or again a projective module of rank 1 over $k[T]$. Since $k[T]$ is a principal ideal domain any projective module is trivial. Hence the first assertion. To see that the S-bundle comes from k we observe that $S(k_s[T]) = S_{k_s}(k_s[T]) = G_m(k_s[T])^n = S(k_s)$. Therefore $H^1(\pi, S(k_s[T])) = H^1(\pi, S(k_s))$.

Suppose now G is a connected reductive group and $B \to \mathbf{A}$ a G-bundle. Let $p: \operatorname{Spec} k \to \mathbf{A}$ be a k-rational point of $\mathbf{A}$. Then the pull back p^*B is a G-bundle on Spec k. Twisting G by this principal homogeneous space, *i.e.* forming the associated bundle of p^*B for the adjoint action of G on itself ([15] I §5.3) we obtain a new connected reductive k-group G', a k-form of G. We can also twist the bundle B by the pull back of p^*B by the structure morphism $\mathbf{A} \to \operatorname{Spec} k$ thereby obtaining a G'-bundle B' on $\mathbf{A}$. It is easy to see that the restriction of B' to the point p is trivial and that B comes from Spec k if and only if B' does. Thus we have proved the following claim.

CLAIM 2.8. To prove Theorem 1.1 we can assume without loss of generality that the bundle B is trivial when restricted to a given k-rational point of $\mathbf{A}$.

Next let G, $B \to \mathbf{A}$ be as above and $H = G/[G,G]$. Let $\varphi: G \to H$ be the natural projection. We then obtain from B by extension of structure group a H-bundle $B_H \to \mathbf{A}$ which is trivial over a k-rational point of $\mathbf{A}$. Since the theorem holds for tori (Proposition 2.7) and H is a torus, B_H is trivial. This means that B admits a reduction of structure group to $[G, G]$ (cf. [14], [13]). This proves the following claim.

CLAIM 2.9. It is enough to prove theorem 1.1 for G-bundles B such that G is semisimple and B restricted to a k-rational point of $\mathbf{A}$ is trivial.

Our method of proof of the theorem will be to show that G-bundles satisfying the assumptions of the above claim and the theorem admit a reduction of structure group to a suitable maximal torus T of G. Then we can appeal to Proposition 2.7 to conclude the theorem. For this purpose we will need the following lemma (2.10 below).

Let $\underline{G} \to X$ be a semisimple group scheme over X. A subgroup scheme $\underline{S} \subset \underline{G}$ is called a *maximal torus subgroup scheme* (or a *maximal subtorus*) if for any $x \in X(\bar{k})$ the fibre $\underline{S}_x$ is a maximal torus of the fibre $\underline{G}_x$ of $\underline{G}$.

Let G be a semisimple group over k and $\tilde{G} \to G$ a covering group (a central isogeny). Let $E \to X$ be a G-bundle. Let T be a maximal torus of G and $\tilde{T}$ the maximal torus of $\tilde{G}$ which is the inverse image of T. Let $E(\tilde{G}) \to X$ be the group scheme given by the associated bundle with fibre $\tilde{G}$ for the adjoint action of G on $\tilde{G}$.

LEMMA 2.10. With the above notation suppose that there is an embedding $\varphi: X \times_k \tilde{T} \to E(\tilde{G})$ of the constant torus scheme $X \times_k \tilde{T}$ as a maximal sub-torus of $E(\tilde{G})$ with property that X is covered by étale morphisms $f: U \to X$ such that there is an allowable trivialisation (see §2.5) $\psi: U \times_k \tilde{G} \to f^*E(\tilde{G})$ whose restriction to $U \times_k \tilde{T}$ coincides with $f^*\varphi$. Then φ gives rise to a natural reduction of structure group of E to T.

Proof. Let $f_i: U_i \to X$ be a covering of X by étale morphisms and $\sigma_i: U_i \to f_i^* E$ be sections giving allowable trivialisations, again denoted by σ_i, $\sigma_i: U_i \times \tilde{G} \to f_i E(\tilde{G})$ extending $f_i^* \varphi$. Let $\bar{\sigma}_i: U_i \to f_i^* E(G/T)$ be the composite of σ_i with the projection $f_i^* E(G) \to f_i(E(G/T))$. Since any inner automorphism of $\tilde{G}_{k_s}$ which is identity on $\tilde{T}$ comes from T_{k_s} it follows that in the fibre product $U_i \times_X U_j$ the pull backs of $\bar{\sigma}_i$ and $\bar{\sigma}_j$ coincide locally in the étale topology and hence by étale descent on the whole of $U_i \times_X U_j$. Again by étale descent the $\bar{\sigma}_i$ patch up to give a section $\bar{\sigma}: X \to E(G/T) = E/T$ which is the required reduction of structure group to T.

DEFINITION 2.11. Let X be a k-scheme and $\bar{X} = X_{\bar{k}}$. Let $J \subset \mathcal{O}_{\bar{X}}$ be the ideal sheaf of nilpotent elements. Let $I \subset \mathcal{O}_X$ be the intersection of all the ideal sheaves of $\mathcal{O}_X$ whose extensions to $\mathcal{O}_{\bar{X}}$ contain J. We then call the closed k-subscheme X_0 of X corresponding to I the *separably reduced* of X. (Leema 2.13 below offers some justification for this terminology).

Remarks 2.12. Note that the extension of I to $\mathcal{O}_{\bar{X}}$ contains J and hence X_0 is an absolutely reduced scheme. Further if Y is an absolutely reduced scheme then any k-morphism $Y \to X$ factors through X_0. If the base field k is perfect any reduced scheme is absolutely reduced and the separably reduced of X coincides with the reduced of X.

LEMMA 2.13. With the above notation, $X_0(k_s) = X(k_s)$ and X_0 is characterised by the property that $X_0\ (\bar{k})$ is the Zariski closure of $X\,(k_s)$ in $X\,(\bar{k})$.

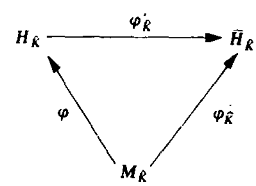
Proof. This follows from the fact that a reduced k-scheme Y is absolutely reduced if and only if $Y(k_s)$ is dense in $Y(\bar{k})$ (see [1], theorem 14.4 Chap. AG).

3. Construction of a group scheme over P

Let G be a connected semisimple group over k and $\tilde{G}$ its simply connected covering. Let $B \to \mathbf{A}$ be a G-bundle which becomes trivial on $\mathbf{A}_{k_s}$. Let $H = E(\tilde{G}) \to \mathbf{A}$ be the associated group scheme for the adjoint action of G on $\tilde{G}$. We will now extend the group scheme H to a smooth group scheme $\bar{H} \to \mathbf{P}$ over the projective line $\mathbf{P} = \mathbf{A} \cup \{\infty\}$, though $\bar{H}_\infty$ the fibre at ∞ may not be reductive.

Let R be the local ring of $\mathbf{P}$ at the point ∞ and $\hat{R}$ its completion with respect to the maximal ideal. Let K be the quotient field of R and $\hat{K}$ its completion (with respect to the valuation corresponding to ∞). Note that $\hat{K}$ is the quotient field of $\hat{R}$.

LEMMA 3.1. Suppose $M \to \operatorname{Spec} \hat{R}$ is a smooth affine group scheme and $\varphi: M_{\hat{K}} \to H_{\hat{K}}$ an isomorphism of $\hat{K}$-groups where $M_{\hat{K}}$ is the base change of M by $\hat{R} \to \hat{K}$ and $H_{\hat{K}}$ that of H by $R \hookrightarrow K \hookrightarrow \hat{K}$. Then there is a smooth group scheme $\bar{H} \to \mathbf{P}$, such that there are isomorphisms of group scheme $\varphi': H \to \bar{H}|\mathbf{A}$ and $\varphi'': M \to \bar{H}_{\hat{R}}$ making the diagram commutative.



Proof. Consider the morphisms $\mathbf{A} \hookrightarrow \mathbf{P}$ and $\operatorname{Spec} \hat{R} \to \mathbf{P}$. Since $\hat{R}$ is flat over R these two morphisms give a covering of $\mathbf{P}$ in the faithfully flat quasi-compact topology. The fibre product $\mathbf{A} \times_{\mathbf{P}} \operatorname{Spec} \hat{R}$ is $\operatorname{Spec} K$. Since $M \to \mathbf{A}$ and $N \to \operatorname{Spec} \hat{R}$ are affine the lemma is an immediate consequence of faithfully flat descent ([12] I, theorem 2.23 p. 19).

PROPOSITION 3.12. There is a smooth affine group scheme $\overline{H} \to \mathbf{P}$ extending $H \to \mathbf{A}$ with the following properties.

(a) There is a finite Galois extension L of k such that the base change $\overline{H}_L \to \mathbf{P}_L$ admits a constant torus subgroup scheme $\varphi: \mathbf{P}_L \times_L S \to \overline{H}_L$ where S is a maximal torus of $\tilde{G}_L$ which is split.

(b) On $\mathbf{A}_L$ there is an allowable trivialisation (see §2.5) $\mathbf{A}_L \times_k \tilde{G} \to H_L$ extending $\varphi|_{\mathbf{A}_L \times_L S}$.

The rest of this section will be devoted to proving this proposition. We will obtain the extension $\overline{H}_L$ by making use of the theory of Bruhat-Tits for groups over local fields ([5] §5, also [3], [4]). To get the constant subtorus we will also make use of the theorem of Grothendieck-Harder ([6], [9], [13]) on G-bundles over $\mathbf{P}$.

By assumption $B_{k_s} \to \mathbf{A}_{k_s}$ is trivial. Since all the schemes involved are of finite type over k we can find a finite Galois extension L over k such that (i) $B_L \to \mathbf{A}_L$ is trivial and (ii) $\tilde{G}$ has a maximal torus S defined and split over L.

Let $R' = R \otimes_k L$ be the local ring of $\mathbf{P}_L$ at ∞ and $K' = K \otimes_k L$ be its quotient field. Let $\hat{R}'$ and $\hat{K}'$ be the respective completions with respect to the valuation corresponding to ∞. Note that $\hat{R}' = \hat{R} \otimes_k L$ and $\hat{K}' = \hat{K} \otimes_k L$ and the Galois group of $\hat{K}'$ over $\hat{K}$ is the same as the Galois group π of L over k.

It follows from Lemma 3.1 that to get an extension of H_L to $\mathbf{P}_L$ it is enough to construct a smooth affine group scheme $M \to \operatorname{Spec} \hat{R}'$ together with an isomorphism $\varphi: H_{\hat{K}'} \to M_{\hat{K}'}$ of the generic fibres.

Consider the semisimple simply connected split group $H_{\hat{K}'}$ over the complete local field $\hat{K}'$. Let $P \subset H_{\hat{K}'}(\hat{K}')$ be a parahoric subgroup of $H_{\hat{K}'}$ (see [3] §2). Then according to Bruhat-Tits if A is the subring of the coordinate ring of $H_{\hat{K}'}$ consisting of functions which take values in $\hat{R}'$ on P then $M' = \operatorname{Spec} A \to \operatorname{Spec} \hat{R}'$ is a smooth affine group scheme such that the natural map $A \otimes_{\hat{R}'} \hat{K}'$ into the coordinate ring of $H_{\hat{K}'}$ is an isomorphism. Thus the generic fibre of M' is canonically isomorphic to $H_{K'}$ and therefore M' gives an extension $\overline{H}' \to \mathbf{P}_L$ of H_L (Lemma 3.1).

If moreover the parahoric subgroup P is invariant under the natural action of the Galois group π on $H_{\hat{K}'}(\hat{K}')$ then π acts on M' compatibly with its action on $H_{\hat{K}'}$. Hence π then acts on $\overline{H}'$. By Galois descent it then follows that M' and $\overline{H}'$ descend to $M \to \operatorname{Spec} \hat{R}$ and $\overline{H} \to \mathbf{P}$ respectively.

The fixed point theorem of Bruhat-Tits ([5], §5) guarantees the existence of a π-invariant parahoric subgroup P_0. Let $H'_0 \to \mathbf{P}_L$ and $\overline{H}_0 \to \mathbf{P}$ be the extensions corresponding to P_0.

Now fix a section $\sigma: \mathbf{A}_L \to B_L$. Then σ gives an allowable trivialisation $\tilde{\sigma}: \mathbf{A}_L \times_k \tilde{G} \to H_L$. Its restriction $\tilde{\sigma} | \mathbf{A}_L \times_{\mathrm{L}} S$ gives a constant subtorus of H_L. Let I be the apartment of the Bruhat-Tits building corresponding to the split torus $\tilde{\sigma}(S_{\hat{K}'})$. Then if P is a parahoric subgroup belonging to I by [4] we have a canonical embedding of the constant torus $S_{\hat{R}'} = \operatorname{Spec} \hat{R}' \times_L S$ in M' such that the diagram below commutes.

$$\begin{array}{ccc} S_{\hat{K}'} & \hookrightarrow & M' \\ \uparrow & & \uparrow \\ S_{\hat{K}'} & \xrightarrow{\tilde{\sigma}} & H_{\hat{K}'} = M_{\hat{K}'} \end{array}$$

As in 3.1 this implies that the subtorus $\tilde{\sigma}$: $\mathbf{A}_L \underset{L}{\times} S \to H_L$ extends to φ: $\mathbf{P}_L \underset{L}{\times} S \to \bar{H}'$.

Now by the conjugacy theorem for parahoric subgroups ([3] §3) we can find a parahoric subgroup P_1 belonging to I such that $P_0 = \operatorname{Int} g(P_1)$ where $g \in H_{\hat{K}'}(\hat{K}')$. Since parahoric subgroups are open and $H_{K'}(K')$ is dense in $H_{\hat{K}'}(\hat{K}')$ we can take g in $H_{K'}(K')$. Then there exist sections $\alpha: \mathbf{A}_L \to H_L$, $d \in S_{K'}$ and $\beta \in P_1$ such that $g = \alpha_{K'} d\beta$. This is essentially a consequence of the theorem of Grothendieck-Harder that any G_L-bundle on $\mathbf{P}_L$ admits a reduction of structure group to a split maximal torus (see [13], theorem 4.2).

Let $P_2 = \operatorname{Int} d(P_1)$. Then P_2 belongs to I and hence we have the constant subtorus $\varphi_2: \mathbf{P}_L \underset{L}{\times} S \to \bar{H}'_2$ where the latter is the extension of H_L corresponding to P_2. Since $P_0 = \operatorname{Int} \alpha(P_2)$ it is easily seen that $\operatorname{Int} \alpha: H_L \to H_L$ extends to $\psi: \bar{H}'_2 \to \bar{H}'_0$. It now follows that the extension $\bar{H}_0 \to \mathbf{P}$ and the constant subtorus $\varphi = \psi \cdot \varphi_2: \mathbf{P}_L \underset{L}{\times} S \to \bar{H}'_0$ satisfy the conditions (a) and (b) of the proposition. This completes the proof of Proposition 3.2.

In §4 we shall show how to pass from φ, which is defined only over L, to a subtorus defined over k using the group of global section of $\bar{H}$.

4. Sections of $\bar{H}$

4.1 Let X be a projective scheme over k and $f: \underline{G} \to X$ be a group scheme of finite type, affine over X. Consider the functor which associates to each k-scheme Y the group of global sections of the product group scheme $\varphi \times id: \underline{G} \underset{k}{\times} Y \to X \times Y$. By [9] §1.4 or [7] we know that this functor is representable by a group scheme $\Gamma(\underline{G})$ of finite type over k. We also have the evaluation morphism (the universal global section) ev:

$$\begin{array}{ccc} \Gamma(\underline{G}) \times X & \xrightarrow{ev} & \underline{G} \\ & {\scriptstyle p_2} \searrow \quad \swarrow {\scriptstyle \varphi} & \\ & X & \end{array}$$

4.2 If $\underline{N} \subset \underline{G}$ is a subgroup scheme then the global section functor of $\underline{N}$ is a subfunctor of that of $\underline{G}$. Therefore $\Gamma(\underline{N})$ is a subgroup scheme of $\Gamma(\underline{G})$.

4.3 If N is an affine k-group scheme then the global sections functor of the constant group scheme $N \underset{k}{\times} X \to X$ is clearly represented by N itself, since X is projective.

4.4 Now we apply the above discussions to the group scheme $\bar{H} \to \mathbf{P}$ constructed in the previous section (Proposition 3.2). Let Γ' be the group of global sections of $\bar{H}$. Let Γ be the separably reduced of Γ' (see §2.11). Since Γ is a group scheme and is absolutely reduced it is a smooth group scheme.

By the rank of a k-group scheme we mean the dimension of a maximal subtorus defined over the algebraic closure $\bar{k}$ of k.

By Proposition 3.2 $\bar{H}$ has the constant maximal torus subgroup scheme $\mathbf{P}_L \underset{L}{\times} \tilde{S}$. Therefore by the remarks in §§4.2 and 4.3 (and §2.12) $\tilde{S}$ is naturally embedded in Γ_L. This shows that rank $\Gamma \geqslant$ rank G. In fact we prove that rank $\Gamma =$ rank G in the proposition below.

Proposition 4.5. $\tilde{S} \subset \Gamma_L$ is a maximal torus of Γ

Proof. Consider the Lie algebra bundle $V \to \mathbf{P}$ of the smooth group scheme $\bar{H} \to \mathbf{P}$. The fibres of V are of constant rank and $V \to \mathbf{P}$ is a vector bundle with Lie algebra structures on fibres. Let $\sigma \in \Gamma(\bar{k})$ be a $\bar{k}$-rational point of Γ. Then σ gives a section $\mathbf{P}_{\bar{k}} \to \bar{H}_{\bar{k}}$ and by the adjoint action gives a section $Ad\,\sigma : \mathbf{P}_{\bar{k}} \to \operatorname{End} V_{\bar{k}}$. Since the ith coefficient of the characteristic polynominal of an endomorphism is invariant under inner conjugation it gives rise to a function $a_i : \operatorname{End} V_{\bar{k}} \to \bar{k}$. Composing with $Ad\,\sigma$ we then get a regular function $a_i\, Ad\,\sigma : \mathbf{P}_{\bar{k}} \to \bar{k}$. This must be a constant since $\mathbf{P}_{\bar{k}}$ is complete. Thus the characteristic polynominal of $Ad\,\sigma(x)$ is the same for all $x \in \mathbf{P}(\bar{k})$.

Now let $S \subset \Gamma_{\bar{k}}$ be a maximal torus of $\Gamma_{\bar{k}}$ and $x \in \mathbf{A}(\bar{k})$. We then have the evaluation morphism $ev_x : S_{\bar{k}} \to \bar{H}_x$. Suppose for some $\sigma \in S(\bar{k})$ we have $ev_x(\sigma) = \sigma(x)$ is the identity. Then $Ad\,\sigma(x) = id.$ for any $y \in \mathbf{P}^1(\bar{k})$, $Ad\,\sigma(y)$ is semisimple since it is the image of the semisimple element $\sigma \in S(\bar{k})$. Further it has the same characteristic polynominal as $Ad\,\sigma(x)$. Therefore it must be the identity. This shows that on $\mathbf{A}$ the section $\sigma : \mathbf{A}_{\bar{k}} \to B_{\bar{k}}(\tilde{G})$ factors through the kernel $B_{\bar{k}}(F) \subset B_{\bar{k}}(\tilde{G})$ where F is the kernel of $\tilde{G} \to G$. Since F is a finite group scheme the reduced of its identity component is the trivial group $\{1\}$. Therefore the reduced of the identity component of $B_{\bar{k}}(F)$ is $B_{\bar{k}}(\{1\})$. Since $\mathbf{A}_{\bar{k}}$ is reduced and connected the morphism $\sigma : \mathbf{A}_{\bar{k}} \to B_{\bar{k}}(F)$ then factors through $B_{\bar{k}}(\{1\})$ which shows that $\sigma = id$.

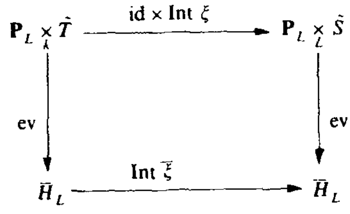
Thus we have proved that $ev_x : S_{\bar{k}} \to B_{\bar{k}}(\tilde{G})$ is injective on $\bar{k}$-valued points. Hence it must be an isogeny and in particular the image of ev_x is a torus of the same dimension as S. This shows that rank $\Gamma \leqslant$ rank G and hence $\tilde{S} \subset \Gamma_L$ must be a maximal torus.

Proposition 4.6. Suppose $B \to \mathbf{A}$ is such that for a k-rational point x_0 of $\mathbf{A}$ the restriction $B_{x_0} \to \operatorname{Spec} k$ is trivial. Then there is a maximal torus $\tilde{T}$ of $\tilde{G}$ defined over k such that

(a) We have an embedding $\psi : \mathbf{P} \underset{k}{\times} \tilde{T} \to \bar{H}$ (where $\bar{H}$ is the extension constructed in Proposition 3.2).

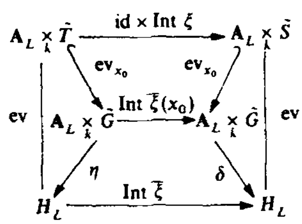
(b) When base changed to a suitable finite Galois extension L of k, $\psi_L : \mathbf{A}_L \underset{k}{\times} \tilde{T} \to H$ is the restriction of an allowable trivialisation $\mathbf{A}_L \underset{k}{\times} \tilde{G} \to H_L$.

Proof. The separably reduced Γ of the group of global sections of B is a smooth group scheme over k. Hence by Grothendieck's theorem ([2] or [8] Expose XIV) it has a maximal torus $\tilde{T}$ defined over k. By making the field L considered earlier in Proprosition 3.2 larger if necessary we can assume that $\tilde{T}_L$ and the maximal torus $\tilde{S}$ of Proposition 4.5 are conjugate by an element ξ of $\Gamma(L)$. Let $\bar{\xi}$ denote the section of $\mathbf{P}_L \to \bar{H}_L$ corresponding to ξ. We then have the commutative diagram.



Since the second vertical arrow is an embedding and the horizontal arrows are isomorphism it follows that the first vertical arrow is also an isomorphism. Since it is the pull back to L of $\mathbf{P} \underset{k}{\times} \tilde{T} \to \bar{H}$ we have proved that the latter is an embedding.

On $\mathbf{A}_L$ let $\delta: \mathbf{A}_L \underset{k}{\times} \tilde{G} \to H_L$ be the allowable trivialisation extending $ev: \mathbf{A}_L \times \tilde{S} \to H_L$ given by Proposition 3.2. Since by assumption $B_{x_0} \to \operatorname{Spec} k$ is trivial there is an allowable trivialisation $B_{x_0}(\tilde{G}) \approx \tilde{G}$. We make this identification of $B_{x_0}(\tilde{G})$ with $\tilde{G}$. We then have the commutative diagram



where η is defined to be $\operatorname{Int} \xi^{-1} \circ \delta \circ \operatorname{Int} \bar{\xi}(x_0)$. Thus η provides the allowable trivialisation extending $\mathbf{A}_L \underset{k}{\times} \tilde{T} \to H_L$. The point in this is that the torus $\tilde{T}$ which *a priori* sits in Γ gets identified as a subtorus of $\tilde{G}$ by evaluation at x_0 and the morphism of tori $\operatorname{Int} \xi: \tilde{T} \to \tilde{S}$, when $\tilde{T}$ and $\tilde{S}$ are considered as subtori of $\tilde{G}$ coincides with the restriction of the inner automorphism $\operatorname{Int} \bar{\xi}(x_0)$ of $\tilde{G}$. This completes the proof of Proposition 4.6.

We can now quickly run through the argument needed to finish the proof of theorem 1.1.

Let $B \to \mathbf{A}$ be a G-bundle which becomes trivial on $\mathbf{A}_{k_s}$, with G connected reductive. By Claim 2.9 we can assume that G is semisimple and B restricted to some k-rational point of $\mathbf{A}$ is trivial. Then Proposition 4.6 and Lemma 2.10 show that B admits a reduction of structure group to a maximal torus T of G. Since by Proposition 2.7 any T-bundle on $\mathbf{A}$ is the pull back of a T-bundle on Spec k this completes the proof of theorem 1.1.

Acknowledgements

The authors thank Gopal Prasad and Harder for their interest and suggestions.

References

[1] Borel A 1969 *Linear algebraic groups* (New York: Benjamin)
[2] Borel A and Springer T A 1968 *Tohoku Math. J.* **22** 443
[3] Bruhat F and Tits J 1967 *Proceedings: Conference on local fields* (Driebergen 1966) (Heidelberg: Springer Verlag) pp. 23–36
[4] Bruhat F and Tits J 1966 *C.R. Acad. Sci.* **263** 822
[5] Bruhat F and Tits J 1984 *Pub. Math. IHES* **60** 1
[6] Grothendieck A 1957 *Am. J. Math.* **79** 121
[7] Grothendieck A 1961 *Seminaire Bourbaki Exp.* 221
[8] Grothendieck A and Demazure M 1962 *Schémas en Groupes II* (Heidelberg: Springer Verlag).
[9] Harder G 1968 *Invent. Math.* **6** 107
[10] Knebusch M 1970 *Sitzungsber. Heidelb. Akad. Wiss. Math.* **3** 90
[11] Knus M A, Ojanguren M and Saltman D J 1975 *Brauer Groups* (Evanston 1975) (Heidelberg: Springer Verlag) pp. 25–29
[12] Milne J S 1980 *Étale cohomology* (Princeton: Princeton University Press)
[13] Ramanathan A 1983 *Invent. Math.* **71** 165
[14] Serre J P 1958 Anneaux de Chow et applications Seminaire Chevalley
[15] Serre J P 1962 *Cohomologie galoisienne* (Heidelberg: Springer Verlag)
[16] Steinberg R 1965 *Pub. Math. IHES* **25** 281