Proc. Indian Acad. Sci. (Math. Sci.), Vol. 92, No. 3, December 1983, pp. 157–166.

MHD slow motion past a sphere

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MS received 27 December 1982; revised 12 December 1983

Abstract. The slow motion of an incompressible, viscous electrically conducting fluid, in the presence of a uniform aligned magnetic field, past a sphere is studied. Solutions obtained by Chester, using Stokes' approximations, and by Blerkom and Ludford, using Oseen' approximations, are reviewed. Expressions for stream functions are obtained for MHD Stokes' flow and Oseen' flow respectively.

Keywords. Slow motion; magnetohydrodynamics.

1. Introduction

Parallel to the Stokes' and Oseen' approximate studies of slow motion in ordinary fluid dynamics (OFD), efforts have been made to study the effects of magnetic field on the flow of a viscous electrically conducting fluid past bodies at low values of viscous and magnetic Reynolds numbers. Magnetohydrodynamics (MHD) flow past a sphere in the presence of an externally applied magnetic field has been studied by Chester [5], Blerkom [2], Ludford [7] and Mathon and Ranger [9]. The case in which the magnetic field originates in the body itself has been investigated by Ludford and Murray [8], Chopra and Singer [6], Barthel and Lykoudis [1] and others. MHD flow past a circular cylinder has been studied by Yosinobu and Kakutani [12], Bramley [3], Shanti Swarup and Sinha [11] and others. Some of these works have been summarized in the book by Cabannes [4].

This shows that the work of Chester [5] for MHD Stokes' flow and that of Blerkom [2] and Ludford [7] for MHD Oseen' flow is of fundamental interest and need to be reviewed. It is noted that the expressions for the dimensionless functions ϕ_1, ϕ_2, velocity v and the pressure p are misprinted in the paper of Chester [5], whereas slight correction in the expression for integrand, in calculating drag on the sphere, in Ludford [7] paper is needed. Further, we have added the expressions for the stream function in both the cases.

2. Formulation of the problem

Consider the streaming motion of an incompressible, viscous, electrically conducting fluid past a sphere, which is assumed to have the same magnetic permeability as the fluid. At infinity, the streaming motion and the magnetic field are assumed to be uniform and their directions are parallel to the positive x-axis. The flow is assumed to be steady.

The equations governing the motion in non-dimensional form, in the usual way, are

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\mathrm{Re}(\mathbf{v} \cdot \nabla)\mathbf{v} + \mathrm{Re}(\hat{i} \cdot \nabla)\mathbf{v} = -\nabla p + \nabla^2 \mathbf{v} + \mathrm{Re}_h^2 \mathbf{J} \times \mathbf{H} \tag{2}$$

$$\mathrm{Re}_\sigma \nabla \times (\mathbf{v} \times \mathbf{H}) + \mathrm{Re}_\sigma \nabla \times (\hat{i} \times \mathbf{H}) = \nabla \times (\nabla \times \mathbf{H}) \tag{3}$$

where

$$\mathbf{J} = \mathbf{v} \times \mathbf{H} + \hat{i} \times \mathbf{H} \tag{4}$$

with the boundary conditions

$$\left.\begin{array}{l} r = 1 : \mathbf{v} = -\hat{i} \\ r = \infty : \mathbf{v} = 0, \ p = 0 \end{array}\right\} \tag{5}$$

where

$$\left.\begin{array}{l} \mathrm{Re} = \rho U_\infty a/\mu \ \text{(viscous Reynolds number)} \\ \mathrm{Re}_h = \mu_e H_\infty a \sqrt{\sigma/\mu} \ \text{(Hartmann number)} \\ \mathrm{Re}_\sigma = \mu_e \sigma a U_\infty \ \text{(Magnetic Reynolds number)} \end{array}\right\} \tag{6}$$

ρ is the mass density, σ is the electrical conductivity, μ_e is the magnetic permeability, and μ is the viscosity of the fluid. U_∞ is the speed of the uniform stream at infinity parallel to the x-axis, H_∞ is the intensity of the applied aligned external magnetic field and a is the radius of the sphere.

3. MHD Stokes' flow

As in OFD Stokes' flow, we assume that Re is small. In addition we assume that Re_σ is also small. In many practical problems the first condition implies the second but the converse need not be true (MHD boundary layers).

Neglecting the terms containing Re_σ in (3), we get

$$\nabla \times (\nabla \times \mathbf{H}) = 0 \tag{7}$$

This shows that in MHD Stokes' flow the magnetic field is independent of the fluid velocity. The problem is now reduced to the solution of (1) and (2) for $\mathbf{v}$ and p, with $\mathbf{J}$ given by (4) and with a prescribed value of $\mathbf{H} = \hat{i}$.

Neglecting the terms containing Re in (2), the MHD Stokes' equations are

$$\nabla \cdot \mathbf{v} = 0 \tag{8}$$

$$-\nabla p + \nabla^2 \mathbf{v} - \mathrm{Re}_h^2 \{\mathbf{v} - (\mathbf{v} \cdot \hat{i})\hat{i}\} = 0 \tag{9}$$

Chester [5] assumed the following forms for $\mathbf{v}$ and p.

$$\mathbf{v} = \exp(\mathrm{Re}_h x)\nabla\phi_1 + \exp(-\mathrm{Re}_h x)\nabla\phi_2, \tag{10}$$

$$p = \mathrm{Re}_h\left(\exp(\mathrm{Re}_h x)\frac{\partial\phi_1}{\partial x} - \exp(-\mathrm{Re}_h x)\frac{\partial\phi_2}{\partial x}\right), \tag{11}$$

where ϕ_1 and ϕ_2 are dimensionless functions to be determined. As $\mathrm{Re}_h \to 0$, both ϕ_1 and ϕ_2 are of the order of $1/\mathrm{Re}_h$ so that the limiting value of the pressure is non-zero, which should be the case for non-magnetic Stokes' flow. Moreover, the singular terms in (10) cancel each other, so that a finite limiting value of the velocity is also obtained.

With these forms of $\mathbf{v}$ and p, (8) and (9) are satisfied provided that

$$\nabla^2\phi_1 + \mathrm{Re}_h\frac{\partial\phi_1}{\partial x} = 0, \tag{12}$$

$$\nabla^2\phi_2 - \mathrm{Re}_h\frac{\partial\phi_2}{\partial x} = 0. \tag{13}$$

The general solutions of ϕ_1 and ϕ_2, keeping the symmetry in view, which vanish at infinity are obtained as, Milne-Thomson [10],

$$\phi_1 = \exp((-\mathrm{Re}_h/2)x)\sum_{n=0}^{\infty} C_n\left(\frac{\partial}{\partial x}\right)^n\left(\frac{1}{r}\exp((-\mathrm{Re}_h/2)r)\right), \tag{14}$$

$$\phi_2 = \exp((\mathrm{Re}_h/2)x)\sum_{n=0}^{\infty}(-1)^{n+1} C_n\left(\frac{\partial}{\partial x}\right)^n\left(\frac{1}{r}\exp((-\mathrm{Re}_h/2)r)\right). \tag{15}$$

These expressions and the corresponding expressions for $\mathbf{v}$ and p from (10) and (11) are misprinted in Chester [5].

Let

$$\mathbf{v} = \{v_1, v_2, v_3\},\ \mathbf{r} = \{x_1, x_2, x_3\} \tag{16}$$

Then from (10), taking $x = x_1$, substituting ϕ_1 and ϕ_2 from (14) and (15) and retaining small quantities of Re_h to the first order, we get

$$\begin{aligned} v_1 &= 2C_1\left(\frac{3x_1^2}{r^5} - \frac{1}{r^3}\right) + \mathrm{Re}_h\left[C_2\left(\frac{1}{r^3} + \frac{6x_1^2}{r^5} - \frac{15x_1^4}{r^7}\right)\right. \\ &\quad \left. - C_0\left(\frac{1}{r} + \frac{x_1^2}{r^3} - \frac{\mathrm{Re}_h}{2}\right)\right] + 0(\mathrm{Re}_h^2), \end{aligned} \tag{17}$$

$$\begin{aligned} v_2 &= \frac{6C_1x_1x_2}{r^5} + \mathrm{Re}_h\left[-C_0\frac{x_1x_2}{r^3} + C_2\left(\frac{3x_1x_2}{r^5} - \frac{15x_1^3x_2}{r^7}\right)\right] \\ &\quad + 0(\mathrm{Re}_h^2), \end{aligned} \tag{18}$$

$$\begin{aligned} v_3 &= \frac{6C_1x_1x_3}{r^5} + \mathrm{Re}_h\left[-C_0\frac{x_1x_3}{r^3} + C_2\left(\frac{3x_1x_3}{r^5} - \frac{15x_1^3x_3}{r^7}\right)\right] \\ &\quad + 0(\mathrm{Re}_h^2). \end{aligned} \tag{19}$$

Applying the boundary conditions (5) and equating to zero the like powers of x_1, to a first order in Re_h, we find

$$\left.\begin{aligned} C_0\,\mathrm{Re}_h &= \frac{6}{8 - 3\mathrm{Re}_h} \approx \frac{3}{4}\left(1 + \frac{3}{8}\mathrm{Re}_h\right), \\ C_1 &= \frac{1}{8 - 3\mathrm{Re}_h} \approx \frac{1}{8}\left(1 + \frac{3}{8}\mathrm{Re}_h\right), \\ C_2 &= 0. \end{aligned}\right\} \tag{20}$$

Hence,

$$v_i = \frac{3}{4}\left(1 + \frac{3}{8}\mathrm{Re}_h\right)\frac{x_1 x_i}{r^3}\left(\frac{1}{r^2} - 1\right)$$

$$-\frac{1}{4}\left(1 + \frac{3}{8}\mathrm{Re}_h\right)\left(\frac{3}{r} + \frac{1}{r^3} - \frac{3\mathrm{Re}_h}{2}\right)\delta_{i1} + 0(\mathrm{Re}_h^2) \tag{21}$$

and

$$p = -\frac{3}{2}\left(1 + \frac{3}{8}\mathrm{Re}_h\right)\frac{x_1}{r^3} + 0(\mathrm{Re}_h^2), \tag{22}$$

where δ_{i1} is the Kronecker delta.

3.1 *Drag in* MHD *Stokes' flow*

The total drag experienced by the sphere is calculated as

$$D = D_S\left[1 + \frac{3}{8}\mathrm{Re}_h + 0(\mathrm{Re}_h^2)\right] \tag{23}$$

where, $\quad D_S = 6\pi\mu U_\infty a$ (Stokes' drag in OFD). (24)

In fact Chester calculated the drag upto the third power of Re_h and reported that

$$D = D_S\left\{1 + \frac{3}{8}\mathrm{Re}_h + \frac{7}{960}\mathrm{Re}_h^2 - \frac{43}{7680}\mathrm{Re}_h^3 + 0(\mathrm{Re}_h^4)\right\} \tag{25}$$

However, for $\mathrm{Re}_h < 1$, the relation (23) gives sufficiently accurate results.

3.2 *Stream function in* MHD *Stokes' flow*

The character of the motion is most concisely expressed in terms of the Stokes' stream function ψ, which is given by

$$(\psi/U_\infty a^2) = -\int_0^\theta r^2(v_r + \cos\theta)\sin\theta\,\mathrm{d}\theta \tag{26}$$

where v_r is the dimensionless radial component of the fluid velocity and is given by

$$v_r = \exp(\mathrm{Re}_h x_1)\frac{\partial\phi_1}{\partial r} + \exp(-\mathrm{Re}_h x_1)\frac{\partial\phi_2}{\partial r} \tag{27}$$

Substituting (27) in (26) and carrying out some mathematical computations, we finally get

$$\frac{\psi}{U_\infty a^2} = -\frac{1}{2}\left(1 + \frac{3}{8}\mathrm{Re}_h\right)\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)r^2\sin^2\theta + 0(\mathrm{Re}_h^2). \tag{28}$$

If we put $\mathrm{Re}_h = 0$, we get Stokes' result of OFD.

4. MHD Oseen's flow

In the Stokes' flow studied in § 3, we neglected the disturbance of the magnetic field and the inertia of the fluid. However, using Oseen type approximation, where quadratic terms in the disturbance quantities are neglected in the equations of motion and magnetic field, the results of Chester [5] may be improved.

We linearize (2) and (3) by writing

$$\mathbf{H} = \mathbf{h} + \hat{i} \tag{29}$$

where $\mathbf{h}$ is the dimensionless disturbance in the applied magnetic field.

The boundary conditions on $\mathbf{h}$ and $\mathbf{v}$ are

$$\begin{aligned} &r = 1: h_t \text{ and } \mu_e h_n \text{ be continuous; } \mathbf{v} = -\hat{i} \\ &r = \infty: \mathbf{h} = 0; \mathbf{v} = 0 \end{aligned} \tag{30}$$

where h_t and h_n are, respectively, components of the magnetic field, tangent and normal to the sphere.

From Ampere's law and (29), we conclude that

$$\mathbf{J} \times \mathbf{H} = \frac{1}{\mathrm{Re}_\sigma}(\nabla \times \mathbf{h}) \times (\mathbf{h} + \hat{i}) \tag{31}$$

Substituting (31) in (2) and (29) in (3) and neglecting the small quantities of second order in $\mathbf{v}$ and $\mathbf{h}$, the linearized MHD-Oseen' equations are respectively

$$\mathrm{Re}(\hat{i}\cdot\nabla)\mathbf{v} = -\nabla p + \nabla^2\mathbf{v} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}(\nabla \times \mathbf{h}) \times \hat{i}, \tag{32}$$

$$\mathrm{Re}_\sigma \nabla \times (\mathbf{v} \times \hat{i}) + \mathrm{Re}_\sigma \nabla \times (\hat{i} \times \mathbf{h}) = \nabla \times (\nabla \times \mathbf{h}) \tag{33}$$

which may be written as

$$\nabla^2\mathbf{v} - \mathrm{Re}\frac{\partial \mathbf{v}}{\partial x} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\frac{\partial \mathbf{h}}{\partial x} = \nabla\left(p + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\mathbf{h}\cdot\hat{i}\right), \tag{34}$$

and

$$\nabla^2\mathbf{h} + \mathrm{Re}_\sigma\left(\frac{\partial \mathbf{v}}{\partial x} - \frac{\partial \mathbf{h}}{\partial x}\right) = 0, \tag{35}$$

where $\operatorname{div}\mathbf{h} = 0$ and $\operatorname{div}\mathbf{v} = 0$ have been used.

Taking the divergence of (34), we get

$$\nabla^2\left(p + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\mathbf{h}\cdot\hat{i}\right) = 0 \tag{36}$$

Let,

$$p_0 = p + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\mathbf{h}\cdot\hat{i}, \tag{37}$$

then p_0 satisfies the Laplace equation.

A particular integral of (34) is therefore obtained if we write

$$\left.\begin{aligned} &p_0 = \left(-\mathrm{Re} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\right)\frac{\partial \phi}{\partial x}, \\ &\mathbf{v} = \nabla\phi \text{ and } \mathbf{h} = \nabla\phi, \end{aligned}\right\} \tag{38}$$

where ϕ satisfies

$$\nabla^2\phi = 0. \tag{39}$$

For the complete solution we put

$$\mathbf{v} = \nabla\phi + \mathbf{v}_1,\ \mathbf{h} = \nabla\phi + \mathbf{h}_1, \tag{40}$$

where $\mathbf{v}_1$ and $\mathbf{h}_1$ satisfy the corresponding homogeneous equations

$$\nabla^2\mathbf{v}_1 - \mathrm{Re}\frac{\partial\mathbf{v}_1}{\partial x} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\frac{\partial\mathbf{h}_1}{\partial x} = 0, \tag{41}$$

$$\nabla^2\mathbf{h}_1 + \mathrm{Re}_\sigma\left(\frac{\partial\mathbf{v}_1}{\partial x} - \frac{\partial\mathbf{h}_1}{\partial x}\right) = 0. \tag{42}$$

These equations may be decoupled, if we write

$$\mathbf{v}_1 + \alpha_1\mathbf{h}_1 = (\alpha_2 - \alpha_1)\left[\frac{1}{2k_1}\nabla\chi_1 - \chi_1\hat{i}\right], \tag{43}$$

$$\mathbf{v}_1 + \alpha_2\mathbf{h}_1 = (\alpha_2 - \alpha_1)\left[\frac{1}{2k_2}\nabla\chi_2 - \chi_2\hat{i}\right], \tag{44}$$

where χ_1 and χ_2 satisfy the equations

$$\left(\nabla^2 - 2k_1\frac{\partial}{\partial x}\right)\chi_1 = 0,\ \left(\nabla^2 - 2k_2\frac{\partial}{\partial x}\right)\chi_2 = 0, \tag{45}$$

$$\alpha_1, \alpha_2 = \frac{1}{2\mathrm{Re}_\sigma}[(\mathrm{Re} - \mathrm{Re}_\sigma) \pm \{(\mathrm{Re} - \mathrm{Re}_\sigma)^2 + 4\mathrm{Re}_h^2\}^{1/2}], \tag{46}$$

and

$$k_j = (\mathrm{Re} - \mathrm{Re}_\sigma\alpha_j)/2, \tag{47}$$

are the 'modified' Reynolds numbers.

We now define

$$\mathrm{Re}_H = \frac{\mu_e H_\infty^2}{\rho U_\infty^2} = \frac{\mathrm{Re}_h^2}{\mathrm{Re}\,\mathrm{Re}_\sigma}\ \text{(Magnetic pressure number).} \tag{48}$$

It may be noted from (47) that for $\mathrm{Re}_H < 1$, each of the 'modified' Reynolds numbers is positive and for $\mathrm{Re}_H > 1$ there is a positive and a negative 'modified' Reynolds number. When $\mathrm{Re}_H = 1$, the 'modified' Reynolds numbers are $(\mathrm{Re} + \mathrm{Re}_\sigma)/2$ and zero. These cases will be considered later in the analysis.

Thus the complete solution is given by

$$\mathbf{v} = \nabla\left(\phi + \frac{\alpha_2}{2k_1}\chi_1 - \frac{\alpha_1}{2k_2}\chi_2\right) - (\alpha_2\chi_1 - \alpha_1\chi_2)\hat{i},$$

$$\mathbf{h} = \nabla\left(\phi - \frac{1}{2k_1}\chi_1 + \frac{1}{2k_2}\chi_2\right) + (\chi_1 - \chi_2)\hat{i},$$

$$p = -\mathrm{Re}\frac{\partial\phi}{\partial x} + \frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\left[\left(\frac{1}{2k_1}\frac{\partial\chi_1}{\partial x} - \frac{1}{2k_2}\frac{\partial\chi_2}{\partial x}\right) - (\chi_1 - \chi_2)\right], \tag{49}$$

where ϕ, χ_1 and χ_2 are to be obtained from the differential equations (39) and (45) respectively, whose general solutions, in the case of axial symmetry, outside the sphere, which vanish at infinity, are

$$\phi = \sum_{n=0}^{\infty} A_n(\partial/\partial x)^n (1/r), \tag{50}$$

$$\chi_1 = \exp(k_1 x) \sum_{n=0}^{\infty} B_n(\partial/\partial x)^n (\exp(\mp k_1 r)/r), \tag{51}$$

$$\chi_2 = \exp(k_2 x) \sum_{n=0}^{\infty} C_n(\partial/\partial x)^n (\exp(-k_2 r)/r), \tag{52}$$

where the $\mp$ sign in χ_1 correspond to $k_1 \gtrless 0$.

Solutions of the type $\exp[-|k_1|r + k_1 x]/r$ and $\exp[-|k_2|r + k_2 x]/r$ lead to wakes in the directions of increasing $k_1 x$ and $k_2 x$ respectively. As pointed out earlier k_2 is essentially positive so that there is always a downstream wake. On the other hand, $k_1 \gtrless 0$, according as $\mathrm{Re}_H \lesssim 1$. Thus, when the free stream velocity U_∞ is greater than the Alfvén velocity $(H_\infty \sqrt{\mu_e/\rho})$ the second wake is also downstream, but when it is less, it is upstream.

Inside the sphere,

$$\mathbf{h} = \nabla\psi, \tag{53}$$

where

$$\psi = \sum_{n=1}^{\infty} D_n r^{2n+1} (\partial/\partial x)^n (1/r). \tag{54}$$

To determine the unknown constants, A_n, B_n, C_n and D_n we apply the boundary conditions (30) on the surface of the sphere $r = 1$ and assume that the sphere has the same magnetic permeability as that of the fluid, we find that

$$A_0 = -\frac{3(k_1 - k_2)}{4(\alpha_2 - \alpha_1)k_1 k_2}\left[1 + \frac{3(\pm\alpha_2 k_1 - \alpha_1 k_2)}{4(\alpha_2 - \alpha_1)}\right] \tag{55}$$

$$B_0 = C_0 = \frac{3}{2(\alpha_2 - \alpha_1)}\left[1 + \frac{3(\pm\alpha_2 k_1 - \alpha_1 k_2)}{4(\alpha_2 - \alpha_1)}\right], \tag{56}$$

and

$$A_1 = 1 + (B_0/2)[(\alpha_1 - \alpha_2) + (\pm\alpha_2 k_1 - \alpha_1 k_2)], \tag{57}$$

correct to the first order in k_1 and k_2. The determination of other constants become increasingly complicated.

4.1 *Drag in* MHD *Oseen' flow*

We now calculate the drag on the sphere. The drag D due to the pressure and viscous forces on the sphere is given by

$$D = -\mu U_\infty a \int (lp + m\zeta - n\eta)_{r=1}\, \mathrm{d}S, \tag{58}$$

where (l, m, n) are the direction-cosines of the outward drawn normal to the sphere

S: $r = 1$, while η and ζ are the y and z components of curl $\mathbf{v}$. The drag due to Maxwell stresses is zero; it is proportional to the flux of $\mathbf{h}$ through the surface of the sphere.

Keeping (49) in view, the integrand of (58) may be written as

$$l\left[-\mathrm{Re}\frac{\partial\phi}{\partial x}+\frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\left\{\left(\frac{1}{2k_1}\frac{\partial\chi_1}{\partial x}-\frac{1}{2k_2}\frac{\partial\chi_2}{\partial x}\right)-(\chi_1-\chi_2)\right\}\right]$$
$$+m\left[\alpha_2\frac{\partial\chi_1}{\partial y}-\alpha_1\frac{\partial\chi_2}{\partial y}\right]+n\left[\alpha_2\frac{\partial\chi_1}{\partial z}-\alpha_1\frac{\partial\chi_2}{\partial z}\right], \tag{59}$$

the values are to be taken at $r = 1$.

On the sphere, from (43), (44) and (40) we find

$$\frac{1}{2k}\nabla\chi-\chi\hat{i}=\frac{1}{\alpha_1-\alpha_2}[\hat{i}-\alpha\mathbf{h}+(\alpha+1)\nabla\phi], \tag{60}$$

for $k = k_1, k_2$ etc. This expression, in the right side, differs by a factor $R_M(=\mathrm{Re}_\sigma)$ of reference [7], which is wrongly introduced.

Hence,

$$D=\mu U_\infty a\left(\mathrm{Re}-\frac{\mathrm{Re}_h^2}{\mathrm{Re}_\sigma}\right)\int\left(\frac{\partial\phi}{\partial r}\right)_{r=1}\mathrm{d}S$$
$$=-4\pi\mu U_\infty a(\mathrm{Re}-(\mathrm{Re}_h^2/\mathrm{Re}_\sigma))A_0, \tag{61}$$

where A_0 is given by (55). Substituting the values of α_1, α_2, k_1 and k_2 in (55), we find that

$$A_0=-\frac{3}{2\mathrm{Re}(1-\mathrm{Re}_H)}\left[1+\frac{3}{8}K\,\mathrm{Re}\right], \tag{62}$$

$$K=\begin{cases}1 \text{ for } k_1\geqslant 0, \text{ i.e., } \mathrm{Re}_H\leqslant 1\\ \dfrac{2\mathrm{Pr}_m\mathrm{Re}_H+1-\mathrm{Pr}_m}{\{(1-\mathrm{Pr}_m)^2+4\mathrm{Pr}_m\mathrm{Re}_H\}^{1/2}} \text{ for } k_1\leqslant 0, \text{ i.e. } \mathrm{Re}_H\geqslant 1\end{cases} \tag{63}$$

$$\mathrm{Pr}_m=(\mathrm{Re}_\sigma/\mathrm{Re})=\mu_e\sigma\nu \text{ (Magnetic Prandtl number)} \tag{64}$$

Hence,

$$D=D_S(1+\tfrac{3}{8}K\,\mathrm{Re}) \tag{65}$$

Thus, when $\mathrm{Re}_H \leqslant 1$ i.e. the free stream velocity is greater or equal to the Alfvén velocity the magnetic field has no influence on the drag and it is simply the Oseen' drag of OFD. It is only affected when $\mathrm{Re}_H > 1$. In the purely viscous case $\mathrm{Pr}_m = 0$, $\mathrm{Re}_H = 0$, hence $K = 1$ and we recapture Oseen' drag. On the other hand, when Re and Re_σ are small and are of the same order of magnitude we may take $\mathrm{Pr}_m = 1$ and in that case (63) gives $K = \mathrm{Re}_h/\mathrm{Re}$, which reduces (65) to the MHD Stokes' drag (24).

4.2 *Stream function in MHD Oseen' flow*

In the present case, we have

$$v_r=\frac{\partial\phi}{\partial r}+\frac{\alpha_2}{2k_1}\frac{\partial\chi_1}{\partial r}-\frac{\alpha_1}{2k_2}\frac{\partial\chi_2}{\partial r}-(\alpha_2\chi_1-\alpha_1\chi_2)\cos\theta \tag{66}$$

Substituting (66) in (26), we get
For $k_1 > 0$

$$\frac{\psi}{U_\infty a^2} = A_0(1-\cos\theta) - \frac{A_1}{r}\sin^2\theta + B_0\left(\frac{\alpha_2}{k_1} - \frac{\alpha_1}{k_2}\right)$$
$$-\frac{B_0(1+\cos\theta)}{2}\left[\frac{\alpha_2}{k_1}\exp(-k_1 r(1-\cos\theta))\right.$$
$$\left.-\frac{\alpha_1}{k_2}\exp(-k_2 r(1-\cos\theta))\right] - \frac{r^2\sin^2\theta}{2} + \dots \tag{67}$$

For $k_1 < 0$

$$\frac{\psi}{U_\infty a^2} = A_0(1-\cos\theta) - \frac{A_1}{r}\sin^2\theta + \frac{B_0\alpha_2}{2k_1}(1-\cos\theta)\exp(k_1 r(1+\cos\theta))$$
$$+\frac{B_0\alpha_1}{2k_2}\left[(1+\cos\theta)\exp(-k_2 r(1-\cos\theta)) - 2\right]$$
$$-\frac{r^2\sin^2\theta}{2} + \dots \tag{68}$$

In the non-magnetic case, $k_1 > 0$, where $\mathrm{Re}_h \to 0$, we have

$$\begin{aligned} k &= k_1 = k_2 = \frac{\mathrm{Re}}{2} \\ A_0 &= -\frac{3}{2\mathrm{Re}}\left(1+\frac{3}{8}\mathrm{Re}\right) \\ B_0(\alpha_2-\alpha_1) &= \frac{3}{2}\left(1+\frac{3}{8}\mathrm{Re}\right) \\ A_1 &= 1 + \frac{3}{4}\left(1+\frac{3}{8}\mathrm{Re}\right)\left(\frac{\mathrm{Re}}{2}-1\right) \end{aligned} \tag{69}$$

Therefore, the stream function (67) reduces to

$$\frac{\psi}{U_\infty a^2} = \frac{3}{2\mathrm{Re}}\left(1+\frac{3}{8}\mathrm{Re}\right)(1+\cos\theta)[1-\exp(-kr(1-\cos\theta))]$$
$$-\left(\frac{1}{4}+\frac{3}{32}\mathrm{Re}+\frac{9}{64}\mathrm{Re}^2\right)\frac{\sin^2\theta}{r} - \frac{r^2\sin^2\theta}{2}. \tag{70}$$

For $\mathrm{Re} \ll 1$, it is the same as given by Milne-Thomson [10] in OFD.

5. Conclusion

In MHD Stokes flow there is a wake downstream unlike OFD Stokes' flow for $\mathrm{Re} \ll 1$, which modifies the OFD Stokes' drag expression in a similar manner as by OFD Oseen' flow to the Stokes' flow with Re replaced by Re_h ($\mathrm{Re}_h \ll 1$). In MHD Oseen's flow, when the free stream velocity is greater or equal to Alfvén velocity the formation of the wake is only downstream and the magnetic field has no influence on the Oseen drag of OFD. On the other hand, when the free stream velocity is less than the Alfvén velocity we have both a wake upstream and a wake downstream which modifies the OFD Oseen's drag expression and increases the drag due to the presence of the magnetic field.

Acknowledgement

The authors are thankful to the referee for his valuable suggestions. This work has been carried out with the financial support of CSIR in the form of a Junior Research Fellowship awarded to one of the authors (Miss Rama Kumari).

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