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Some theorems in temperature-rate-dependent theory of thermoelasticity

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Abstract. A theorem on the uniqueness of solutions, a generalised Hamilton's principle and a reciprocal theorem for dynamical mixed boundary value problems are obtained in the context of a linear anisotropic thermoelasticity theory which predicts a finite speed of propagation of thermal signals.

Keywords. Generalised thermoelasticity; second-sound effect; uniqueness of solution; Hamilton's principle; reciprocal theorem.

1. Introduction

The thermoelasticity theory first proposed by Green and Lindsay [9] and later by Suhubi [15] has aroused much interest in recent years. Unlike the coupled thermoelasticity theory [5], this theory includes the temperature rate among the constitutive variables and consequently predicts a finite speed for the propagation of thermal signals. Since thermal signals propagating with finite speeds have actually been observed in solids [1, 2], this theory is more general and physically more realistic than the coupled theory, and several problems revealing interesting phenomena which characterize this theory are contained in references [3, 4, 6–8].

The purpose of the present paper is to prove three main theorems, *viz.* (i) a uniqueness theorem (ii) a variational principle of Hamilton-type and (iii) a reciprocal theorem of Betti-Rayleigh-type for linearized anisotropic thermoelastic interactions, by employing the equations obtained in [9]. In § 2 we summarize the governing equations and formulate an initial, mixed boundary value problem. In §3, we obtain the equation of energy balance in terms of the generalized free energy function introduced by Biot in [5], and in § 4 we employ it, to establish a uniqueness theorem. Unlike in [9] and [10], our proof of the theorem rests on the positive definiteness of the energy function. In § 5 we establish a Hamilton-type variational principle for the field equations. Whereas the corresponding principle in the coupled theory involves two functionals [14], our principle involves just one. In § 6 we establish a reciprocal theorem which includes as special cases several important theorems obtained in earlier works, including the celebrated Betti-Rayleigh theorem of classical elasticity. We propose to report some significant applications of our reciprocal theorem in a separate communication.

2. Basic equations

In the context of the theory proposed by Green and Lindsay [9], the field equations for linear thermoelastic interactions in a homogeneous and anisotropic solid are

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

$$t_{ij} = k_{ijrs} e_{rs} + a_{ij}(\theta + \alpha\dot{\theta}) \tag{2}$$

$$t_{ji,j} + \rho_0 F_i = \rho_0 \ddot{u}_i, \quad t_{ij} = t_{ji} \tag{3}$$

$$q_i = -\theta_0 (b_i \dot{\theta} + k_{ij}\theta_{,j}) \tag{4}$$

$$\rho_0 \eta = a + \mathrm{d}\theta + h\dot{\theta} - a_{ij}e_{ij} - b_i\theta_{,i} \tag{5}$$

$$\frac{\rho_0 r}{\theta_0} + k_{ij}\theta_{,ij} - \mathrm{d}\dot{\theta} - h\ddot{\theta} + a_{ij}\dot{e}_{ij} + 2b_i\dot{\theta}_{,i} = 0 \tag{6}$$

Unless stated to the contrary, the notation and the symbols in these equations and those to follow are as explained in [9].

With the aid of (1), equations (2) and (6) may be rewritten as

$$t_{ij} = k_{ijrs} u_{r,s} + a_{ij}(\theta + \alpha\dot{\theta}) \tag{7}$$

$$(\rho_0 r/\theta_0) + k_{ij}\theta_{,ij} - d\dot{\theta} - h\ddot{\theta} + a_{ij}\dot{u}_{i,j} + 2b_i\dot{\theta}_{,i} = 0 \tag{8}$$

Eliminating t_{ij} from (3) and (7), we obtain

$$k_{ijrs} u_{r,sj} + a_{ij}(\theta + \alpha\dot{\theta})_{,j} + \rho_0 F_i = \rho_0 \ddot{u}_i \tag{9}$$

If we assume that at time $t = 0$, the body is at rest in its initial undeformed state and is at the reference temperature θ_0, zero temperature-velocity and zero entropy, then the following initial conditions are to be satisfied:

$$u_i(x,0) = \dot{u}_i(x,0) = \theta(x,0) = \dot{\theta}(x,0) = \eta(x,0) = 0, \quad x \in B, \tag{10}$$

B being the initial configuration of the body.

Further, if we assume that for $t \geqslant 0$, (i) the surface forces are prescribed on a part ∂B_1 and the displacements on the remaining part ∂B_2 of the boundary surface ∂B of B, and (ii) the temperature is prescribed on a part ∂B_3 and the heat flux on the remaining part ∂B_4 of ∂B, then the following conditions are to be satisfied:

$$\left.\begin{aligned} t_{ji}n_j &= T_i(x,t), \ x \in \partial B_1 \\ u_i &= U_i(x,t), \ x \in \partial B_2 \end{aligned}\right\}, \quad t \geqslant 0 \tag{11}$$

$$\left.\begin{aligned} \theta &= \Theta(x,t), \ x \in \partial B_3 \\ q_i n_i &= Q(x,t), \ x \in \partial B_4 \end{aligned}\right\}, \quad t \geqslant 0 \tag{12}$$

In these equations $T_i(x,t)$, $U_i(x,t)$, $\Theta(x,t)$ and $Q(x,t)$ are prescribed functions in their respective domains. With $F_i(x,t)$ and $r(x,t)$ as prescribed functions for $x \in B$ and $t \geqslant 0$, the problem of determining the field variables $u_i(x,t)$, $\theta(x,t)$ and $t_{ij}(x,t)$ for $x \in B$ and $t > 0$ by solving the equations (7)–(9) under the conditions (10)–(12) constitutes an initial mixed boundary value problem. We will refer to this problem as the problem associated with the system: $\{(F_i, r); (T_i, U_i, \Theta, Q); (u_i, t_{ij}, \theta)\}$.

3. Energy equation

We now obtain the equation of energy balance in terms of the generalized free energy function V introduced by Biot [5] through the equation

$$V = \varepsilon - \eta\theta_0. \tag{13}$$

The energy function considered by Green and Lindsay [9] is given by

$$\psi = \varepsilon - \eta\phi \tag{14}$$

and has the following explicit form:

$$\rho_0\psi = \sigma_0 - a(\theta + \alpha\dot{\theta}) - \tfrac{1}{2}\mathrm{d}\theta^2 - e\theta\dot{\theta} - \tfrac{1}{2}f\dot{\theta}^2 + \alpha b_i\dot{\theta}\theta_{,i} + a_{ij}e_{ij}(\theta + \alpha\dot{\theta})$$
$$+ \tfrac{1}{2}\alpha k_{ij}\theta_{,i}\theta_{,j} + \tfrac{1}{2}k_{ijrs}e_{ij}e_{rs} \tag{15}$$

The scalar function ϕ appearing in (14) is given by ([9], eqn. (4.4)).

$$\phi = \theta_0 + \theta + \alpha\dot{\theta} + \beta\theta\dot{\theta} + \tfrac{1}{2}\gamma\dot{\theta}^2 \tag{16}$$

Eliminating ε from (13) and (14) and substituting for ψ, ϕ and η from (15), (16) and (5), we obtain the following quadratic form expansion for the Biot's energy function V:

$$\rho_0 V = \tfrac{1}{2}k_{ijrs}e_{ij}e_{rs} - b_i\theta\theta_{,i} + \tfrac{1}{2}\alpha k_{ij}\theta_{,i}\theta_{,j} + \tfrac{1}{2}\mathrm{d}\theta^2 + h\theta\dot{\theta} + \tfrac{1}{2}\alpha h\dot{\theta}^2 \tag{17}$$

In obtaining this expression we have made use of the initial conditions (10) and the restrictions on the material constants [[9], eqn (4.8)], and have neglected terms of third and higher degrees in the field variables.

If K denotes the kinetic energy per unit mass, i.e. if

$$K = \tfrac{1}{2}\dot{u}_i\dot{u}_i, \tag{18}$$

we obtain with the aid of (7) and (9) and the divergence theorem,

$$(\mathrm{d}/\mathrm{d}t)\int_m K\,\mathrm{d}m = \int_m [F_i\dot{u}_i - (1/\rho_0)\{k_{ijrs}u_{r,s} + a_{ij}(\theta + \alpha\dot{\theta})\}\dot{u}_{i,j}]\,\mathrm{d}m$$
$$+ \int_{\partial B} t_{ij}n_j\dot{u}_i\,\mathrm{d}A \tag{19}$$

where m is the mass included in B.
Eliminating $\int_m a_{ij}(\theta + \alpha\dot{\theta})\dot{u}_{i,j}\,\mathrm{d}m$ from (8) and (19) and using (4) and (17), we obtain

$$(\mathrm{d}/\mathrm{d}t)\int_m (K + V)\,\mathrm{d}m + N = \int_m [F_i\dot{u}_i + (r/\theta_0)(\theta + \alpha\dot{\theta})]\,\mathrm{d}m$$
$$+ \int_{\partial B}[t_{ij}\dot{u}_j - (q_i/\theta_0)(\theta + \alpha\dot{\theta})]\eta_i\,\mathrm{d}A \tag{20}$$

where

$$N = \int_B [(\mathrm{d}\alpha - h)\theta^2 + 2b_i\theta\theta_{,i} + k_{ij}\theta_{,i}\theta_{,j}]\,\mathrm{d}B \tag{21}$$

Equation (20) is the desired equation of energy balance. Equation (4.17) of Green and Lindsay [9] is an alternative form of a particular case of this equation.

It must be noted that because of the thermodynamical restrictions on the material constants obtained in [[9], eqn (4.9)], the integrand in the right side of (21) is non-negative and accordingly, we have

$$N \geqslant 0 \tag{22}$$

Further, we note that if the time derivatives are replaced by variations, (20) and (21) yield a variational principle analogous to the Biot's variational principle [[5], § 9] of coupled thermoelasticity.

4. Uniqueness theorem

We now employ (20) to establish the following

THEOREM

If the Biot's energy function V is positive definite, then there exists atmost one solution for the problem associated with the system:

$$\{(F_i, r); (T_i, U_i, \Theta, Q); (u_i, t_{ij}, \theta)\}.$$

Proof: To establish the theorem it is sufficient to show that for $F_i \equiv r \equiv 0$ in B and $T_i \equiv U_i \equiv \Theta \equiv Q \equiv 0$ in their respective domains on ∂B, the solution is trivial. For $F_i \equiv r \equiv 0$ and homogeneous boundary conditions, (20) simplifies to

$$(\mathrm{d}/\mathrm{d}t) \int_m (K + V) \mathrm{d}m = -N \tag{23}$$

Since the right side is non-positive, because of (22), it follows that $\int_m (K + V) \mathrm{d}m$ is a non-increasing function of time. At $t = 0$ we have $K = 0 = V$, in view of the initial conditions (10); consequently we should have

$$\int_m (K + V) \mathrm{d}m \leqslant 0 \quad \text{for all} \quad t \geqslant 0.$$

Since $K \geqslant 0$ by definition, it follows that

$$K + V = 0 \quad \text{or} \quad K = 0 = V \quad \text{for all} \quad t \geqslant 0, \tag{24}$$

provided V is positive definite.
Equation (24) readily yield the trivial solution

$$u_i(x, t) = \theta(x, t) = 0, \quad \text{for all} \quad x \in B, t \geqslant 0.$$

This completes the proof.

It may be mentioned here that Green and Lindsay [9] and Green [10] proved the above theorem by imposing restrictions on the individual material constants α, h and k_{ij}. The positive definiteness condition imposed by us on V takes care of all these restrictions and is in conformity with the nature of V in the coupled theory [13].

In the particular case when $\alpha = h = b_i = 0$, our energy equation (20) and the uniqueness theorem reduce to those obtained by Ionescu-Cazimir [12] in the context of the coupled theory.

5. Variational principle

We obtain below a generalized Hamilton's principle in the form of the following

THEOREM

If t_1 and t_2 are two arbitrary instants of time, then the field equations (8) and (9) form a set of necessary and sufficient conditions for the variational equation

$$\begin{aligned} &\delta \int_{t_1}^{t_2} \mathrm{d}t \left[\int_m \{K - \psi + \theta(\partial\psi/\partial\theta) + F_i u_i + (\alpha/\theta_0) r\theta\} \,\mathrm{d}m \right. \\ &\left. \quad + \int_{\partial B_2} T_i u_i \,\mathrm{d}s - (\alpha/\theta_0) \int_{\partial B_4} Q\theta \,\mathrm{d}A \right] = 0 \end{aligned} \tag{25}$$

to hold for arbitrary variations δu_i and $\delta\theta$ in u_i and θ respectively, which in addition to being compatible with the kinematic constraints satisfy the conditions

$$\delta u_i(x, t_1) = \delta u_i(x, t_2) = \delta\theta(x, t_1) = \delta\theta(x, t_2) = 0, \quad x \in B \tag{26}$$

$$\delta u_i = 0 \text{ on } \partial B_1, \ \theta = 0 \text{ on } \partial B_3, \text{ for all } t \geqslant 0, \tag{27}$$

the functions F_i, r, $(\partial\psi/\partial\theta)$, T_i, Q and t being kept unchanged.

Proof: From the defn. of K, *viz*., (18), we obtain

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathrm{d}t \int_m K\,\mathrm{d}m &= \int_m [\dot{u}_i \delta u_i |_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{u}_i \delta u_i \mathrm{d}t]\,\mathrm{d}m \\ &= -\int_{t_1}^{t_2} \mathrm{d}t \int_B \rho_0 \ddot{u}_i \delta u_i\,\mathrm{d}B,\end{aligned} \tag{28}$$

on using (26).
Also,

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \int_m (F_i u_i + (\alpha/\theta_0) r\theta)\,\mathrm{d}m = \int_{t_1}^{t_2} \mathrm{d}t \int_B \rho_0 (F_i \delta u_i + (\alpha/\theta_0) r\delta\theta)\mathrm{d}B \tag{29}$$

and

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathrm{d}t [\int_{\partial B_2} T_i u_i \,\mathrm{d}A - \alpha \int_{\partial B_4} Q\theta\,\mathrm{d}A] = \int_{t_1}^{t_2} \mathrm{d}t [\int_{\partial B_2} T_i \delta u_i \,\mathrm{d}A \\ - \alpha \int_{\partial B_4} Q\delta\theta\,\mathrm{d}A]\end{aligned} \tag{30}$$

Further,

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathrm{d}t \int_m (\psi - [\partial\psi/\partial\theta]\theta)\,\mathrm{d}m = \int_{t_1}^{t_2} \mathrm{d}t \int_B \rho_0 ([\partial\psi/\partial e_{ij}]\delta e_{ij} + [\partial\psi/\partial\dot{\theta}]\delta\dot{\theta} \\ + [\partial\psi/\partial\theta_{,i}]\delta\theta_{,i})\,\mathrm{d}B\end{aligned}$$

Substituting for ψ from (15) in the rightside of the above equation and simplifying the resulting expression with (1), (2), (4), (26), (27), the boundary conditions (11), (12) and the divergence theorem, we obtain

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathrm{d}t \int_m (\psi - [\partial\psi/\partial\theta]\theta)\mathrm{d}m = \int_{t_1}^{t_2} \mathrm{d}t \,[\int_{\partial B_2} T_i \delta u_i \mathrm{d}A - \int_B \{k_{ijrs} u_{r,sj} \\ + a_{ij}(\theta + \alpha\dot{\theta})_{,j}\}\delta u_i \mathrm{d}B \\ + \alpha \int_B (\mathrm{d}\dot{\theta} + h\ddot{\theta} - 2b_i \dot{\theta}_{,i} - a_{ij}\dot{u}_{i,j} \\ - k_{ij}\theta_{,ij})\delta\theta\,\mathrm{d}B \\ - (\alpha/\theta_0) \int_{\partial B_4} Q\delta\theta\mathrm{d}A]\end{aligned} \tag{31}$$

Equations (28)–(31) yield

$$\begin{aligned}&\delta \int_{t_1}^{t_2} \mathrm{d}t [\int_m \{K - \psi + \theta(\partial\psi/\partial\theta) + F_i u_i + (\alpha/\theta_0) r\theta\}\,\mathrm{d}m \\ &\quad + \int_{\partial B_2} T_i u_i \mathrm{d}A - (\alpha/\theta_0)\int_{\partial B_4} Q\theta\mathrm{d}A] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \int_B [\{k_{ijrs} u_{r,sj} + a_{ij}(\theta + \alpha\dot{\theta})_{,j} + \rho_0(F_i - \ddot{u}_i)\}\delta u_i \\ &\quad + \alpha\{(\rho_0/\theta_0) r + k_{ij}\theta_{,ij} - \mathrm{d}\dot{\theta} - h\ddot{\theta} + 2b_i \dot{\theta}_{,i} + a_{ij}\dot{u}_{i,j}\}\delta\theta]\,\mathrm{d}B\end{aligned}$$

From this equation, it is obvious that the variational equation (25) holds if and only if (8) and (9) hold. This completes the proof.

It is worth noting that while the generalized Hamilton's principle in coupled thermoelasticity involves *two* variational equations [14], the principle obtained above involves just *one* variational equation (*viz*., equation (25)).

6. Reciprocal theorem

We now consider two initial, mixed boundary-value problems associated with the two systems

$$\{(F_i^{(\lambda)}, r^{(\lambda)});\ (T_i^{(\lambda)}, U_i^{(\lambda)}, \Theta^{(\lambda)}, Q^{(\lambda)});\ (u_i^{(\lambda)}, \theta^{(\lambda)})\},\ \lambda = 1, 2$$

and suppose that $t_{ij}^{(\lambda)}$ and $q_i^{(\lambda)}$ are the corresponding stresses and heat flux respectively. For λ, $\mu = 1, 2$, $\lambda \neq \mu$, if

$$\begin{aligned} L_{\lambda\mu} = {} & \int_B [\rho_0 (F_i^{(\lambda)} * \hat{u}_i^{(\mu)}) - (\rho_0/\theta_0)\{r^{(\lambda)} * \theta^{(\mu)} + \alpha(r^{(\lambda)} * \hat{\theta}^{(\mu)})\} \\ & - 2b_i\{\theta_{,i}^{(\lambda)} * \hat{\theta}^{(\mu)} + \alpha(\theta_{,i}^{(\lambda)} * \hat{\hat{\theta}}^{(\mu)})\}]\,\mathrm{d}B + \int_{\partial B_1} (T_i^{(\lambda)} * \hat{u}_i^{(\mu)})\,\mathrm{d}A \\ & - \int_{\partial B_2} U_i^{(\lambda)} * \hat{t}_{ij}^{(\mu)} n_j\,\mathrm{d}A - (1/\theta_0)\int_{\partial B_3} \{\Theta^{(\lambda)} * q_i^{(\mu)} + \alpha(\Theta^{(\lambda)} * \hat{q}_i^{(\mu)})\} n_i\,\mathrm{d}A \\ & + \int_{\partial B_4} \{Q^{(\lambda)} * \theta^{(\mu)} + \alpha(Q^{(\lambda)} * \hat{\theta}^{(\mu)})\}\,\mathrm{d}A, \end{aligned} \tag{32}$$

where

$$\begin{aligned} f * g &= \int_0^t f(x, t-\tau) g(x,\tau)\,\mathrm{d}\tau \\ f * \hat{g} &= \int_0^t f(x, t-\tau)(\partial g/\partial \tau)(x,\tau)\,\mathrm{d}\tau \\ f * \hat{\hat{g}} &= \int_0^t f(x, t-\tau)(\partial^2 g/\partial \tau^2)(x,\tau)\,\mathrm{d}\tau, \end{aligned} \tag{33}$$

we prove the *reciprocal theorem* that

$$L_{12} = L_{21} \tag{34}$$

Proof: By hypothesis, the functions associated with the two problems considered are governed by (4) and (7)–(9) and the boundary conditions (11) and (12). Taking the Laplace transform with respect to t of these equations under the initial conditions (10), we obtain the following equations for the transformed functions

$$\text{In } B\text{: } \bar{q}_i^{(\lambda)} = -\theta_0[pb_i\bar{\theta}^{(\lambda)} + k_{ij}\bar{\theta}_{,j}^{(\lambda)}] \tag{35}$$

$$\bar{t}_{ij}^{(\lambda)} = k_{ijrs}\bar{u}_{r,s}^{(\lambda)} + a_{ij}(1+\alpha p)\bar{\theta}^{(\lambda)} \tag{36}$$

$$\begin{aligned} & (\rho_0/\theta_0)\bar{r}^{(\lambda)} + k_{ij}\bar{\theta}_{,ij}^{(\lambda)} - \mathrm{d}p\bar{\theta}^{(\lambda)} - hp^2\bar{\theta}^{(\lambda)} \\ & + a_{ij}p\bar{u}_{i,j}^{(\lambda)} + 2b_ip\bar{\theta}_{,i}^{(\lambda)} = 0 \end{aligned} \tag{37}$$

$$k_{ijrs}\bar{u}_{r,sj}^{(\lambda)} + a_{ij}(1+\alpha p)\bar{\theta}_{,j}^{(\lambda)} + \rho_0\bar{F}_i^{(\lambda)} = \rho_0 p^2 \bar{u}_i^{(\lambda)} \tag{38}$$

and

$$\left.\begin{aligned} &\text{on } \partial B_1: \bar{t}_{ij}^{(\lambda)} n_j = \bar{T}_i^{(\lambda)} \\ &\text{on } \partial B_2: \bar{u}_i^{(\lambda)} = \bar{U}_i^{(\lambda)} \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} &\text{on } \partial B_3: \bar{\theta}^{(\lambda)} = \bar{\Theta}^{(\lambda)} \\ &\text{on } \partial B_4: \bar{q}_i^{(\lambda)} n_i = \bar{Q}^{(\lambda)} \end{aligned}\right\} \tag{40}$$

In these equations $\bar{f} = \bar{f}(x, p)$ denotes the Laplace transform of $f = f(x, t)$.

From equations (36) and (38) we obtain

$$\begin{aligned} & \int_B [\rho_0(\bar{F}_i^{(1)}\bar{u}_i^{(2)} - \bar{F}_i^{(2)}\bar{u}_i^{(1)}) - (1+\alpha p)a_{ij}(\bar{\theta}^{(1)}\bar{u}_{i,j}^{(2)} - \bar{\theta}^{(2)}\bar{u}_{i,j}^{(1)})]\,\mathrm{d}B \\ & \quad = \int_B (\bar{t}_{ji}^{(2)}\bar{u}_i^{(1)} - \bar{t}_{ji}^{(1)}\bar{u}_i^{(2)})_{,j}\,\mathrm{d}B \end{aligned} \tag{41}$$

From (35) and (37) we obtain

$$\int_B [p\{a_{ij}(\bar{u}^{(1)}_{i,j}\bar{\theta}^{(2)} - \bar{u}^{(2)}_{i,j}\bar{\theta}^{(1)}) + 2b_i(\bar{\theta}^{(1)}_{,i}\bar{\theta}^{(2)} - \bar{\theta}^{(2)}_{,i}\bar{\theta}^{(1)})\} + (\rho_0/\theta_0)(\bar{r}^{(1)}\bar{\theta}^{(2)} - \bar{r}^{(2)}\bar{\theta}^{(1)})]\mathrm{d}B = (1/\theta_0)\int_B (\bar{q}^{(1)}_i\bar{\theta}^{(2)} - \bar{q}^{(2)}_i\bar{\theta}^{(1)})_{,i}\mathrm{d}B \quad (42)$$

Converting the volume integrals in the right sides of (41) and (42) into surface integrals using the divergence theorem, applying the boundary conditions (39) and (40) and eliminating the common terms, we obtain the following Laplace transform form of the reciprocal theorem:

$$\begin{aligned}&\int_B [\rho_0 p(\bar{F}^{(1)}_i\bar{u}^{(2)}_i - \bar{F}^{(2)}_i\bar{u}^{(1)}_i) - (\rho_0/\theta_0)(1+\alpha p)(\bar{r}^{(1)}\bar{\theta}^{(2)} - \bar{r}^{(2)}\bar{\theta}^{(1)})\\ &- 2b_i p(1+\alpha p)(\bar{\theta}^{(1)}_{,i}\bar{\theta}^{(2)} - \bar{\theta}^{(2)}_{,i}\bar{\theta}^{(1)})]\mathrm{d}B + p\int_{\partial B_1}(\bar{T}^{(1)}_i\bar{u}^{(2)}_i - \bar{T}^{(2)}_i\bar{u}^{(1)}_i)\mathrm{d}A\\ &+ p\int_{\partial B_2}(\bar{t}^{(1)}_{ij}\bar{U}^{(2)}_i - \bar{t}^{(2)}_{ij}\bar{U}^{(1)}_i)n_j\mathrm{d}A + ([1+\alpha p]/\theta_0)\\ &\times \int_{\partial B_3}(\bar{q}^{(1)}_i\bar{\Theta}^{(2)} - \bar{q}^{(2)}_i\bar{\Theta}^{(1)})n_i\mathrm{d}A\\ &+ (1+\alpha p)\int_{\partial B_4}(\bar{Q}^{(1)}\bar{\theta}^{(2)} - \bar{Q}^{(2)}\bar{\theta}^{(1)})\mathrm{d}A = 0 \end{aligned} \quad (43)$$

Inverting this equation with the aid of the convolution theorem for Laplace transforms, we obtain the desired result (34). This completes the proof.

It is interesting to note that the extra constant h, the presence of which is responsible for the inclusion of the temperature-acceleration term in the heat equation (8), does not influence the reciprocal theorem. If the material has a centre of symmetry at every point, but otherwise is anisotropic, we have $b_i = 0$, and our theorem then reduces to that obtained in [8]. If we set $\alpha = b_i = 0$ in (32), our reciprocal theorem (34) reduces to that in the classical coupled thermoelasticity theory. Ionescu-Cazimir [11] obtained a reciprocal theorem in coupled thermoelasticity for prescribed temperature on the entire boundary; obviously, his theorem corresponds to the case $\alpha = b_i = 0$ and $\partial B_3 = \partial B$ of our theorem (34).

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