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Mean-value of the Riemann zeta-function on the critical line

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Abstract. This is an expository article. It is a collection of some important results on the mean-value of $|\zeta(\frac{1}{2}+it)|$.

Keywords. Mean value; zeta-function; critical line.

1. Introduction

This article is a collection of some important results on the mean value of $|\zeta(\frac{1}{2}+it)|$ and its powers. Section 2 deals with the mean square and §3 with the mean fourth power and the mean twelfth power. Section 4 deals with the general $(2k)$th power where $k > 0$ and §5 gives other important results.

2. Mean square

In 1918 Hardy and Littlewood [12] proved that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \,dt \sim T\log T.$$

In 1926 this was improved by Ingham [18] to

$$\frac{1}{2\pi}\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \,dt = \frac{T}{2\pi}\log\frac{T}{2\pi} + (2\gamma-1)\frac{T}{2\pi} + E(T),$$

where $E(T) = 0\,(T^{1/2}\log T)$. More complicated argument due to Titchmarsh [33] showed that $E(T) = 0\,(T^{5/12}(\log T)^2)$. The final improvement of this theorem was published in 1978 by Balasubramanian [3] who proved that $E(T) = 0(T^{\frac{1}{3}})$, $E(T) = 0(T^{27/82})$, and so on. It should be mentioned that subsequent to this discovery Good [10] gave a variant of Balasubramanian's proof. It should also be mentioned that Heath-Brown's powerful method [14] which gives the mean fourth power with error term also gives the theorem of Balasubramanian.

Titchmarsh [34, 35] had introduced a new mean value. He had proved that, as $\delta \to 0$

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 \exp(-\delta t)\,dt \sim \frac{1}{\delta}\log\frac{1}{\delta}$$

and

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 \exp(-\delta t)\,dt \sim \frac{1}{2\pi^2\delta}\left(\log\frac{1}{\delta}\right)^4.$$

Kober [22] proved that

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 \exp(-2\delta t)\,\mathrm{d}t = \frac{\gamma - \log(4\pi\delta)}{2\sin\delta} + \sum_{n=0}^{N} c_n \delta^n + 0(\delta^{N+1}),$$

where the 0-constant depends on N. A simpler proof was given by Atkinson [1]. Regarding Ω theorems for $E(T)$ Good [11] developed a method to prove that $E(T) = \Omega(T^{1/4})$. Heath-Brown [15] developed a more powerful method which actually gives

$$\int_0^T (E(t))^2\,\mathrm{d}t = cT^{3/2} + 0(T^{5/4}(\log T)^2),$$

where $c = \frac{2}{3}(2\pi)^{-1/2}(\zeta(\frac{3}{2}))^4(\zeta(3))^{-1}$.

3. Mean fourth power

In 1922 Hardy and Littlewood [13] proved that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t = 0(T(\log T)^4).$$

This was improved in 1926 by Ingham [18] to

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t = \frac{T}{2\pi^2}(\log T)^4 + 0\ (T(\log T)^3)$$

by using the approximate functional equation for $(\zeta(s))^2$. A simpler proof of this fact was given recently by Ramachandra [24] using the theorem of Montgomery and Vaughan. Subsequently Heath-Brown [14] showed by deep arguments that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t = TP(\log T) + 0(T^{\frac{7}{8}+\varepsilon}),$$

where $P(x)$ is a polynomial of degree 4 in x. The coefficient of x^4 is $1/2\pi^2$ and that of x^3 is

$$2\pi^{-2}[4\gamma - 1 - \log(2\pi) - 12\pi^{-2}\zeta'(2)].$$

He also proved another deep result [16] which asserts that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{12}\,\mathrm{d}t = 0(T^2(\log T)^{17}).$$

Previously Atkinson [2] had proved that

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 \exp(-\delta t)\,\mathrm{d}t = \frac{1}{\delta}Q\left(\log\frac{1}{\delta}\right) + 0\left(\left(\frac{1}{\delta}\right)^{\frac{8}{9}+\varepsilon}\right)$$

where $Q(x)$ is a polynomial of degree 4 in x. The coefficient of x^4 is $1/2\pi^2$ and that of x^3 is

$$-\frac{1}{\pi^2}\left[2\log(2\pi) - 6\gamma + \frac{24\zeta'(2)}{\pi^2}\right].$$

For the fourth power mean Iwaniec [19] proved the following deep result. Let $I_r (r = 1, 2, \ldots, R)$ be R disjoint intervals of length T_0 satisfying $T^{1/2} \leqslant T_0 \leqslant T$. Then

$$\sum_{r=1}^{R} \int_{I_r} |\zeta(\tfrac{1}{2}+it)|^4 \, dt \ll_\varepsilon (RT_0 + T(R/T_0)^{1/2})T^\varepsilon.$$

Choosing $R = 1, T_0 = T^{2/3}$ we obtain

$$\int_T^{T+T^{2/3}} |\zeta(\tfrac{1}{2}+it)|^4 \, dt \ll T^{\frac{2}{3}+\varepsilon}.$$

The deep result of Iwaniec also implies

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{12} \, dt \ll T^{2+\varepsilon}.$$

This can be seen as follows. It suffices to prove that

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{12} \, dt \ll T^{2+\varepsilon}.$$

The contributions to the twelfth power moment from those t with $|\zeta(\frac{1}{2}+it)| \leqslant T^{\frac{1}{8}+\varepsilon}$ are together $\ll T^{2+9\varepsilon}$ using $\int_0^T |\zeta(\frac{1}{2}+it)|^4 \, dt \ll T(\log T)^4$. So one has to consider the range

$$T^{\frac{1}{8}+\varepsilon} \leqslant |\zeta(\tfrac{1}{2}+it)| \leqslant T^{\frac{1}{6}+\varepsilon}$$

since $\mu(\frac{1}{2}) \leqslant \frac{1}{6}$. We divide the range $[T^{\frac{1}{8}+\varepsilon}, T^{\frac{1}{6}+\varepsilon}]$ into $[T^{\frac{1}{8}+\varepsilon}, 2T^{\frac{1}{8}+\varepsilon})$, $[2T^{\frac{1}{8}+\varepsilon}, 4T^{\frac{1}{8}+\varepsilon})$, $[4T^{\frac{1}{8}+\varepsilon}, 8T^{\frac{1}{8}+\varepsilon})$, ... the last interval either terminates at $T^{\frac{1}{6}+\varepsilon}$ or goes just a little beyond. In any typical interval $[V, 2V)$ we have $V \leqslant |\zeta(\frac{1}{2}+it)| < 2V$. For any given V we consider the set $G(V)$ of all t with $V \leqslant |\zeta(\frac{1}{2}+it)| < 2V$. We choose a small $\delta > 0$ to be fixed later. We next divide $[T, 2T]$ into disjoint intervals $[T, T+V^{4-\delta})$, $[T+V^{4-\delta}, T+2V^{4-\delta})$, $[T+2V^{4-\delta}, T+3V^{4-\delta})$, ... the last interval either terminating at $2T$ or going just a little beyond $2T$. Out of these we retain only those which have at least one point of $G(V)$. With every such interval $[A, B)$ we associate the interval I: $(A-(\log T)^2, B+(\log T)^2)$. The intervals I are not disjoint. But every point of $[A, B)$ is interior to at most two intervals I. Hence if we put $T_0 = V^{4-\delta} + 2(\log T)^2$ we have

$$\sum_I \int_I |\zeta(\tfrac{1}{2}+it)|^4 \, dt \ll (R\,V^{4-\delta} + T(R/V^{4-\delta})^{1/2})T^\varepsilon.$$

By convexity each of the integrals on the left $\geqslant V^4 (\log T)^{-2}$. Hence

$$RV^4 (\log T)^{-2} \ll (RV^{4-\delta} + T(R/V^{4-\delta})^{1/2}T^\varepsilon.$$

We now put $\delta = 16\varepsilon$. It follows that

$$RV^4 (\log T)^{-2} \ll (R/V^{4-\delta})^{1/2} T^{1+\varepsilon}.$$

Hence $R \ll \dfrac{T^{2+\varepsilon}}{V^{12-\delta}} \ll \dfrac{T^{2+100\varepsilon}}{V^{12}}$. On the other hand by Iwaniec's result we have also

$$\sum_I \int_I |\zeta(\tfrac{1}{2}+it)|^4 \, dt \ll_\varepsilon (RT_0 + T(R/T_0)^{1/2})T^\varepsilon.$$

Here the first term on the right side can be ignored and the left side is

$$\geqslant \sum_I \mu(I) V^4$$

where $\mu(I)$ is the measure of $G(V) \cap I$. Hence

$$\sum_I \mu(I) \ll \frac{1}{V^4} T^{1+\varepsilon}(R/T_0)^{1/2}$$

and so

$$\sum_I \mu(I) V^{12} \ll V^8 T^{1+\varepsilon}(T^{2+100\varepsilon}/V^{16-\delta})^{1/2} \ll T^{2+100\varepsilon}$$

i.e.

$$\int_{G(V)} |\zeta(\tfrac{1}{2}+it)|^{12}\,\mathrm{d}t \ll T^{2+100\varepsilon}$$

Summing up over all V the twelfth power moment stated in the beginning is proved.

4. Mean $(2k)$th power

Titchmarsh [36] was the first to prove the following result. For every fixed positive integer k,

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k} \exp(-\delta t)\,\mathrm{d}t \gg_k \frac{1}{\delta}\left(\log\frac{1}{\delta}\right)^{k^2}.$$

This gives

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t = \Omega((\log T)^{k^2}).$$

Put

$$M(k,T,T+H) = \frac{1}{H}\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t$$

where k is a positive real number and

$$T \geqslant H \geqslant 100 \log\log T \geqslant C_0$$

where C_0 is a large positive constant. Ramachandra [25] was the first to prove that if $2k$ is a positive integer then $M(k,T,T+H) \gg_k (\log H)^{k^2}$. This gives

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \gg_k (\log T)^{k^2}.$$

Using Gabriel's convexity theorem this was upheld by Heath-Brown [17] for all rational $k > 0$. Regarding upper bounds Ramachandra [26] was the first to prove that

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|\,\mathrm{d}t \ll (\log T)^{1/4}.$$

In fact he [26] proved that $M(\tfrac{1}{2},T,T+T^{1/2+\varepsilon}) \ll (\log T)^{1/4}$ without any hypothesis and $M(\tfrac{1}{2},T,T+T^{1/4+\varepsilon}) \ll (\log T)^{1/4}$ on Riemann hypothesis. Heath-Brown [17] proved again by using Gabriel's convexity theorem that $M\left(\frac{1}{m}, T, 2T\right) \ll (\log T)^{1/m^2}$ where $m \geqslant 1$ is an integer. Jutila [21] proved that

$$(\log T)^{1/m^2} \ll M\left(\frac{1}{m}, T, 2T\right) \ll (\log T)^{1/m^2}$$

where the constants implied by the Vinogradov symbol $\ll$ are independent of m. He applied this to show that the measure of the set $|\zeta(\frac{1}{2}+it)| \geqslant V$ is

$$\leqslant \exp\left(-\frac{(\log V)^2}{\log\log T}\left(1+0\left(\frac{\log V}{\log\log T}\right)\right)\right)$$

where $1 \leqslant V \leqslant \log T$.

Regarding lower bounds Ramachandra [27] was the first to prove

$$M(k, T, T+H) \gg (\log H)^{k^2} (\log\log H)^{-C}$$

where $k > 0$ is any real constant and C depends only on k. Using Heath-Brown's idea he [28] proved

$$M(k, T, T+H) \underset{k}{\gg} (\log H)^{k^2}$$

where $k > 0$ is any rational constant. Using the same idea he proved [29, 30] that for irrational k

$$M(k, T, T+H) \gg g\left(\frac{\log H}{\log\log H}\right)^{k^2},$$

where g is a function of H which tends to infinity with H.

Ramachandra [31] has proved (see also Balasubramanian and Ramachandra [4]) that if k is an integer subject to $1 \leqslant k \leqslant \log H$, we have uniformly in k, T, H

$$M(k, T, T+H) \gg \sum_{n \leqslant H/100} \frac{(d_k(n))^2}{n}\left(1-\frac{\log n}{\log H}+\frac{1}{\log\log H}\right).$$

Put $\Lambda = (\text{RHS})^{1/2k}$. The quantity Λ has been studied as a function of k by Balasubramanian. It was shown by Balasubramanian and Ramachandra [5] that the maximum of Λ as k varies is attained in $1 \leqslant k \leqslant \log H$ and that $\log\max\Lambda \gg \left(\frac{\log H}{\log\log H}\right)^{1/2}$. By an ingeneous argument Balasubramanian [6] showed that

$$\log\max\Lambda \sim \alpha\left(\frac{\log H}{\log\log H}\right)^{1/2},$$

where $\alpha = \frac{1}{2} \max_{l>0,\ l \text{ is real}} \{(\exp(-2l) + \int_{2l}^{\infty} \frac{e^{-\theta}}{\theta}\, d\theta)(2l\exp(2l)^{1/2})\}$.

Richert showed by his pocket calculator that $\alpha = 0.75\ldots.$ Thus

$$\max_{T \leqslant t \leqslant T+H} |\zeta(\tfrac{1}{2}+it)| > \exp\left(\frac{3}{4}\left(\frac{\log H}{\log\log H}\right)^{1/2}\right)$$

where $T \geqslant H \geqslant 100 \log\log T$ and $T > C_0$. For the earlier history of the problem see [32] and [5]. This is an improvement on the result

$$\max_{0 \leqslant t \leqslant T} |\zeta(\tfrac{1}{2}+it)| > \exp\left(\frac{1}{20}\left(\frac{\log T}{\log\log T}\right)^{1/2}\right)$$

for $T \geqslant T_0$ due to Montgomery [23] who proved this on the assumption of the Riemann hypothesis.

5. Other important results

Let

$$M(s) = \sum_{m \leqslant M} a_m m^{-s}$$

and

$$S(T, M) = \int_0^T |\zeta(\tfrac{1}{2}+it) M(it)|^2 \, \mathrm{d}t.$$

Iwaniec [21] was the first to prove that

$$S(T, M) \ll T^{1+\varepsilon} \sum_{m \leqslant M} |a_m|^2$$

for $M \leqslant T^{1/2}$ unconditionally and for $M \leqslant T^{4/7}$ conditionally. In the case $a_m = \mu(m)$ Heath-Brown has proved this result unconditionally for $M \leqslant T^{8/15}$ (see p. 306 of [8]). In a series of papers Deshouillers and Iwaniec [8, 9] have proved this unconditionally when $a_m = \Lambda(m)$ or $a_m = \mu(m)$ for $M \leqslant T^{4/7}$.

Iwaniec [20] was the first to prove that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 |M(it)|^2 \, \mathrm{d}t \ll T^{1+\varepsilon} \sum_{m \leqslant M} |a_m|^2$$

for $M \leqslant T^{1/10}$. Deshouillers and Iwaniec prove this for $M \leqslant T^{1/5}$. Asymptotic formulae are also important (*e.g.* for the discussion of the zeros on the critical line). In this connection Balasubramanian and Conrey [7] have proved that

$$\int_0^T |\zeta(\tfrac{1}{2}+it) M(\tfrac{1}{2}+it)|^2 \, \mathrm{d}t \sim T \sum_{m_1, m_2 \leqslant M} \frac{a_{m_1} a_{m_2}}{[m_1, m_2]} \left(\log \frac{T(m_1, m_2)}{2\pi m_1 m_2} + 2\gamma - 1\right)$$

Provided $a_m = \mu(m) f(m)$ where f is a smooth function $\ll 1$ and $M \leqslant T^{\frac{8}{15}-\varepsilon}$. Using this and refining the method of Levinson they proved that 38% of the zeros lie on the critical line.

In the theory of the Riemann zeta-function an indispensible book is [37]. This has been very useful in the preparation of this article.

References

[1] Atkinson F V 1939 *Q. J. Oxford* **10** 122
[2] Atkinson F V 1941 *Proc. London Math. Soc.* **47** 174

[3] Balasubramanian R 1978 *Proc. London Math. Soc.* **36** 540
[4] Balasubramanian R and Ramachandra K *Acta Arith.* (to appear)
[5] Balasubramanian R and Ramachandra K 1977 *Proc. Indian Acad. Sci.* (*Math. Sci.*) **A86** 341
[6] Balasubramanian R (to appear)
[7] Balasubramanian R and Conrey J B (to appear)
[8] Deshouillers J -M and Iwaniec H 1984 *Acta Arith.* **43** 305
[9] Deshouillers J -M and Iwaniec H 1982 *Mathematika* **29** 202
[10] Good A 1977 *Comment. Math. Helv.* **52** 35
[11] Good A 1977 *Invent. Math.* **41** 233
[12] Hardy G H and Littlewood J E 1918 *Acta Math.* **41** 119
[13] Hardy G H and Littlewood J E 1922 *Proc. London Math. Soc.* **21** 39
[14] Heath-Brown D R 1979 *Proc. London Math. Soc.* **38** 385
[15] Heath-Brown D R 1978 *Mathematika* **25** 177
[16] Heath-Brown D R 1978 *Q. J. Math.* (*Oxford*) **29** 443
[17] Heath-Brown D R 1981 *J. London Math. Soc.* **24** 65
[18] Ingham A E 1926 *Proc. London Math. Soc.* **27** 273
[19] Iwaniec H (to appear)
[20] Iwaniec H 1980 *J. London Math. Soc.* **22** 39
[21] Jutila M 1983 *Bull. London Math. Soc.* **15** 513
[22] Kober H 1936 *Compositio Math.* **3** 174
[23] Montgomery H L 1977 *Comment Math. Helv.* **52** 511
[24] Ramachandra K 1975 *J. London Math. Soc.* **10** 482
[25] Ramachandra K 1980 *Hardy-Ramanujan J.* **3** 1
[26] Ramachandra K 1980 *Ann. Acad. Sci. Fennicae* AI **5** 145
[27] Ramachandra K 1978 *Hardy-Ramanujan J.* **1** 1
[28] Ramachandra K 1981 *Proc. of the Int. Colloq. on number theory* Budapest (to appear)
[29] Ramachandra K 1981 *Int. Conf. on analytic number theory* Vinogradov's 90th birthday celebrations, Steklov Institute, Moscow, (to appear)
[30] Ramachandra K 1983 *Hardy-Ramanujan J.* **6** 1
[31] Ramachandra K 1981 *Recent progress in analytic number theory* (New York: Academic Press) vol 1 303–318
[32] Ramachandra K 1974 *J. London Math. Soc.* **8** 683
[33] Titchmarsh E C 1934 *Q. J. Maths* (*Oxford*) **5** 195
[34] Titchmarsh E C 1928 *Proc. London Math. Soc.* **27** 137
[35] Titchmarsh E C 1937 *Q. J. Math.* (*Oxford*) **8** 107
[36] Titchmarsh E C 1928 *Proc. London Math. Soc.* **28** 70
[37] Titchmarsh E C 1951 *The theory of the Riemann zeta function* (Oxford: Clarendon Press)