Proc. Indian Acad. Sci. (Math. Sci.), Vol. 93, Nos 2 & 3, December 1984, pp. 67–77

On the Rogers-Ramanujan continued fraction

K G RAMANATHAN
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

Abstract. In the "Lost" note book, Ramanujan had stated a large number of results regarding evaluation of his continued fraction

$$R(\tau) = \frac{\exp 2\pi i\tau/5 \exp(2\pi i\tau) \exp(4\pi i\tau)}{1+ \quad\quad 1+ \quad\quad 1+} \cdots$$

for certain values of τ. It is shown that all these results and many more have their source in the Kronecker limit formula.

Keywords. Continued fractions; Kronecker limit formula; Dirichlet series.

1. The continued fraction

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\cdots, |x| < 1$$

was introduced by Rogers [7] in his work on expansions of infinite products. He proved that

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\cdots = \prod_{n=1}^{\infty} \frac{(1-x^{5n-1})(1-x^{5n-4})}{(1-x^{5n-2})(1-x^{5n-3})}.$$

Nearly twenty years later, in 1912, Ramanujan rediscovered this continued fraction and evaluated it for various values of x. If we put $\tau = x + iy$, x, y real and $y > 0$ and put

$$R(\tau) = \frac{\exp(2\pi i\tau/5)}{1+}\,\frac{\exp(2\pi i\tau)}{1+}\,\frac{\exp(4\pi i\tau)}{1+}\cdots, \tag{1}$$

then Ramanujan proved

$$\frac{1}{R(\tau)} - 1 - R(\tau) = \frac{\eta(\tau/5)}{\eta(5\tau)}, \tag{2}$$

where $\eta(\tau)$ is Dedekind's modular form

$$\eta(\tau) = \exp(\pi i\tau/12) \prod_{n=1}^{\infty} (1 - \exp(2n\pi i\tau)).$$

Ramanujan's proof of (2) is found in his unpublished manuscripts in the Oxford

Mathematical Library. He also showed that

$$\left(\frac{1}{R(\tau)}\right)^5 - 11 - (R(\tau))^5 = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6. \tag{3}$$

(See [4]). From (2) it follows that $R(\tau)$ is an elliptic modular function. Indeed it belongs to the principal congruence subgroup

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } 5)$$

of the modular group [[2], p. 383]. If we put $(\tau+5)/2$ instead of τ we then have

$$-S(\tau) = R\left(\frac{\tau+5}{2}\right) = -\left(\frac{\exp(\pi i\tau/5)}{1-}\,\frac{\exp(\pi i\tau)}{1+}\,\frac{\exp(2\pi i\tau)}{1-}\cdots\right). \tag{4}$$

Analogous to (2) and (3) one has

$$\frac{1}{S(\tau)} + 1 - S(\tau) = \frac{\eta(\tau/5)f(\tau/5)}{\eta(5\tau)f(5\tau)}, \tag{5}$$

$$\left(\frac{1}{S(\tau)}\right)^5 + 11 - (S(\tau))^5 = \left(\frac{\eta(\tau)f(\tau)}{\eta(5\tau)f(5\tau)}\right)^6, \tag{6}$$

where $f(\tau)$ is Schläfli's modular function

$$f(\tau) = \exp[-(\pi i/24)]\,\frac{\eta\left(\frac{1+\tau}{2}\right)}{\eta(\tau)}. \tag{7}$$

It is well-known that $f(\tau)$ and $\eta(\tau)$ satisfy the transformation formulae

$$f(-1/\tau) = f(\tau);\ \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau), \tag{8}$$

where $(-i\tau)^{1/2}$ is positive for τ on the imaginary positive axis.

In his Note books [[5] Vol. II, p. 204] Ramanujan stated the two following formulae

$$\left(\frac{\sqrt{5}+1}{2} + R(i\alpha)\right)\left(\frac{\sqrt{5}+1}{2} + R(i\beta)\right) = \sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)$$

$$\left(\frac{\sqrt{5}-1}{2} + S(i\alpha)\right)\left(\frac{\sqrt{5}-1}{2} + S(i\beta)\right) = \sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right) \tag{9}$$

where $\alpha > 0$, $\beta > 0$ and $\alpha\beta = 1$. A proof of this was given by Watson [9]. Recently we gave [3] a simple proof of these and in addition proved two new formulae

$$\left[\left(\frac{\sqrt{5}+1}{2}\right)^5 + (R(i\alpha))^5\right]\left[\left(\frac{\sqrt{5}+1}{2}\right)^5 + (R(i\beta))^5\right] = 5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^5 \tag{10}$$

$$\left[\left(\frac{\sqrt{5}-1}{2}\right)^5 + (S(i\alpha))^5\right]\left[\left(\frac{\sqrt{5}-1}{2}\right)^5 + (S(i\beta))^5\right] = 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5 \tag{11}$$

whenever $\alpha > 0$, $\beta > 0$ and $\alpha\beta = 1/5$.

Ramanujan stated (10) in his unpublished manuscripts but (11) is not to be found anywhere though Ramanujan must certainly have known it.

In the "Lost" Note book Ramanujan [6] states a number of results on the continued fractions $R(\tau)$ and $S(\tau)$. Indeed on one page he states:

$$\text{“}F(x) = \frac{x^{1/5}}{1-}\,\frac{x}{1+}\,\frac{x^2}{1-}\,\frac{x^3}{1+}\cdots$$

$$F(\exp(-\pi/\sqrt{5})) = \left\{-\left(\frac{\sqrt{5}-1}{2}\right)^5 + \left[1+\left(\frac{\sqrt{5}-1}{2}\right)^{10}\right]^{1/2}\right\}^{1/5}$$

$$\begin{cases} F(\exp(-\pi/\sqrt{15})) = \\ F(\exp(-\pi\sqrt{3/5})) = \end{cases}$$

$$F(\exp(-\pi/5)) =$$

$$\begin{cases} F(\exp(-\pi/\sqrt{35})) = [5\sqrt{5}-7+(35(5-2\sqrt{5}))^{1/2}]^{1/5}) \\ F(\exp(-\pi\sqrt{7/5})) = [-5\sqrt{5}-7+(35(5+2\sqrt{5}))^{1/2}]^{1/5} \end{cases}$$

$$\begin{cases} F(\exp(-\pi/\sqrt{45})) = \\ F(\exp(-\pi\sqrt{9/5})) = \end{cases}$$

....”.

It is obvious from the way he brackets the results, that they are related to (11) which Ramanujan must have known.

Our object in this note is not only to prove these results and other results of Ramanujan's stated in the Note books, but also to show that the real source of such results is to be found in the Kronecker limit formula. We thus obtain many other results not given by Ramanujan.

2. Clearly $S(i\alpha) = F(\exp(-\pi\alpha)), \alpha > 0.$

We shall therefore first deal with the two simple cases given by Ramanujan

$$\alpha = \frac{1}{\sqrt{5}}, \quad \alpha = \frac{1}{5}.$$

If we put $\alpha = \beta = \dfrac{1}{\sqrt{5}}$ in (11) we at once get

$$\left(\frac{\sqrt{5}-1}{2}\right)^5 + (S(i/\sqrt{5}))^5 = \left[5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5\right]^{1/2}$$

which shows

$$\left[\frac{\exp(-\pi/5\sqrt{5})}{1-}\,\frac{\exp(-\pi/\sqrt{5})}{1+}\,\frac{\exp(-2\pi/\sqrt{5})}{1-}\cdots\right]^5 = -\left(\frac{\sqrt{5}-1}{2}\right)^5 +$$

$$+\left\{\left[\left(\frac{\sqrt{5}+1}{2}\right)^5 + \left(\frac{\sqrt{5}-1}{2}\right)^5\right]\left(\frac{\sqrt{5}-1}{2}\right)^5\right\}^{1/2}.$$

Hence we have

$$\frac{\exp(-\pi/5\sqrt{5})}{1-}\,\frac{\exp(-\pi/\sqrt{5})}{1+}\,\frac{\exp(-2\pi/\sqrt{5})}{1-}\cdots$$

$$=\left[-\left(\frac{\sqrt{5}-1}{2}\right)^5+\left(1+\left(\frac{\sqrt{5}-1}{2}\right)^{10}\right)^{1/2}\right]^{1/5} \tag{12}$$

Next let us take $\alpha=\beta=1$ in (9). Then

$$S(i)=\left(\frac{5-\sqrt{5}}{2}\right)^{1/2}-\frac{\sqrt{5}-1}{2}$$

In (11) let us take $\alpha=1,\ \beta=1/5$. Then

$$(S(i/5))^5=\frac{5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5}{\left(\frac{\sqrt{5}-1}{2}\right)^5+\left\{\left(\frac{5-\sqrt{5}}{2}\right)^{1/2}-\left(\frac{\sqrt{5}-1}{2}\right)\right\}^5}-\left(\frac{\sqrt{5}-1}{2}\right)^5.$$

We could simplify this and obtain finally

$$\frac{\exp(-\pi/25)}{1-}\,\frac{\exp(-\pi/5)}{1+}\,\frac{\exp(-2\pi/5)}{1-}\cdots=\left\{\left(\frac{\sqrt{5}+1}{2}\right)^5\right.$$

$$\left.-\left(\frac{\sqrt{5}-1}{2}\right)^5\left[\left(\frac{5+\sqrt{5}}{2}\right)^{1/2}-1\right]^5\right\}\Big/1+\left[\left(\frac{5+\sqrt{5}}{2}\right)^{1/2}-1\right]^5. \tag{13}$$

3. Let $n>0$ be an odd integer and let us take $\tau=i/\sqrt{5n}$ in (6). Using (8) we obtain

$$\left[S\left(\frac{i}{\sqrt{5n}}\right)\right]^{-5}+11-\left[S\left(\frac{i}{\sqrt{5n}}\right)\right]^5=5^3\left[\frac{\eta\left(\frac{1+\sqrt{-5n}}{2}\right)}{\eta\left(\frac{1+\sqrt{-5n/5}}{2}\right)}\right]^6. \tag{14}$$

Let us now assume that $-5n$ is a fundamental discriminant *i.e.* that the imaginary quadratic field $F=Q(\sqrt{-5n})$ has discriminant $-5n$. Furthermore let every genus of ideal classes of F have only one class in it. Such discriminants are listed in [1].

Let us assume for simplicity that F has class number 2. From the tables in [1] we see that

$$n=3,\ 7,\ 23,\ 47 \tag{15}$$

are numbers with this property. The two ideal classes in $F=Q(\sqrt{-5n})$ will be represented by the ideals A and B with minimal bases

$$A=\left(1,\frac{1+\sqrt{-5n}}{2}\right),\ B=\left(1,\frac{1+\sqrt{-5n/5}}{2}\right) \tag{16}$$

and with norms 1 and 1/5 respectively. Since the class number is 2, there is only one non-trivial genus character χ and this corresponds to the decomposition

$$-5n = -n \cdot 5$$

(Note that since $-5n \equiv 1 \pmod 4$, $-n \equiv 1 \pmod 4$). The L-series of F corresponding to this character has the property (Siegel [8])

$$L_F(1,\chi) = \frac{4\pi}{u_n \cdot \sqrt{5n}} \cdot h(-n) \cdot \log \frac{\sqrt{5}+1}{2} \tag{17}$$

where u_n is the number of roots of unity in F and $h(-n)$ is the class number of $Q(\sqrt{-n})$.
The Kronecker limit formula gives ([18] p. 71)

$$\log\left|\eta\left(\frac{1+\sqrt{-5n}}{2}\right)\right|^2 - \log\frac{1}{\sqrt{5}}\left|\eta\left(\frac{1+\sqrt{-5n/5}}{2}\right)\right|^2$$

$$= -\frac{2h(-n)}{u_n}\log\frac{\sqrt{5}+1}{2}. \tag{18}$$

Combining this with (14) we have the

THEOREM 1. If $-5n$ is the discriminant of the imaginary quadratic field $F = Q(\sqrt{-5n})$, where n is odd and F has class number two, then

$$\frac{\exp(-\pi/5\sqrt{5n})}{1-}\,\frac{\exp(-\pi/\sqrt{5n})}{1+}\,\frac{\exp(-2\pi/\sqrt{5n})}{1-}\cdots$$

$$= \left\{\frac{1}{2}(\alpha + (\alpha^2+4)^{1/2})\right\}^{1/5}$$

where $\quad \alpha = 11 - 5\sqrt{5}\left(\dfrac{\sqrt{5}-1}{2}\right)^{6h(-n)/u_n}.$

$h(-n)$ is the class number of $Q(\sqrt{-n})$ and u_n the number of roots of unity in it.
We can obtain a companion result either by using (11) or by proceeding as for the previous theorem:

THEOREM 1′: Under the same conditions as for theorem 1,

$$\frac{\exp[(-\pi/5)(\sqrt{n/5})]}{1-}\,\frac{\exp[-\pi(\sqrt{n/5})]}{1+}\,\frac{\exp[-2\pi(\sqrt{n/5})]}{1-}\cdots$$

$$= \frac{1}{2}(\beta + (\beta^2+4)^{1/2})^{1/5} \tag{20}$$

where $\quad \beta = 11 - 5\sqrt{5}\left(\dfrac{\sqrt{5}+1}{2}\right)^{6h(-n)/u_n}.$

We now give some examples which include Ramanujan's given in §1.

(i) Let $n = 3$. Then $h(-3) = 1$ and $u_3 = 6$. This gives

$$\frac{\exp(-\pi/5\sqrt{15})}{1-}\,\frac{\exp(-\pi/\sqrt{15})}{1+}\,\frac{\exp(-2\pi/\sqrt{15})}{1-}\cdots$$

$$=\left[\frac{-3+5\sqrt{5}+(30(5+\sqrt{5}))^{1/2}}{4}\right]^{1/5}.$$

$$\frac{\exp[(-\pi/5)(\sqrt{3/5})]}{1-}\,\frac{\exp(-\pi\sqrt{3/5})}{1+}\,\frac{\exp(-2\pi\sqrt{3/5})}{1-}\cdots$$

$$=\left[\frac{-3-5\sqrt{5}+(30(5-\sqrt{5}))^{1/2}}{4}\right]^{1/5}.$$

(ii) Let now $n = 7$, then $h(-7) = 1$ and $u_7 = 2$. This gives the Ramanujan results

$$\frac{\exp(-\pi/5\cdot\sqrt{35})}{1-}\,\frac{\exp(-\pi/\sqrt{35})}{1+}\,\frac{\exp(-2\pi/\sqrt{35})}{1-}=\cdots$$

$$=[-7+5\sqrt{5}+(35(5-2\sqrt{5}))^{1/2}]^{1/5}$$

$$\frac{\exp[(-\pi/5)(\sqrt{7/5})]}{1-}\,\frac{\exp(-\pi\sqrt{7/5})}{1+}\,\frac{\exp(-2\pi\sqrt{7/5})}{1-}\cdots$$

$$=[(-7-5\sqrt{5}+(35(5+2\sqrt{5}))^{1/2}]^{1/5}$$

(iii) Let now $n = 23$ which is given in (15). Then from [1] we see that $h(-23) = 3$ and $u_{23} = 2$;

$$\frac{\exp(-\pi/5\sqrt{115})}{1-}\,\frac{\exp(-\pi/\sqrt{115})}{1+}\,\frac{\exp(-2\pi/\sqrt{115})}{1-}\cdots$$

$$=[-207+95\sqrt{5}+3((1955-874\sqrt{5})5))^{1/2}]^{1/5}$$

$$\frac{\exp[(-\pi/5)(\sqrt{23/5})]}{1-}\,\frac{\exp(-\pi\sqrt{23/5})}{1+}\,\frac{\exp(-2\pi\sqrt{23/5})}{1-}\cdots$$

$$=[-207-95\sqrt{5}+3(5(1955+874\sqrt{5}))^{1/2}]^{1/5}$$

One can consider the case where $Q(\sqrt{-5n})$, with n odd has class number 4 and with four genera, for example

$$n = 39, 87, 111, 119, 159, 287.$$

However we shall omit dealing with this here since an analogous situation with n even is studied in the next section from which the method would become obvious.

4. We shall consider the case of $R(i\alpha)$ where

$$\alpha = \frac{1}{\sqrt{10\,m}},\ m > 0$$

and $10\,m$ is an idoneal number of Euler. In this case $F = Q(\sqrt{-10\,m})$ has one class in each genus. If we use (8) we have

$$\left(R\left(\frac{i}{\sqrt{10\,m}}\right)\right)^{-5} - 11 - \left(R\left(\frac{i}{\sqrt{10\,m}}\right)\right)^{5} = 5^3 \cdot \left[\frac{\eta(\sqrt{-10\,m})}{\eta(\sqrt{-10\,m/5})}\right]^6. \quad (21)$$

Since $F = Q(\sqrt{-10m})$ has one class in each genus, it follows from the table [[1] (p. 426–427)], that

$$10\,m = 10, 30, 70, 130, 190, 210, 330 \quad (22)$$

We have taken only the cases where m is odd. Of these numbers 10 is the only number for which $Q(\sqrt{-10})$ has class number 2 (with of course 2 genera). The numbers 30, 70, 130 and 190 are all associated with fields having 4 genera with one class in each genus.

We shall consider only the two cases 10 and 30 which illustrate our methods.

The two classes in $F = Q(\sqrt{-10})$ are represented by the ideals

$$A = (1, \sqrt{-10}),\ B = (1, \sqrt{-10}/5)$$

with norms 1 and 1/5. There is one non-trivial genus character χ corresponding to the decomposition

$$-40 = -8.5$$

and so the L-series of F has the value

$$L_F(1, \chi) = \frac{4\pi}{2\sqrt{40}} \log \frac{\sqrt{5}+1}{2}.$$

and by the Kronecker limit formula,

$$\log \eta(\sqrt{-10})^2 - \log \frac{1}{\sqrt{5}}\eta\left(\frac{\sqrt{-10}}{5}\right)^2 = \log \frac{\sqrt{5}-1}{2}. \quad (23)$$

From (21) and (23) we obtain

$$\left(R\left(\frac{i}{\sqrt{10}}\right)\right)^{-5} - 11 - \left(R\left(\frac{i}{\sqrt{10}}\right)\right)^{5} = 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^3.$$

This gives at once

$$\frac{\exp(-2\pi/5\sqrt{10})}{1+}\frac{\exp(-2\pi/\sqrt{10})}{1+}\frac{\exp(-4\pi/\sqrt{10})}{1+}\cdots$$
$$= [-18 + 5\sqrt{5} + 3(\sqrt{10}(5-2\sqrt{5}))^{1/2}]^{1/5}.$$

Similarly

$$\frac{\exp[(-2\pi/5)(\sqrt{2/5})]}{1+}\frac{\exp(-2\pi\sqrt{2/5})}{1+}\frac{\exp(-4\pi\sqrt{2/5})}{1+}\cdots$$
$$= [-18 - 5\sqrt{5} + 3(\sqrt{10}(5+2\sqrt{5}))^{1/2}]^{1/5}$$

The field $F = Q(\sqrt{-30})$ has 4 classes and 4 genera. The four classes are represented by the ideals A, B, C, D with bases $A=(1,\sqrt{-30})$, $B=(1,\sqrt{-30}/2)$, $C=(1,\sqrt{-30}/3)$, $D=(1,\sqrt{-30}/5)$. For any character χ of the group of genera ([8] p. 62–71).

$$\left.\begin{aligned}&\log F(A)+\chi(B)\log F(B)+\chi(C)\log F(C)+\chi(D)\log F(D)\\ &\qquad=\frac{-\sqrt{120}}{2\pi}L_F(1,\chi)\end{aligned}\right\}\tag{24}$$

Let χ_2, χ_3 and χ_5 be the three non-trivial characters, corresponding respectively to the decompositions

$$\begin{aligned}&\chi_2\colon\ -120=-15.8\\ &\chi_3\colon\ -120=-3.40\\ &\chi_5\colon\ -120=-24.5\end{aligned}$$

From their definition ([8] p. 60)

$$\left.\begin{aligned}&\chi_2(A)=1,\chi_2(B)=1,\chi_2(C)=-1,\chi_2(D)=-1\\ &\chi_3(A)=1,\chi_3(B)=-1,\chi_3(C)=1,\chi_3(D)=-1\end{aligned}\right\}\tag{25}$$

From (24) and (25), it follows that

$$\log F(A)-\log F(D)=-\frac{\sqrt{120}}{4\pi}(L_F(1,\chi_2)+L_F(1,\chi_3)).$$

However

$$L_F(1,\chi_2)=\frac{4\pi}{2\sqrt{120}}h(-15)\,h(8)\log(\sqrt{2}+1)$$

$$L_F(1,\chi_3)=\frac{4\pi}{6\sqrt{120}}h(-3)\cdot h(40)\log(3+\sqrt{10})$$

and therefore

$$\begin{aligned}\left(R\left(\frac{i}{\sqrt{30}}\right)\right)^{-5}-11-\left(R\left(\frac{i}{\sqrt{30}}\right)\right)^{5}&=5^3\left(\frac{\eta(\sqrt{-30})}{\eta(\sqrt{-30/5})}\right)^6\\ &=5\sqrt{5}(\sqrt{2}+1)^6(3+\sqrt{10})^2.\end{aligned}\tag{26}$$

The solution of the quadratic equation in $(R(i/\sqrt{30}))^5$ leads to the value of $R(i/\sqrt{30})$.

It is now obvious how one should proceed in other cases where $Q(\sqrt{-5n})$ or $Q(\sqrt{-10n})$ have 4 or 8 or even 16 genera with one class in each genus. It might be interesting to evaluate $R(i/\sqrt{10n})$ or $S(i/\sqrt{5n})$ in other cases related to quadratic binary forms with one class in each genus.

5. We conclude this note by proving another of Ramanujan's statements which is however incomplete and which is given in the Note books ([5] Vol 1, p. 311). This concerns the evaluation of $R(2i\alpha)$ for $\alpha = 1, 2, 4$. Let us take the continued fraction

$$R(2i) = \frac{\exp(-4\pi/5)}{1+}\frac{\exp(-4\pi)}{1+}\frac{\exp(-8\pi)}{1+}\cdots$$

In order to evaluate this, observe that since by (2)

$$\frac{1}{R(2i)} - 1 - R(2i) = \frac{\eta(2i/5)}{\eta(10i)},$$

we have only to find the value of $\eta(2i/5)/\eta(10i)$. Clearly if we put

$$\frac{\eta(2i/5)}{\eta(10i)} = 2C - 1$$

then

$$R(2i) = (C^2+1)^{1/2} - C. \tag{27}$$

Let, in the usual notation of elliptic functions

$$\frac{K'}{K} = \frac{1}{5}, \frac{L'}{L} = 5$$

and k, k' and l, l' the respective associated moduli. Then

$$\frac{\eta(i/5)}{\eta(5i)} = (K/L)^{1/2}\cdot(kk'/ll')^{1/6}$$

Let us further assume that

$$\frac{K_0'}{K_0} = 2\frac{K'}{K}, \frac{L_0'}{L_0} = 2\frac{L'}{L}$$

and k_0, k_0' and l_0, l_0' the associated moduli. Then

$$\frac{\eta(2i/5)}{\eta(10i)} = (K_0/L_0)^{1/2}\cdot(k_0k_0'/l_0l_0')^{1/6}.$$

On the other hand, by the formulae of Jacobi-Legendre,

$$K_0 = \frac{1}{2}K(1+k'), L_0 = \frac{1}{2}L(1+l')$$

and

$$k_0 = \frac{1-k'}{1+k'}, k_0' = \frac{2\sqrt{k'}}{1+k'}, l_0 = \frac{1-l'}{1+l'}, l_0' = \frac{2\sqrt{l'}}{1+l'}.$$

Therefore we finally have

$$\frac{\eta(2i/5)}{\eta(10i)} = (K/L)^{1/2}(k/l)^{1/3}\cdot(l'/k')^{1/12}$$

Since from (8), $\eta(i/5) = \sqrt{5}\,\eta(5i)$, we have

$$(K/L)^{1/2} = \sqrt{5}(ll'/kk')^{1/6}$$

Substituting we finally get

$$\frac{\eta(2i/5)}{\eta(10i)} = \sqrt{5}\cdot(k/l)^{1/6}\cdot(l'/k')^{1/12}. \tag{28}$$

In order to find k, k', l and l', notice that since $K'/K = 1/5$,

$$f(i/5) = 2^{1/6}/(kk')^{1/12}.$$

On the other hand $f(i/5) = f(5i) = 2^{1/6}/(ll')^{1/12}$. From the tables of Weber and Ramanujan

$$f(5i) = (\sqrt{5}+1)/2^{3/4}$$

which gives

$$k^2 = \frac{1}{2}\left[1+\left(1-\left(\frac{\sqrt{5}-1}{2}\right)^{24}\right)^{1/2}\right] \tag{29}$$

(It is interesting to note that this result is given by Ramanujan [[5] Vol. 1, p. 287]). One can simplify this using the fact that

$$\frac{\sqrt{5}+1}{2}\cdot\frac{\sqrt{5}-1}{2} = 1.$$

Thus

$$\begin{aligned} k^2 &= \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^6\left\{\left(\frac{\sqrt{5}+1}{2}\right)^6 + \left[\left(\frac{\sqrt{5}+1}{2}\right)^{12} - \left(\frac{\sqrt{5}-1}{2}\right)^{12}\right]^{1/2}\right\} \\ &= \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^6(9+4\sqrt{5}+12\sqrt[4]{5}) \\ &= \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^6(3+2\sqrt[4]{5})^2. \end{aligned}$$

Since $k'^2 = 1-k^2$, we have

$$k'^2 = \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^6(3-2\sqrt[4]{5})^2.$$

Now $f(i/5) = f(5i)$ and so $k^2 \neq l^2$; but since $kk' = ll'$, we have $k^2 = l'^2$ and $k'^2 = l^2$. Taking k, k', l, l' positive we have

$$\frac{\eta(2i/5)}{\eta(10i)} = \sqrt{5}\cdot(k/k')^{1/4} = \sqrt{5}\left(\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}\right)^{1/4}.$$

It is easy to verify that

$$(\sqrt[4]{5}\pm 1)^4 = 2(1+\sqrt{5})(3\pm 2\sqrt[4]{5}).$$

We thus finally arrive at Ramanujan's result (not completely stated), THEOREM 2. If $(\sqrt[4]{5}+1/\sqrt[4]{5}-1)\sqrt{5}=2C+1$, then

$$\frac{\exp(-4\pi/5)}{1+}\,\frac{\exp(-4\pi)}{1+}\,\frac{\exp(-8\pi)}{1+}\cdots=(C^2+1)^{1/2}-C.$$

References

[1] Borevich Z I and Shafarevich I R 1966 *Number theory* (New York: Academic Press)
[2] Fricke R and Klein F 1892 Vorlesungen über die theorie der elliptischen Modulfunktionen Bd2
[3] Ramanathan K G 1984 *Acta Arith.* **44** 209
[4] Ramanathan K G 1980 *Proc. Indian Acad. Sci* (*Math. Sci.*) **89** 133
[5] Ramanujan S 1957 *Note books, I and II* printed facsimile (Bombay: TIFR)
[6] Ramanujan S *Lost Note book* (unpublished manuscripts in the library, Trinity College, Cambridge, England)
[7] Rogers L J 1894 *Proc. London Math. Soc.* **25** 318
[8] Siegel C L 1980 *Advanced analytic number theory. Studies in mathematics* (Bombay: TIFR)
[9] Watson G N 1929 *J. London Math. Soc.* **4** 39
[10] Whittaker E T and Watson G N 1946 *A course of modern analysis* (Cambridge University Press) 4th edn
[11] Weber H 1908 *Lehrb. Algebra* III Braunschweig