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Couette flow of a stratified fluid with suction varying along a boundary

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Abstract. A viscous incompressible fluid between two plane boundaries is stratified by maintaining the planes at different temperatures. The upper plane moves with a uniform velocity. The suction/injection mechanism with constant injection velocity at the upper plane and suction velocity varying sinusoidally along the lower plane with a wave number k is introduced at the boundaries. The steady linearised equations are solved using similarity variables for the velocity components. The wave number k is shown to be effective in controlling the boundary layer thickness.

Keywords. Boundary layer control; suction; stratification; wave number; Couette flow.

1. Introduction

The flow of a viscous incompressible fluid bounded by one or two infinite planes with porous walls has gained considerable importance in view of its applications to reduce boundary layer growth. Several investigations have shown that the suction of the fluid and heat transfer serve as effective mechanisms in the boundary layer control. Stuart [3] studied the oscillatory flow past an infinite porous wall with constant suction. The free convection flow and heat transfer along a porous plane are described by Gersten and Gross [1] and Singh *et al* [2]. Venkatasiva Murthy and Ponnuraj [4] studied the Couette flow of a viscous stratified fluid between two parallel porous planes.

In this paper, the Couette flow of a viscous incompressible temperature stratified fluid is studied. The two plane boundaries are maintained at different temperatures. The lower plane is at rest and the upper plane moves with a uniform velocity. The upper plane is subjected to a constant injection and the lower plane to a suction whose magnitude varies sinusoidally with the distance along the boundary, over a mean suction. The solution of the steady linear equations is obtained using similarity variables for velocity components following Venkatasiva Murthy and Ponnuraj [4]. The wave number k appearing in the variable suction is effective in controlling the thickness of the boundary layer.

2. Governing equations and solution

The two-dimensional flow of a viscous stratified fluid between two infinite planes $z' = \pm L/2$ is considered with reference to the rectangular cartesian coordinate axes $0x'z'$. In the basic state the planes are maintained at different temperatures T_1 and T_2. The fluid is subjected to constant suction and injection at the lower and upper planes respectively. The temperature difference between the planes causes an exponential density distribution in the fluid. If (u_e, v_e, w_e) are the velocity components, ρ_e the density and T_e the temperature in the basic state, then

$$u_e = v_e = 0, \quad w_e = -W' \text{ (constant)}.$$

$$T_e = T_0 + \left(\frac{T_1 - T_2}{2}\right)\frac{\cosh(\rho_0 C_p L W'/2K) - \exp(\rho_0 C_p W' z'/K)}{\sinh(\rho_0 C_p L W'/2K)},$$

$$\rho_e - \rho_0 = -\frac{\varepsilon\rho_0}{T_0}(T_e - T_0),$$

$$T_0 = (T_1 + T_2)/2.$$

The pressure p_e in the basic state satisfies the equation

$$\frac{\partial p_e}{\partial z} = -\rho_e g.$$

T_0 and ρ_0 are the mean values and ε is a small constant. The disturbance in the fluid, over this basic state is created when (i) the upper plane moves with velocity $\varepsilon U'$ and (ii) the normal velocities at the permeable planes $z' = \pm L/2$ are maintained at $-W'$, $-W'(1 - \varepsilon\cos k'x')$ respectively. The rate of injection at the upper plane equals the rate of suction at the lower plane over a wave length $2\pi/k'$. The equations of motion for a steady flow are

$$\rho'\bar{q}' \cdot \nabla\bar{q}' = -\nabla p' + \mu\nabla^2\bar{q}' - \rho' g\bar{k}.$$

$$\nabla \cdot (\rho'\bar{q}') = 0.$$

$$\rho' C_p \bar{q}' \cdot \nabla T' = -K\nabla^2 T' + \theta$$

$$\rho' - \rho_e = -\frac{\varepsilon\rho_0}{T_0}(T' - T_e)$$

where θ is the viscous dissipation function and $\bar{k}$ is the unit vector in the z' direction. The flow is two-dimensional and the velocity $\bar{q}'$ has components $(u', 0, w')$. The variables ρ', p' and T' are the density, pressure and temperature respectively. The constant μ is the coefficient of viscosity. The boundary conditions are

$$u' = \varepsilon U', \quad w' = -W', \quad T' = T_1 \quad \text{at } z' = L/2$$

$$u' = 0, \quad w' = -W'(1 - \varepsilon\cos k'x'), \quad T' = T_2 \text{ at } z' = -L/2.$$

Since the stratification parameter ε is small ($\varepsilon \ll 1$) for all stable stratifications, we introduce the nondimensional variables as follows

$$u' = \varepsilon V u, \tag{1}$$

$$w' = -W' + \varepsilon V w, \tag{2}$$

$$p' = p_e + \frac{\varepsilon\rho_0 g L}{T_0}(T_1 - T_2)p, \tag{3}$$

$$T' = T_e + \varepsilon(T_1 - T_2)\phi, \tag{4}$$

where $V = [(T_1 - T_2)gL/T_0]^{1/2}$ is a constant which is of the dimension of velocity. We also write $x' = Lx$, $z' = Lz$. With these nondimensional variables and under the

Boussinesq approximation, the linearised equations of motion in the nondimensional form are

$$u_x + w_z = 0, \tag{5}$$

$$E\nabla^2 u - p_x + \beta u_z = 0, \tag{6}$$

$$E\nabla^2 w - p_z + \beta w_z = 0, \tag{7}$$

$$\frac{E}{P}\nabla^2\phi + \beta\phi_z = \frac{P\beta}{2E}\,\frac{\exp(-zP\beta/E)}{\sin h\,(P\beta/2E)}\,w, \tag{8}$$

where $E = \mu/\rho_0 VL$ is the inverse of Reynolds number, $P = \mu C_p/K$ is the Prandtl number, $\beta = W'/V$ is a suction parameter and $k = k'L$ is a wave number.
The continuity equation (5) suggests the form for the velocities u and w as

$$\left.\begin{aligned} w(x,z) &= -f(z)\cos kx,\\ u(x,z) &= f'(z)\frac{\sin kx}{k} + g(z), \end{aligned}\right\} \tag{9}$$

Obtaining p from (7) and substituting in (6), the form for p is

$$p(x,z) = p_0(z)\cos kx + p_1 x + p_2, \tag{10}$$

where p_1 and p_2 are constants. Equation (8) gives the form for ϕ as

$$\phi(x,z) = \phi_0(z)\cos kx \tag{11}$$

The boundary conditions are

$$\left.\begin{aligned} &f = f' = \phi_0 = 0, \quad g = U \quad \text{on } z = 1/2,\\ &f = -\beta, \quad f' = g = \phi_0 = 0 \quad \text{on } z = -1/2 \end{aligned}\right\} \tag{12}$$

where $U = U'/V$.
Equations (9)–(11) are substituted in (6) and (7) to obtain the following differential equations for f and g.

$$f^{iv} + \frac{\beta}{E}f''' - 2k^2 f'' - \frac{\beta k^2}{E}f' + k^4 f = 0, \tag{13}$$

$$g'' + \frac{\beta}{E}g' = p_1, \tag{14}$$

$$\phi_0'' + \frac{P\beta}{E}\phi_0' - k^2\phi_0 = -\frac{P^2\beta}{2E^2}\,\frac{\exp(-zP\beta/E)}{\sin h\,(PB/2E)}\,f \tag{15}$$

The solutions of (13)–(15) subject to boundary conditions (12) are

$$f = C_1\exp(m_1 z) + C_2\exp(m_2 z) + C_3\exp(kz) + C_4\exp(-kz), \tag{16}$$

$$g = A + B\exp(-2Nz) + \frac{p_1 E}{\beta}z, \tag{17}$$

$$\phi_0 = C_5\exp(m_1' z) + C_6\exp(m_2' z) + \psi(z), \tag{18}$$

where

$$\psi(z) = \frac{-P^2\beta}{2E^2 \sinh (P\beta/2E)} \sum_{i=1}^{4} \frac{C_i \exp(Q_i z)}{Q_i^2 + \frac{P\beta}{E} Q_i - k^2} \tag{19}$$

$$Q_{1,2} = m_{1,2} - P\beta/E,$$

$$Q_{3,4} = \pm k - P\beta/E,$$

$$m_{1,2} = -N \pm (N^2 + k^2)^{1/2},$$

$$m'_{1,2} = -PN \pm (P^2N^2 + k^2)^{1/2},$$

$$N = \beta/2E$$

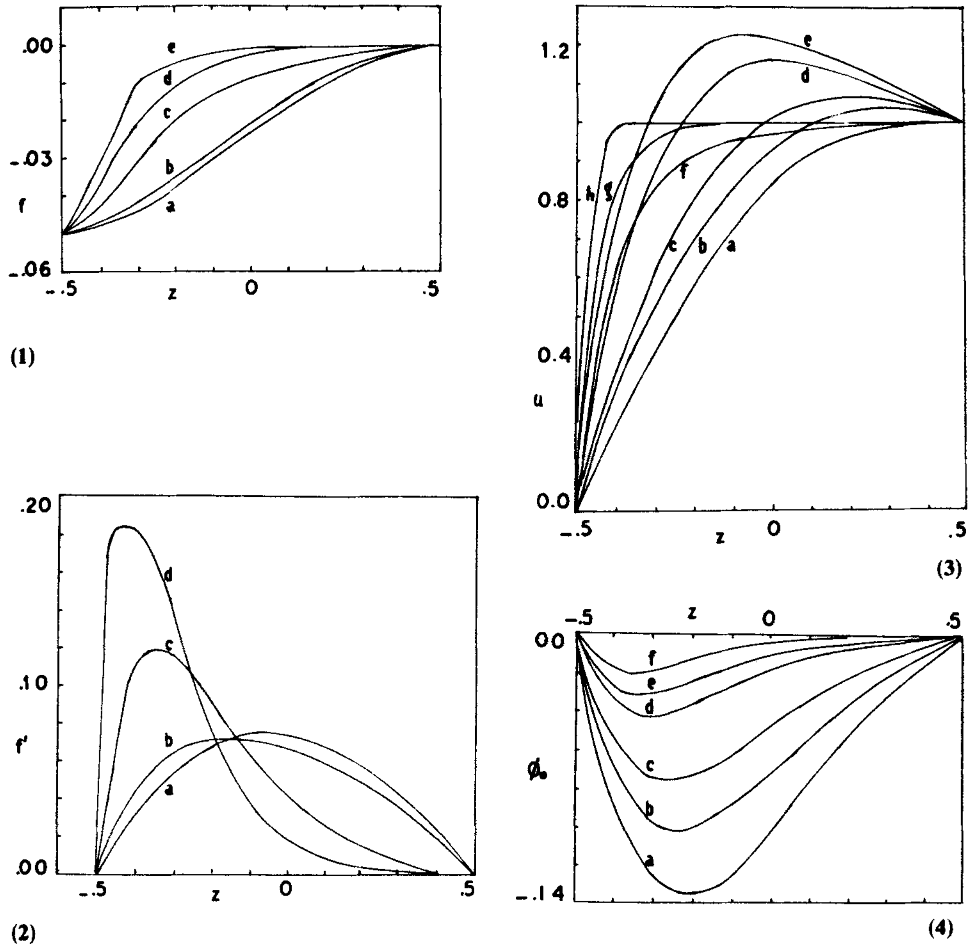
The other constants are not presented to save space.

3. Discussion

The solution given in (16) and (17) shows the existence of boundary layers of thickness orders k^{-1} and $[-N + (N^2 + k^2)^{1/2}]^{-1}$ near the upper plane and boundary layers of thickness orders k^{-1} and $[N + (N^2 + k^2)^{1/2}]^{-1}$ near the lower plane. When k is small ($k^2 \ll N^2$) the boundary layer depths at $z = 1/2$ are of orders $1/k$ and β/Ek^2 and the latter penetrates deeper than the former. Thus the effective depth of penetration is of order β/Ek^2. This layer need not be confined to small thickness near the boundary. If $k^2 \ll N^2$, the boundary layers at $z = -1/2$ have thickness orders E/β and $1/k$. If $k^2 \gg N^2$ the effective boundary layer thickness at both the planes is of order $1/k$ showing that the wave number k controls the thickness of the boundary layer. The solution (18) for temperature shows the dependence of the thermal boundary layer thickness on the relative values of the parameters P, N and k. However, large values of k correspond to thin thermal boundary layers.

For $k = 0$, the suction velocity at the lower plane is a constant $-W(1-\varepsilon)$ and the continuity equation shows the form for velocity components as $w = -f(z)$, $u = xf'(z) + g(z)$. Thus the horizontal velocity u changes linearly with x when $k = 0$. This is in contrast to the case $k \neq 0$ wherein the velocity components are given by (9). The horizontal velocity oscillates with x in the presence of k. The solution for the case $k = 0$ is developed by Venkatasiva Murthy and Ponnuraj [4]. The solution for f has parabolic dependence on z whereas for $k \neq 0$ the present solution for f consists of all exponential modes.

The behaviour of velocity and temperature and their dependence on the parameters is shown in figures 1–4. Calculations are done by prescribing $\partial p/\partial x = 0$ at $x = 0$ ($p_1 = 0$). The functions f and f' are plotted in figures 1 and 2. These figures demonstrate how the wave number k affects the thickness of the boundary layer. The profiles in figure 2 are nearly parabolic when $k = 3\pi/10, 9\pi/10$ but are rapidly distorted to steeply changing profiles near the boundary $z = -1/2$ as k increases. The curves representing f and f' in figures 1 and 2 rapidly approach zero in a layer near the lower plane and remain nearly zero throughout the channel when k is 3π or more. Figure 3 represents the curves of the horizontal velocity u as the parameter E changes. As E decreases from 0·05 to 0·01 the curves steepen and the maximum velocity attained increases beyond unity. But as E further decreases from 0·01 to 0·001 the figure reveals that the velocity tends to unity from below and the curves flatten. Thus the boundary layer thins as E



Figures 1–4. **1.** The function f drawn for various k when $\beta = E = 0{\cdot}05$ **a.** $k = 5\pi/10$ **b.** $k = 9\pi/10$ **c.** $k = 2\pi$ **d.** $k = 3\pi$ **e.** $k = 4\pi$. **2.** The function f' drawn for various k when $\beta = E = 0{\cdot}05$ **a.** $k = 3\pi/10$ **b.** $k = 9\pi/10$ **c.** $k = 2\pi$ **d.** $k = 3\pi$. **3.** Velocity u drawn for various E when $\beta = 0{\cdot}05$, $k = \pi/10$ $x = 5$, **a.** $E = 0{\cdot}05$ **b.** $E = 0{\cdot}03$ **c.** $E = 0{\cdot}02$ **d.** $E = 0{\cdot}01$ **e.** $E = 0{\cdot}007$ **f.** $E = 0{\cdot}005$ **g.** $E = 0{\cdot}003$ **h.** $E = 0{\cdot}001$. **4.** Temperature ϕ_0 when $\beta = E = 0{\cdot}05$, $P = 2$ **a.** $k = 0{\cdot}05$ **b.** $k = 2$ **c.** $k = 3$ **d.** $k = 5$ **e.** $k = 6$ **f.** $k = 8$.

decreases and the velocity u is almost constant except in a very thin layer near $z = -1/2$. Figure 4 shows how the temperature ϕ_0 gets arrested as k increases.

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