Proc. Indian Acad. Sci. (Math. Sci.), Vol. 92, No. 1, September 1983, pp. 61–65.

An approximate solvability scheme for a class of nonlinear equations

MOHAN JOSHI

Mathematics Group, Birla Institute of Technology and Science, Pilani 333031, India

MS received 5 January 1982; revised 6 June 1983

Abstract. An approximate solvability scheme for equations of the type $u + K_u(u) = w$, in a closed convex subset A of a Hilbert space X is given. Here, for each $u \in A$, $K_u: X \to X$ is a bounded linear operator.

Keywords. Monotone operator; Leray-Schauder principle; Chandrasekhar's H-equation.

1. Introduction

The existence of solutions of equations of the type

$$u + K_u(u) = w, \tag{1}$$

where for each $u \in A$, $K_u: X \to X$ is a bounded linear operator was earlier discussed [2]. A is a closed convex subset of a Hilbert space X. Approximate solvability of such type of equations has not yet been dealt with. Our main purpose is to give an approximation scheme for (1), which is significant from the computational point of view. As an application of our abstract result, we discuss approximate solvability of nonlinear integral equations of the form

$$u(s) + u(s)\int_0^1 K(s,t)u(t)\mathrm{d}t = w(s). \tag{2}$$

These equations include Chandrasekhar's H-equation

$$1 + u(s)\int_0^1 \frac{s}{s+t}\Psi(t)u(t)\mathrm{d}t = u(s). \tag{3}$$

Equation (3) plays an important role in the theory of radiative transfer in semi-infinite atmosphere [1], and hence the importance of our result needs no explanation.

2. Main result

Throughout this paper '$\to$' denotes strong and '$\rightharpoonup$' denotes weak convergence in a Hilbert space X. $B[O, r]$ denotes the closed ball of radius 'r' in X. For $u, w \in X$, (w, u) refers to inner product between u and w in X.

DEFINITION 2.1.

An operator $T: X \to X$ is said to be monotone if

$$(Tu - Tv, u - v) \geqslant 0 \text{ for all } u, v \text{ in } X.$$

Let $X_1 \subset X_2 \subset \ldots$ be a sequence of finite dimensional subspaces of a Hilbert space X and $P_n: X \to X_n$ a corresponding sequence of linear projections.

DEFINITION 2.2.

A tuple $\{\{X_n\}, \{P_n\}\}$ is called an approximation scheme in the space X if P_n is continuous for every n and for each $x \in X$, $P_n x \to x$ as $n \to \infty$.

In this section we shall give a constructive result concerning the existence of a solution of

$$u + K_u(u) = w \tag{4}$$

as a strong limit of solutions $u_n \in X_n$ of the 'approximate' equation

$$u_n + P_n K_{u_n}(u_n) = w_n. \tag{5}$$

DEFINITION 2.3.

Equation (4) is said to be strongly (weakly) solvable if (5) has a solution $u_n \in X_n$ and there exists a subsequence of $\{u_n\}$ which converges to u strongly (weakly) and u is a solution of (4).

Here, for $u \in X$, K_u is a bounded linear operator on X. In the following K^* denotes the conjugate of the bounded linear operator K.

THEOREM 2.1.

Let X be a Hilbert space with an approximation scheme $\{\{X_n\}, \{P_n\}\}$ and A a closed convex subset of X. Assume that for each $u \in A$, K_u is a bounded linear monotone operator satisfying the following condition:
(a) $u_k \rightharpoonup u$ in A implies that $K^*_{u_k}(v) \to K^*_u(v)$ for all $v \in X$. Then (4) is approximately strongly solvable.

Proof: We first claim that $K_u : X \to X$ is jointly weakly continuous. That is, $u_k \rightharpoonup u$ in A and $v_k \rightharpoonup v$ in X implies that $K_{u_k}(v_k) \rightharpoonup K_u(v)$. Consider $(K_{u_k}(v_k) - K_u(v), x)$, $x \in X$.

$$\begin{aligned}(K_{u_k}(v_k) - K_u(v), x) &= (K_{u_k}(v_k) - K_u(v_k), x) + (K_u(v_k) - K_u(v), x)\\ &= (v_k, K^*_{u_k}(x) - K^*_u(x)) + (K_u(v_k) - K_u(v), x).\end{aligned}$$

As $k \to \infty$ the first term in the right side of the above inequality tends to zero in view of assumption (a) and the second term tends to zero as K_u is a continuous linear operator and hence also weakly continuous. This proves our claim.

We now consider the approximate equation

$$u_n + P_n K_{u_n}(u_n) = w_n.$$

Let T be the operator on X_n defined by

$$Tu = u + P_n K_u(u) - w_n.$$

Define a closed and bounded set C as

$$C = \{u \in A \cap X_n : (w, u) \leqslant (u, u) \leqslant (w, w)\}.$$

Then T is a continuous operator on X_n such that

$$(Tu, u) = (u, u) + (P_n K_u(u), u) - (w_n, u) \geqslant (u, u) - (w, u) \geqslant 0, \text{ for } u \in C.$$

Hence it follows by Leray-Schauder principle that $Tu = 0$ has a solution $u_n \in C$. That is

$$u_n + P_n K_{u_n}(u_n) = w_n.$$

$\{u_n\}$ is a bounded sequence in a Hilbert space and hence there exists a subsequence of it, which we again denote by $\{u_n\}$, such that $u_n \rightharpoonup u$. We claim that u is a solution of (4). As $u_n \rightharpoonup u$ and $w_n \to w$ it sufficies to show that $P_n K_{u_n}(u_n) \rightharpoonup K_u(u)$ as $n \to \infty$.

Consider

$$
\begin{aligned}
&(P_n K_{u_n}(u_n) - K_u(u), x),\ x \in X.\ (P_n K_{u_n}(u_n) - K_u(u), x)\\
&= (P_n K_{u_n}(u_n) - P_n K_u(u), x) + (P_n K_u(u) - K_u(u), x)\\
&= (K_{u_n}(u_n) - K_u(u), x) + (K_{u_n}(u_n) - K_u(u), P_n x - x) + (P_n K_u(u)\\
&\quad - K_u(u), x)\\
&\leqslant (K_{u_n}(u_n) - K_u(u), x) + \|K_{u_n}(u_n) - K_u(u)\|\ \|P_n x - x\| + \|P_n K_u(u)\\
&\quad - K_u(u)\|\ \|x\|
\end{aligned}
$$

$\to 0$ as $n \to \infty$, since $K_{u_n}(u_n) \rightharpoonup K_u(u)$ and $P_n x \to x$. This proves that (4) is approximately weakly solvable. We now show that $\{u_n\}$ actually converges strongly to u.

Consider

$$
\begin{aligned}
\|u_n - u\|^2 &= (u_n - u, u_n - u)\\
&= (w_n - w, u_n - u) + (K_u(u) - P_n K_{u_n}(u_n), u_n - u)\\
&= (w_n - w, u_n - u) + (P_n K_{u_n}(u) - P_n K_{u_n}(u_n), u_n - u)\\
&\quad - (P_n K_{u_n}(u) - P_n K_u(u), u_n - u) - (P_n K_u(u) - K_u(u), u_n - u)\\
&= (w_n - w, u_n - u) - (K_{u_n}(u_n) - K_{u_n}(u), u_n - u)\\
&\quad - (K_{u_n}(u_n) - K_{u_n}(u), u - P_n(u)) - (K_{u_n}(u) - K_u(u), u_n - P_n u)\\
&\quad - (P_n K_u(u) - K_u(u), u_n - u)\\
&\leqslant (P_n w - w, u_n - u) - (K_{u_n}(u_n) - K_{u_n}(u), u - P_n u)\\
&\quad - (K_{u_n}(u) - K_u(u), u_n - P_n u) - (P_n K_u(u) - K_u(u), u_n - u)
\end{aligned}
$$

by monotonicity of K_{u_n}.

The first, second and fourth term tend to 0 in view of $P_n x \to x$ for every $x \in X$ and the third term tend to zero in view of (a) and uniform boundedness principle. This proves our theorem.

3. Application

As an application of our main theorem we obtain an approximate solvability result for nonlinear integral equations of the type

$$u(s) + u(s) \int_0^1 K(s, t) u(t)\, dt = w(s) \tag{6}$$

in the space $L_2[0, 1]$.

We assume that $K(s, t)$ is a Hilbert-Schmidt kernel. So eigenfunctions of K form a complete orthonormal set in $L_2[0, 1]$. Let $e_1, e_2, \ldots$ be eigen functions of K with eigenvalues $\lambda_1, \lambda_2, \ldots$. Define a sequence X_n of finite dimensional subspaces of

$X = L_2[0,1]$ and linear projections $P_n: X \to X_n$ as follows:

$$X_n = [e_1, e_2, \ldots, e_n],\ P_n u = \sum_{k=1}^{n} \alpha_k e_k \text{ where } u = \sum_{k=1}^{\infty} \alpha_k e_k.$$

Then the approximate equation in the finite dimensional space X_n is given by

$$u(s) + P_n[u(s) \textstyle\int_0^1 K(s,t)u(t)\mathrm{d}t] = P_n w \tag{7}$$

where $u = \Sigma_{k=1}^{\infty} \alpha_k e_k$.

Taking inner product with e_n we get $(u, e_n) + (P_n[u(s) \int_0^1 K(s,t)\, u(t)\mathrm{d}t], e_n) = (P_n w, e_n)$, which gives

$$\alpha_n + \left(\left(\sum_{k=1}^{n} \alpha_k e_k(s)\right)\left(\sum_{j=1}^{n} \lambda_j \alpha_j e_j(s)\right), e_n\right) = \beta_n, \tag{8}$$

where $w = \Sigma_{k=1}^{\infty} \beta_k e_k$. Writing r_{jkn} for $\int_0^1 e_k(s) e_j(s) e_n(s)\mathrm{d}s$ in (8), we get an equivalent system of nonlinear equations

$$\alpha_n + \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_j \alpha_j \alpha_k r_{jkn} = \beta_n, \quad n \geqslant 1. \tag{9}$$

Thus solvability of the approximate equation (7) is equivalent to the solvability of (9). One can now use the known techniques to solve the nonlinear system given by (9). For various reasons we skip the details regarding the computational aspect of (9). We have the following theorem giving the approximate solvability of (6).

THEOREM 3.1

Suppose that

(a) $K(s,t) \geqslant 0$ *a.e.* on $[0,1] \times [0,1]$,

(b) *ess* sup $\int_0^1 K^2(s,t)\mathrm{d}t < \infty$,

(c) $w(s) \geqslant 0$ *a.e.* on $[0,1]$.

Then (6) is approximately strongly solvable in $L_2[0,1]$.

Proof: Let $A = \{u \in L_2 : u(s) \geqslant 0 \text{ a.e. on } [0,1]\}$ and let

$$K_u(v)(s) = v(s) \int_0^1 K(s,t)u(t)\mathrm{d}t.$$

Then for each $u \in A$, K_u is a bounded linear operator on L_2. Also, we have $(K_u(v), v) = \int_0^1 (\int_0^1 K(s,t)u(t)\mathrm{d}t) v^2(t)\mathrm{d}t$

$$\geqslant 0 \quad \text{for all} \quad v \in L_2.$$

That is $\{K_u\}$ is monotone for each $u \in A$. We now verify that the hypothesis (a) of Theorem 2.1 is satisfied.

$$\begin{aligned} K_u^*(v) &= v(s) \int_0^1 K(s,t)u(t)\mathrm{d}t \\ &= \int_0^1 K_v(s,t)u(t)\mathrm{d}t, \end{aligned}$$

where the new kernel $K_v(s, t) = v(s)K(s, t)$.

In view of assumption (b) of Theorem 3.1, it follows that $K_v(s, t)$ is a Hilbert-Schmidt kernel and hence the integral operator generated by it is completely continuous. That is $u_n \rightharpoonup u$ implies that

$$\int_0^1 K_v(s, t)u_n(t)\mathrm{d}t \to \int_0^1 K_v(s, t)u(t)\mathrm{d}t,$$

which in turn implies that $K_{u_n}^*(v) \to K_u^*(v)$. Thus the family $\{K_u\}$, $u \in A$ of linear operators on X satisfies all conditions of Theorem 2.1 and hence (6) is approximately solvable.

As a corollary of this theorem we obtain an approximiate solvability result for Chandrasekhar's H-equation

$$1 + u(s) \int_0^1 \frac{s}{s+t} \Psi(t)u(t)\mathrm{d}t = u(s). \tag{10}$$

Here, the known function $\Psi(t)$ is assumed to be non-negative, bounded and measurable. Since equation (10) is not given in the standard form we first state a lemma which is useful in this direction and for the proof refer [1].

LEMMA.

Suppose that $\int_0^1 \Psi(t)\mathrm{d}t \leqslant 1/2$ and that $u \in L_2$ is a positive solution of the equation

$$u(s)\left\{1 - 2\int_0^1 \Psi(t)\mathrm{d}t\right\}^{1/2} + u(s)\int_0^1 \frac{t}{s+t}\Psi(t)u(t)\mathrm{d}t = 1. \tag{11}$$

Then $\int_0^1 \Psi(s)u(s)\mathrm{d}s = 1 - (1 - 2\int_0^1 \Psi(s)\mathrm{d}s)^{1/2}$ and u is also a solution of (10).

Equation (11) is now in the standard form (6) with

$$K(s, t) = \left[\frac{t}{c(s+t)}\right]\Psi(t) \text{ and } w(s) = \frac{1}{c}. \tag{12}$$

Here $c = [1/(1 - 2\int_0^1 \Psi(t)\mathrm{d}t)^{1/2}]$ (without loss in generality we can assume that $\int_0^1 \Psi(t)\mathrm{d}t < 1/2$, one can tackle the case $\int_0^1 \Psi(t)\mathrm{d}t = 1/2$ as a limiting case of strict inequality). Thus K and w given by (12) satisfy all the requirements of Theorem 3.1 and hence we get the following solvability result for Chandrasekhar's equation (10).

THEOREM 3.2.

Let $\int_0^1 \Psi(t)\mathrm{d}t \leqslant 1/2$, then Chandrasekhar's H-equation is approximately strongly solvable.

References

[1] Chandrasekhar S 1960 *Radiative transfer* (Dover: New York)

[2] Joshi M C and Srikanth P N 1978 *Proc. Indian Acad. Sci.* (*Math. Sci.*) **A87** 169