Proc. Indian Acad. Sci. (Math. Sci.), Vol. 92, No. 3, December 1983, pp. 191–193.

On Erdös-Lax theorem

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MS received 19 December 1983

Abstract. An improved version of the generalized Erdös-Lax theorem is stated and proved.

Keywords. Erdös-Lax theorem; generalization; maximum modulus principle; zero of polynomials.

1. Introduction

Let $P(z) = \sum_{v=0}^{n} c_v z^v$ be a polynomial of degree n and $P'(z)$ denote its derivative. Let also $\|P\| = \max_{|z|=1} |P(z)|$ and $\|P'\| = \max_{|z|=1} |P'(z)|$. Concerning the estimate of $\|P'\|$, the following result is known.

THEOREM A

When all the zeros of $P(z)$ lie on $|z| = K > 1$, then

$$\frac{n}{1+K^n}\|P\| \leqslant \|P'\| \leqslant \frac{n}{1+K}\|P\|. \tag{1}$$

The upper bound for $k = 1$ is now known as Erdös-Lax Theorem (Lax [4]). For the general case, see Govil [2], Govil and Rahman [3], Malik [5] and Turan [6]. We obtain an improvement of this theorem.

2. Proof

THEOREM 1

If $P(z)$ has no zeros in $|z| < K$, $K \geqslant 1$ and $P(z) = c_0 + \sum_{v=\mu}^{n} c_v z^v$ then

$$\|P'\| \leqslant \frac{n}{1+K^\mu}\|P\|. \tag{2}$$

There is equality in (2) for $P(z) = (z^\mu + K^\mu)^{n/\mu}$ where n is a multiple of μ.

Proof of Theorem 1. Since $P(z) \neq 0$ in $|z| < K$ from Laguerre's Theorem [1] we have

$$\xi P'(z) \neq zP'(z) - nP(z) \tag{3}$$

for $|\xi| < K, |z| < K$. For any fixed z, we can appropriately choose the arg ξ in (3) to get

$$|\xi|\ |P'(z)| \neq |zP'(z) - nP(z)| \tag{4}$$

From where we have

$$|\xi|\ |P'(z)| < |zP'(z) - nP(z)| \tag{5}$$

for $|\xi| < K$, $|z| < K$ because the otherwise inequality is violated for sufficiently small values of $|\xi|$. Making $|\xi| \to K$ and $|z| \to K$ in (5) one has

$$K|P'(Kz)| \leqslant |KzP'(Kz) - nP(Kz)| \tag{6}$$

for $|z| \leqslant 1$. Since $c_1 = c_2 = \ldots = c_{\mu-1} = 0$, from (6) we get

$$K^{\mu}\left|\sum_{\nu=\mu}^{n} \nu c_{\nu}(Kz)^{\nu-\mu}\right| \leqslant |KzP'(Kz) - nP(Kz)| \tag{7}$$

for $|z| \leqslant 1$. In fact (7) holds for $|z| = 1$ and for $\rho < K$ instead of K. But $\rho zP'(\rho z) - nP(\rho z) \neq 0$ in $|z| < 1$, by the maximum modulus principle it also holds for $|z| < 1$ and then let $\rho \to K$. Taking z/K instead of z in (7) we have

$$K^{\mu}\left|\sum_{\nu=\mu}^{n} \nu c_{\nu} z^{\nu-1}\right| \leqslant |zP'(z) - nP(z)| \tag{8}$$

for $|z| = 1$. Consequently

$$K^{\mu}|P'(z)| \leqslant |zP'(z) - nP(z)| \tag{9}$$

for $|z| = 1$. On the other hand, from the result of De Bruijn [1]: for $|\xi| \leqslant 1$, $|z| \leqslant 1$,

$$\xi\frac{P'(z)}{n} - \frac{zP'(z)}{n} + P(z) \in S \tag{10}$$

where $S = \{P(z): |z| \leqslant 1\}$; hence for $|z| = 1$

$$|P'(z)| + |zP'(z) - nP(z)| \leqslant n\|P\|. \tag{11}$$

From (9) and (11), one has

$$(1 + K^{\mu})\|P'\| \leqslant n\|P\|.$$

Consequently (2).

If $P(z) = \sum_{\nu=0}^{n-\mu} c_{\nu}z^{\nu} + c_n z^n$ has all its zeros in the disk $|z| \leqslant k \leqslant 1$ then $Q(z) = z^n P(1/z)$ satisfies the hypothesis of theorem 1 with $K = 1/k$. Since

$$Q'(z) = nz^{n-1}P(1/z) - z^{n-2}P'(1/z). \tag{12}$$

We conclude that

$$\begin{aligned}\|P'\| &\geqslant n\|P\| - \|Q'\| \\ &\geqslant n\|P\| - \frac{nk^{\mu}}{1+k^{\mu}}\|P\| = \frac{n}{1+k^{\mu}}\|P\|.\end{aligned} \tag{13}$$

Thus we have the following:

COROLLARY

If $P(z) = \sum_{\nu=0}^{n-\mu} c_\nu z^\nu + c_n z^n$ has all its zeros in $|z| \leqslant k$, $k \leqslant 1$ then

$$\|P'\| \geqslant \frac{n}{1+k^\mu} \|P\|. \tag{14}$$

There is equality in (14) for $P(z) = (z^\mu + k^\mu)^{n/\mu}$ where n is a multiple of μ.

3. Conclusion

Finally, as an application of Erdös-Lax Theorem we present the following:

THEOREM 2

If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ with $c_n = 1$ has no zeros in $|z| < 1$ and $\|P\| = 2$, then $P(z) = z^n + \alpha$ where $|\alpha = 1$.

Proof of Theorem 2. Since $\|P\| = 2$ and $P(z)$ has no zeros in $|z| < 1$, from theorem A we have $|P'(z)| \leqslant n$ for $|z| \leqslant 1$. Using a result due to Visser [7]: if $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$, then $|c_0| + |c_n| \leqslant \|P\|$, we get $n + |c_1| \leqslant n$. But this implies that $c_1 = 0$ and so $P'(z) = \sum_{\nu=2}^{n} \nu c_\nu z^{\nu-1}$. Applying that result again to $\frac{P'(z)}{z} = \sum_{\nu=2}^{n} \nu c_\nu z^{\nu-2}$, one concludes that $c_2 = 0$. Consequently, all the coefficients $c_\nu = 0$ except c_n and c_0. Since $c_n = 1$, one must have $c_0 = \alpha$, where $|\alpha| = 1$ because $\|P\| = 2$.

References

[1] De Bruijn N G 1947 Inequalities concerning polynomial in the complex domain *Ned. Akad. Wetensch.* **50** 1265

[2] Govil N K 1973 On the derivative of a polynomial *Proc. Am. Math. Soc.* **41** 543

[3] Govil N K, Rahman Q I and Schmeisser G 1979 On the derivative of a polynomial *Illinois J. Math.* **23** 319

[4] Lax P D 1944 Proof of a conjecture of P Erdös on the derivative of polynomial *Bull. Am. Math. Soc.* **50** 509

[5] Malik M A 1969 On the derivative of a polynomial *J. London. Math. Soc.* **1** 57

[6] Turan P 1939–40 Über die ableitung von polynomen *Compositio Math.* **7** 89

[7] Visser C 1945 A simple proof of certain inequalities concerning polynomials. *Ned. Akad. Wetensch.* **47** 276