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# Formula for primes, twinprimes, number of primes and number of twinprimes

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**Abstract.** Formulae for computing the $n$th prime, twinprime, the number of primes smaller than a given integer, and the number of twinprimes smaller than a given integer are presented. Proofs for the development are also furnished.

**Keywords.** Primes; twinprimes

## 1. Introduction

Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots, p_n$ be the sequence of the first $n$ primes, and,

$$Q = \prod_{i=1}^{n} p_i \tag{1}$$

This paper presents formula to compute the $(n+1)$th prime which is $p_{n+1}$ and other allied results.

Let

$$Q_i = Q/p_i, \quad \text{for} \quad i = 1, 2, 3, \ldots, n \tag{2}$$

$1 \leqslant a_i \leqslant (p_i - 1)$

$$K = n - \sum_{1}^{n} (1/p_i) \tag{3}$$

$0 \leqslant b \leqslant [K]$, where $[K]$ denotes the integral part of $K$ and

$$J = \sum_{1}^{n} a_i Q_i - bQ \tag{4}$$

The following four theorems are first proved.

THEOREM 1.

For $n$ greater than 1,

$$\sum_{a_1=1}^{(p_1-1)} \cdots \sum_{a_n=1}^{(p_n-1)} \sum_{0 \leqslant b \leqslant [K]} X^J = X + (1/X) + f(X) + f(1/X) \tag{5}$$

where $f(X) = \sum a_m X^m$

$$1 < m \leqslant [K] Q \prod_{i=1}^{n} (1 - (1/p_i)),\ (m, Q) = 1, \text{ and}$$

$a_m = 1$ for all $m < Q$ and $(m, Q) = 1$ and $a_m = 0$ for some $m \geqslant Q$.

THEOREM 2.

For $n$ greater than 1,

$$\sum_{a_1=1}^{(p_1-1)} \cdots \sum_{a_n=1}^{(p_n-1)} \sum_{0 \leqslant b \leqslant [K]} X^{J^2} = 2X + 2g(X) \tag{6}$$

where

$$g(X) = \sum a_m X^{m^2}$$

$$1 < m \leqslant [K]Q \prod_{i=1}^{n} (1-(1/p_i)),\ (m, Q) = 1$$

and $a_m$ as defined in Theorem 1.

THEOREM 3.

For $n$ greater than 1,

Let $(a'_1, a'_2, a'_3, \ldots, a'_n)$ be the unique solution of $-2 \equiv \Sigma_1^n a_i Q_i \bmod Q$, with $0 \leqslant a_i \leqslant (p_i - 1)$, $(1, 2, 3, \ldots, n)$.

Then,

$$\sum_{a'_1 \neq a_1 = 1}^{(p_1-1)} \cdots \sum_{a'_n \neq a_n = 1}^{(p_n-1)} \sum_{0 \leqslant b \leqslant [K]} X^{J} = \sum_{(m(m+2), Q) = 1} X^{m} \tag{7}$$

where $m$ on the right side performs summation over the relevant range.

THEOREM 4.

For $n$ greater than 1,

$$\sum_{a_1} \cdots \sum_{a_n} \sum_{b} X^{J^2} = \sum_{(m(m+2), Q) = 1} X^{m^2} \tag{8}$$

where $m$ performs summation over the same set of integers as in Theorem 3.

*Remark* 1. Note that $a'_1 = 0$ and that $1 \leqslant a'_1 \leqslant (p_i - 1)$, $(i = 1, 2, 3, \ldots, n)$.

*Remark* 2. In all the four theorems, the $m$'s which satisfy

$$(p_n + 1) < m < (p_{n+1}^2 - 1) \tag{9}$$

are precisely all the primes in this interval.

## 2. Proofs

The proofs of the theorems 1 and 2 follow from the remarks given below.

First, given any integer $c$ there is a unique solution of

$$\sum_{i=1}^{n} a_i Q_i \equiv c \bmod Q \tag{10}$$

subject to $0 \leqslant a_i \leqslant (p_i - 1)$, $(i = 1, 2, 3, \ldots, n)$. The proofs of the theorems 3 and 4 follow from the remarks given below.

Subject to $1 \leqslant a_i \leqslant (p_i - 1)$ for all $i$, we have secured $(m, Q) = 1$.

If in addition, the condition that $((m+2), Q) = 1$ is to be satisfied, we should have

$$\left(\left(\sum_1^n a_i Q_i - \sum_1^n a'_1 Q_i\right), Q\right) = 1. \tag{11}$$

That is to say $a_i \neq a'_i$ for each $i$. By a similar method, it is possible to ensure

$$\{[m(m+2)(m+6)], Q\} = 1 \tag{12}$$

and so on.

*Remark* 3. In Theorem 2, substituting $(-J^2)$ for $(J^2)$, we get

$$\sum_{a_1=1}^{(p_1-1)} \cdots \sum_{b=1}^{(p_n-1)} \sum_{b=1}^{n} X^{-J^2} = 2(X^{-1} + X^{-p_{n+1}^2} + \ldots) \tag{13}$$

where $X$ is any given positive number.

Therefore,

$$\tfrac{1}{2}(\text{LHS}) - X^{-1} = M = (X^{-p_{n+1}^2} + X^{-p_{n+2}^2} + \ldots .) \tag{14}$$

where LHS stands for the expression on the left side of the equation.

Multiplying both sides of the above equation by $X^{p_{n+1}^2}$, we get

$$M.X^{p_{n+1}^2} = 1 + X^{-p_{n+2}^2 + p_{n+1}^2} + \ldots = 1 + R \text{ (say)}.$$

For values of $X \geqslant 2$, it is clear that $R < 1$. Therefore,

$$1 < M.X^{p_{n+1}^2} < 2 \tag{15}$$

Taking logarithms of both sides of (15), to the base $X$, we get

$$0 < p_{n+1}^2 + \log_X M < 1$$

$$p_{n+1}^2 > -\log_X M > p_{n+1}^2 - 1$$

That is to say, the integral part of $(-\log_X M)$ is equal to $(p_{n+1}^2 - 1)$, and,

$$p_{n+1}^2 = 1 + \left[-\log_X\left(1/2 \sum_{a_1=1}^{(p_1-1)} \cdots \sum_{a_n=1}^{(p_n-1)} \sum_{b=1}^{n} X^{-(\Sigma_1^n a_i Q_i - bQ)^2} - (1/X)\right)\right]. \tag{16}$$

Equation (16) is thus the formula for the $(n+1)$th prime.

$P$ is the first twinprime in $(p_n, p_n^2)$ if $P$ differs from zero by the equation:

$$P^2 = Y^2\left(1 - \left[\sin^2\left(\frac{\pi}{2} - [2^{-(Y^2 - p_n^4)}]\right)\right]\right) \tag{17}$$

where $Y$ is given by the below expression:

$$Y^2 = 1 + \left[-\log_X\left(1/2 \sum_{a'_1 \neq a_1 = 1}^{(p_1-1)} \cdots \sum_{a'_n \neq a_n = 1}^{(p_n-1)} \sum_{b=1}^{n} X^{-(\Sigma_1^n a_i Q_i - bQ)^2} - (1/X)\right)\right]. \tag{18}$$

It could be noted that when $Y < p_n^2$

$$P = Y \tag{19}$$

and when $Y > p_n^2$

$$P = 0.$$

Therefore, the above expression gives only twinprimes.

*Remark* 5. For any $x$ in $(p_n, p_n^2)$, the number of primes not more than $x$ is given by

$$\Pi(x) = \frac{1}{4\pi}\int_0^{2\pi}\prod_{i=1}^{n}\frac{\sin((Q-Q_i)/2)\theta).\sin((n-1)Q\theta/2).\sin([x]\theta/2)\cos(([x]+1)\theta/2)}{\sin(Q_i\theta/2).\sin(Q\theta/2)\sin(\theta/2)} \times \mathrm{d}\theta + n. \quad (20)$$

Noting that the angles in each set by the summation are in arithmetic progression, the above equation can be proved by summing over $a_n, a_{n-1}, a_{n-2}, \ldots, a_1$, from,

$$\Pi(x) - \Pi(p_n) = \frac{1}{4\pi}\int_0^{2\pi}\sum_{a_1=1}^{(p_1-1)}\cdots \sum_{a_n=1}^{(p_n-1)}\sum_{b=1}^{(n-1)}\sum_{t=1}^{[x]}\cos(\Sigma_1^n a_i Q_i - bQ)\theta\cos(t\theta)\,\mathrm{d}\theta. \quad (21)$$

*Remark* 6. If $\Pi_2(x)$ represents the number of twinprimes not exceeding $x$; $p_n < x \leqslant p_n^2$, then,

$$\Pi_2(x) - \Pi_2(p_n) = \frac{1}{4\pi}\int_0^{2\pi}\sum_{a_1\neq a'_1}\cdots \sum_{a_n\neq a'_n}\sum_{b=1}^{(n-1)}\sum_{t=1}^{[x]}\cos(\Sigma_1^n a_i Q_i - bQ)\theta.\cos(t\theta)\,\mathrm{d}\theta. \quad (22)$$

Since all the twinprimes up to $p_n$ are known, $\pi_2(p_n)$ is known.

$$\Pi_2(x) = \frac{1}{4\pi}\int_0^{2\pi}\sum_{a_1\neq a'_1}\cdots\sum_{a_n\neq a'_n}\sum_{b=1}^{(n-1)}\sum_{t=1}^{[x]} \cos(\Sigma_1^n a_i Q_i - bQ)\theta\cos(t\theta)\,\mathrm{d}\theta + \Pi_2(p_n). \quad (23)$$

The twinprime conjecture follows immediately if the right side (23) is greater than $\Pi_2(p_n)$

## 3. Detailed proofs for remarks 4 and 5

Detailed proofs for remarks 4 and 5 with numerical examples, application in connection with $p_{n+1}$, $P$, $\Pi(x)$ and $\Pi_2(x)$ will be furnished in a separate paper.

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