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Weyl fractional calculus and Laplace transform

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Abstract. The Weyl fractional calculus is developed to obtain Laplace transforms of $t^q\phi(t)$ (for all real values of q) where $\phi(t)$ is taken in the form of $f(a\sqrt{(t^2-b^2)})$ and certain other forms. Also, a generating function involving H-function of several variables is established with the help of generalized Taylor series.

Keywords. Weyl fractional calculus; Laplace transform; H-function.

1. Introduction

Let A denote a class of good functions. By a good function f, we mean a function which is everywhere differentiable any number of times and if it and all of its derivatives are $O(x^{-\nu})$, for all ν as x increases without limit (see [4]).

We define the Weyl fractional derivative of a function $\phi(z)$ as follows: Let $\phi \in A$, then

$$ {}_zD^q_\infty\, \phi(z) = \frac{(-1)^q}{\Gamma(-q)} \int_z^\infty (u-z)^{-q-1}\, \phi(u)\, \mathrm{d}u,\ q<0, \tag{1}$$

and

$$ {}_zD^q_\infty\, \phi(z) = \frac{d^r}{dz^r}\left({}_zD^{q-r}_\infty\, \phi(z)\right),\ q \geqslant 0, \tag{2}$$

r being a positive integer such that $r > q$.

We recall the definition of Laplace transform of $f(t)$ in the form:

$$\psi(p) = L[f(t); p] = \textstyle\int_0^\infty \exp(-pt) f(t)\, \mathrm{d}t,\ p>0. \tag{3}$$

Recently the Laplace transform of $t^q f(t)$, for arbitrary q (real), was obtained in [5] in the theory of Weyl fractional calculus. In this paper, we further develop the Laplace transform of $t^q\phi(t)$ in this theory of Weyl fractional calculus, where $\phi(t)$ is taken in terms of $f(a\sqrt{(t^2-b^2)})$, $f^{(n)}(a\sqrt{(t^2-b^2)})$, $f^{(n)}(t)$ etc., and $f^{(n)}$ denotes the nth derivative of f. As pointed out in ([3], p. 16) the importance of the function $f(a\sqrt{(t^2-b^2)})$ lies in the fact that in certain technical problems involving wave motion, *e.g.* the theory of compressional snock waves, of uniform transmission lines, and that of exponential loudspeaker horns, the variable has this form, b being of the same dimension as t.

We define the Laplace transform of the function $f(a\sqrt{(t^2-b^2)})$ to be that of

$$\left.\begin{aligned} F(t) &= f(a\sqrt{(t^2-b^2)}) && \text{when } t>b>0 \\ &= 0 && \text{when } t<b \end{aligned}\right\} \tag{4}$$

so that we have

$$\phi(p) = L[F(t); p] = \int_b^\infty \exp(-pt) f(a\sqrt{(t^2-b^2)})\, dt, \; p > 0. \tag{5}$$

For the definition of the H-function of several variables *viz.*

$$H[x_1, \ldots, x_r] = H^{0,l:(m',n'),\ldots,(m^{(r)},n^{(r)})}_{A,C:[B',D'],\ldots,[B^{(r)},D^{(r)}]}[x_1, \ldots, x_r] \tag{6}$$

and its conditions of convergence etc., we refer to [6]. The appearance of asterisks in (19) and (20) below indicates that the parameters at those places are the same as in the H-function on the right side of (6) at the corresponding places.

2. Main results

THEOREM 1.

If $\phi(p)$ is given by (5), then, for all real q,

$$\text{(i)}\; (-1)^q {}_pD^q_\infty \{\phi(p)\} = L[t^q f(a\sqrt{(t^2-b^2)}); p]; \tag{7}$$

$$\text{(ii)}\; (-1)^q {}_pD^q_\infty (p^n \phi(p)) = L[t^q f^{(n)}(a\sqrt{(t^2-b^2)}); p], \tag{8}$$

where $f^{(n)}$ denotes the nth derivative of $f(a\sqrt{(t^2-b^2)})$ with respect to t, provided that $f^{(r)}(0) = 0$, $[r = 0, 1, \ldots, (n-1)]$;

$$\text{(iii)}\; (-1)^q {}_pD^q_\infty (p^{-n}\phi(p)) = L[t^q K_n(a\sqrt{(t^2-b^2)}); p], \tag{9}$$

where

$$K_n(a\sqrt{(t^2-b^2)}) = \int_0^t \ldots \int_0^t f(a\sqrt{(t^2-b^2)}) \cdot (\mathrm{d}t)^n, \tag{10}$$

provided that $\lim \exp(-pt) K_1^{(r)}(a\sqrt{(t^2-b^2)}) = \lim \exp(-pt) \int_0^t \{[\int_b^\tau \mathrm{d}\tau]^r\}$

$$\times f(a\sqrt{(\tau^2-b^2)})\, \mathrm{d}\tau \to 0$$

as $t \to \infty$, for $r = 0, 1, \ldots, (n-1)$.

THEOREM 2.

If $\psi(p)$ is given by (3), then, for all real q,

$$\text{(i)}\; (-1)^q {}_pD^q_\infty (p^n \psi(p)) = L[t^q f^{(n)}(t); p], \tag{11}$$

provided that $f^{(r)}(0) = 0$, $[r = 0, 1, \ldots, (n-1)]$;

$$\text{(ii)}\; (-1)^q {}_pD^q_\infty [p^{-n}\psi(p)] = L[t^q F_n(t); p], \tag{12}$$

where

$$F_n(t) = \int_0^t \ldots \int_0^t f(t)\, (\mathrm{d}t)^n, \tag{13}$$

provided that $\lim \exp(-pt) F_1^{(r)}(t) = \lim \exp(-pt) \int_0^t \{[\int_0^\tau \mathrm{d}\tau]^r f(\tau)\}\, \mathrm{d}\tau \to 0$ as $t \to \infty$, for $r = 0, 1, \ldots, (n-1)$.

Proofs: To prove (7), we have, in view of (1) and (5), for $q < 0$,

$$\begin{aligned}(-1)^q {}_pD^q_\infty \phi(p) &= \frac{1}{\Gamma(-q)} \int_p^\infty (u-p)^{-q-1} \phi(u)\, \mathrm{d}u \\ &= \frac{1}{\Gamma(-q)} \int_p^\infty (u-p)^{-q-1} \\ &\quad \times \left(\int_b^\infty f(a\sqrt{(t^2-b^2)}) \exp(-ut)\, \mathrm{d}t \right) \mathrm{d}u.\end{aligned}$$

We now change the order of integration, use ([2], p. 202, (11)) to evaluate the inner integral on the right side to get

$$(-1)^q\, {}_pD^q_\infty\, \phi(p) = L[t^q f(a\sqrt{(t^2-b^2)}); p], \text{ for } q < 0.$$

For $q \geqslant 0$, invoking the definition (2), we can write

$$(-1)^q\, {}_pD^q_\infty\, \phi(p) = \frac{d^r}{dp^r}\left(\int_b^\infty \exp(-pt)\, t^{q-r} f(a\sqrt{(t^2-b^2)})\, \mathrm{d}t\right),\ r > q.$$

Differentiating r-times under the sign of integration ([3], p. 172), we again find that

$$(-1)^q\, {}_pD^q_\infty\, \phi(p) = L[t^q f(a\sqrt{(t^2-b^2)}); p], \text{ for } q \geqslant 0.$$

This completes the proof of (7).

The results (8), (9), (11) and (12) are similarly established by using ([3], p. 191, equation (22); p. 24, Theorem 5(a); p. 21, equation (9); p. 24, equation (5)).

If we take $q = -1$ and $f(t) = J_0(t)$ in (7), use ([1], p. 191, equation (9)) so that

$$\phi(p) = R^{-1}\exp(-bR),\ R = \sqrt{(p^2+a^2)} \tag{14}$$

then, we get, on using ([1], p. 191, equation (11)) and (14)

$${}_pD^{-1}_\infty[\exp(-bR)/R] = \exp(-bR)\int_0^\infty \exp(-u)$$
$$\times (u^2+2buR)^{-1/2}\, \mathrm{d}u,\ p > a, \tag{15}$$

where $R = \sqrt{(p^2+a^2)}$.

Remark: The results given in theorems 1 and 2 can further be generalised in view of the following relation, which can be easily established:

If $\psi(p)$ is given by (3), then, for all q,

$$(-1)^q\, {}_pD^q_\infty[\exp(-ap)\psi(p)] = L[t^q f(t-a); p]. \tag{16}$$

3. The generating function

The generalized Taylor's formula in Weyl fractional calculus is given by ([5], p. 189, equation (10)):

$$g(p+t) = \sum_{n=-\infty}^{\infty} \frac{a t^{an+\eta}}{\Gamma(an+\eta+1)}\, {}_pD^{an+\eta}_\infty[g(p)], \tag{17}$$

valid for all t on the circle $|t/p| = 1$, provided that η is an arbitrary complex number, a is such that $0 < a \leqslant 1$ and $g(p) \in A$.

Let

$$f(t) = t^{\sigma-1}\exp(-bt)\, H[a_1 t^{h_1}, \ldots, a_r t^{h_r}],$$
$$h_i > 0,\ i = 1, \ldots, r,\ b \geqslant 0, \tag{18}$$

then

$$\text{(i) } g(p) = L[f(t); p] = (p+b)^{-\sigma} H^{0,l+1:*}_{A+1,C:*}\left[\begin{matrix} S:*; \\ *:*; \end{matrix}\right.$$
$$\left. a_1(p+b)^{-h_1}, \ldots, a_r(p+b)^{-h_r}\right] \tag{19}$$

where

$$S:[1-\sigma:h_1,\dots,h_r],[(a):\alpha',\dots,\alpha^{(r)}]$$

provided that $(p+b)>0$, $\mathrm{Re}\left(\sigma+\sum_{i=1}^{r}h_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$ $(j=1,\dots,m^{(i)})$, and

$$\text{(ii)}\ {}_pD^q_\infty\,(p^\lambda H[a_1p^{-h_1},\dots,a_rp^{-h_r}])$$

$$=(-1)^q\,p^{\lambda-q}\,H^{0,l+1:*}_{A+1,C+1:*}\left[\begin{matrix}[1+\lambda-q:h_1,\dots,h_r],\\ [1+\lambda:h_1,\dots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\alpha',\dots,\alpha^{(r)}]:*;\\ [(b):\beta',\dots,\beta^{(r)}]:*;\end{matrix}\ a_1p^{-h_1},\dots,a_rp^{-h_r}\right] \tag{20}$$

provided $h_i>0$,

$$\mathrm{Re}\left(-\lambda+\sum_{i=1}^{r}h_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0,\ (j=1,\dots,m^{(i)}).$$

(19) and (20) can be easily established.

The following generating function is now obtained on substituting for $g(p)$ from (19) into (17) and using (20) and making slight changes in the parameters therein:

$$(p+t)^{-\sigma}\,H^{0,l+1:*}_{A+1,C+1:*}\left[\begin{matrix}[1-\sigma:h_1,\dots,h_r],[(a):\alpha',\dots,\alpha^{(r)}]:*;\\ *\qquad\qquad :*;\end{matrix}\right.$$

$$\left.a_1(p+t)^{-h_1},\dots,a_r(p+t)^{-h_r}\right]$$

$$=p^{-\sigma}\sum_{n=-\infty}^{\infty}\frac{a(-t/p)^{an+\eta}}{\Gamma(an+\eta+1)}\times$$

$$\times\,H^{0,l+1:*}_{A+1,C:*}\left[\begin{matrix}[1-\sigma-an-\eta:h_1,\dots,h_r],[(a):\alpha',\dots,\alpha^{(r)}]:*;\\ *\qquad\qquad :*;\end{matrix}\right.$$

$$\left.a_1p^{-h_1},\dots,a_rp^{-h_r}\right], \tag{21}$$

valid under conditions mentioned with (17).

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