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# On diffraction of SH-waves by cuts in nonhomogeneous solids

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**Abstract.** The diffraction of SH-waves by an infinite periodic system of cuts in an infinite medium possessing nonhomogeneity has been studied. Assuming that shear modulus and density vary, the problem of diffraction of SH-waves by the periodic system of cuts depends on the solution of dual series equations which ultimately reduces to the solution of an infinite system of algebraic equations.

**Keywords.** Diffraction of elastic waves; cuts and cracks; stress intensity factor; SH-waves.

## 1. Introduction

In most of the problems of diffraction of elastic waves by cuts or cracks the solution has been sought for low or high frequency approximations. Sommerfield [8], Sih and Loeber [5,6], Mal [2,3], Thau and Lu [10] and many others investigated such problems. Parton and Morozov [4] presented the problem of diffraction of shear waves by a periodic system of cracks lying in an infinite homogeneous elastic medium.

Studies on the diffraction by cuts and cracks in nonhomogeneous elastic medium have been scanty. Singh and Dhaliwal [7] considered a static problem of crack in a nonhomogeneous medium in which shear modulus decays exponentially on both sides of the crack. To understand the effect of nonhomogeneity on diffraction of SH-type wave by a periodic system of cuts in an infinite medium we consider the simple case of exponentially varying shear modulus and density on the two sides of the cuts such that their ratio always remains the same. The resulting diffraction problem has been solved and the stress intensity factor (SIF) has been calculated. The effect of nonhomogeneity on SIF is more prominent for smaller wave numbers than for larger ones, as shown by the numerical results.

## 2. Formulation of the problem

There is an infinite row of collinear cracks of equal length which are equally spaced on a line taken as the $x_1$-axis. The length of the crack is $2l$ and the distance between the centres of the crack is $2L$.

We suppose that displacement $(0, 0, W_1)$ is produced by SH-waves coming from infinity (*i.e.* positive $x_2$-side) given by

$$W_1 = w^{(1)}(x_2)\exp(-i\Omega t) \tag{1}$$

where $\Omega$ in the time dependent factor in (1) represents the circular frequency.

Equation of motion in the absence of body forces is

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{2}$$

where $\sigma_{ij}$ is the stress tensor $(i,j = 1,2,3)$; $u_i$ are the components of the displacement vector $(i,j = 1,2,3)$; $\rho$ is the density of the medium; comma denotes covariant differentiation and dot denotes time derivatives.

The non-zero components of stresses are clearly

$$\sigma_{23} = \sigma_{32} = \mu \frac{\partial w^{(1)}}{\partial x_2} \exp(-i\Omega t). \tag{3}$$

We assume the material nonhomogeneity to be of the form

$$\mu = \mu_0 \exp(-\delta x_2); \quad \rho = \rho_0 \exp(-\delta x_2) \tag{4}$$

The only nontrivial equation of the system (2) is

$$\frac{\mathrm{d}^2}{\mathrm{d}x_2^2} w^{(1)} - \delta \frac{\mathrm{d}}{\mathrm{d}x_2} w^{(1)} + \frac{\Omega^2}{c^2} w^{(1)} = 0 \tag{5}$$

where $c^2 = (\mu_0/\rho_0)^{1/2}$.

Hence the shear wave solution of (5) is given by, on replacing $x_1$, $x_2$, $x_3$ by $x, y, z$ respectively,

$$W_1 = w_0 \exp\left[\left(\frac{\delta}{2} - i(\lambda^2 - \delta^2/4)^{1/2}\right) y - i\Omega t\right] \tag{6}$$

where $\lambda^2 = \Omega^2/c^2$

The disturbance field is taken to be of the form $(0, 0, W_2)$ where

$$W_2 = w^{(2)}(x, y) \exp(-i\Omega t) \tag{7}$$

and leads to the equation

$$\frac{\partial^2}{\partial x^2} w^{(2)} + \frac{\partial^2}{\partial y^2} w^{(2)} - \delta \frac{\partial}{\partial y} w^{(2)} = -\lambda^2 w^{(2)}. \tag{8}$$

Defining the nondimensional quantities

$$\xi = x(\pi/L) \text{ and } \eta = y(\pi/L). \tag{9}$$

we assume the solution, periodic in $\xi$, to be of the form

$$w^{(2)} = \cos n\xi \cdot F(\eta). \tag{10}$$

Remembering that $W_2$ should vanish as $\eta \to \pm\infty$ it can be shown easily that considering (10) and (8) we have

$$\begin{aligned} W_2 &= \sum_{n=1}^{\infty} A_n \cos n\xi \exp\left[\left(\frac{\delta L}{2\pi} - (n^2 - \beta^2)^{1/2}\right)\eta - i\Omega t\right], \eta > 0, \\ &= \sum_{n=1}^{\infty} B_n \cos n\xi \exp\left[\left(\frac{\delta L}{2\pi} + (n^2 - \beta^2)^{1/2}\right)\eta - i\Omega t\right], \eta < 0, \end{aligned} \tag{11}$$

where

$$\beta^2 = \alpha^2 - \delta^2 L^2/4\pi^2 \quad \text{and} \quad \alpha^2 = \Omega^2 L^2/c^2\pi^2. \tag{12}$$

The final displacement on superposing the two solutions becomes

$$W = W_1 + W_2 \tag{13}$$

where $W_1$ and $W_2$ are given respectively by (6) and (11).

## 3. Boundary conditions of the problem

The boundary conditions to the problem can be formulated as

(i) $\sigma_{yz}|_{y=0} = 0$ in $0 < x \leqslant l$ (14)

where $\sigma_{yz} = \sigma_{yz}^{(1)} + \sigma_{yz}^{(2)}$

with $\sigma_{yz}^{(1)} = \mu \dfrac{\partial w_1}{\partial y}$ and $\sigma_{yz}^{(2)} = \mu \dfrac{\partial W_2}{\partial y}$

and (ii) $W$ is continuous on $y = 0$ in $l < x \leqslant L$. (15)

## 4. Solution to the problem

We introduce a new constant $C_n$ defined by

$$A_n = (\delta L/2\pi + (n^2-\beta^2)^{1/2})C_n/(n^2-\alpha^2)^{1/2}$$

$$B_n = (\delta L/2\pi - (n^2-\beta^2)^{1/2})C_n/(n^2-\alpha^2)^{1/2}$$

so that (14) gives

$$\sum_{n=1}^{\infty} (n^2-\alpha^2)^{1/2} C_n \cos n\xi = Lp_0/\pi\mu_0,\ 0 < \xi \leqslant \xi_0 \tag{16}$$

where $p_0 = \mu_0 w_0(\delta/2 - i(\lambda^2 - \delta^2/4)^{1/2})$, $\xi_0 = \pi l/L$ and the condition (15) leads to the equation.

$$\sum_{n=1}^{\infty} C_n \left(\frac{n_2^2-\beta^2}{n^2-\alpha^2}\right)^{1/2} \cos n\xi = 0, \quad \xi_0 < \xi \leqslant \pi. \tag{17}$$

Taking

$$C_n \left(\frac{n^2-\beta^2}{n^2-\alpha^2}\right)^{1/2} = D_n, \tag{18}$$

and

$$\frac{n^2-\alpha^2}{(n^2-\beta^2)^{1/2}} = n - \Phi(n) \tag{19}$$

we have from (16) and (17) the dual series equations

$$\sum_{n=1}^{\infty} [n - \Phi(n)] D_n \cos n\xi = Lp_0/\pi\mu_0, \quad 0 < \xi \leqslant \xi_0 \tag{20}$$

$$\sum_{n=1}^{\infty} D_n \cos n\xi = 0, \quad \xi_0 < \xi \leqslant \pi \tag{21}$$

where $\Phi(n)$ is given by

$$\Phi(n) = [n - (n^2-\beta^2)^{1/2}] + \frac{\delta^2 L^2}{4\pi^2(n^2-\beta^2)^{1/2}}.$$

We thus reduce the problem to the determination of $D_n$ from the dual series equations (20) and (21).

Following [4] the solution to the dual series equation (20) and (21) can be found as

$$D_n = \frac{1}{\sqrt{2}} \int_0^{\xi_0} g(t) y_n(\cos t)\,\mathrm{d}t$$

where $\quad y_n(\cos t) = P_n(\cos t) + P_{n-1}(\cos t) \qquad (22)$

$P_n(x)$ being Legendre polynomials the unknown function $g(t)$ appearing in (21) satisfies the integral equation

$$\int_0^{\xi_0} \frac{g(t)\,\mathrm{d}t}{(\cos t - \cos \xi)^{1/2}} = \psi(\xi) \tag{23}$$

where

$$\psi(\xi) = \sec\frac{\xi}{2}\left[\int_0^{\xi_0} g(t)\left(\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{\Phi(n)}{n} y_n(\cos t)\sin n\xi\right)\mathrm{d}t\right] + \frac{Lp_0}{\pi\mu_0}\xi. \tag{24}$$

Following Srivastava [9] the integral equation (24) is solved as

$$g(t) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\sin\xi\,\psi(\xi)\,\mathrm{d}\xi}{(\cos\xi - \cos t)^{1/2}}. \tag{25}$$

Replacing $\psi(\xi)$ by (24) in (25) and simplifying, the unknown function $g(t)$ is found to be of the form

$$g(t) = \tfrac{1}{2}\tan\frac{t}{2}\int_0^{\xi_0} g(\tau)\left[\sum_{n=1}^{\infty}\Phi(n) y_n(\cos t) y_n(\cos\tau)\right]\mathrm{d}\tau + \frac{Lp_0}{\pi\mu_0}\sqrt{2}\tan t/2.$$

Denoting

$$\textstyle\int_0^{\xi_0} g(\tau) y_m(\cos\tau)\,\mathrm{d}\tau = a_m$$

$g(t)$ can be expressed as

$$g(t) = \tfrac{1}{2}\tan\frac{t}{2}\sum_{n=1}^{\infty} a_n\Phi(n) y_n(\cos t) + \frac{Lp_0}{\pi\mu_0}\sqrt{2}\tan t/2. \tag{26}$$

Now using the orthogonality condition of the well-known $y_n$ functions one may at once derive from (26) the following system of algebraic equations

$$a_m = \sum_{n=1}^{\infty} a_n d_{mn} + \frac{Lp_0}{\pi\mu_0}\sqrt{2} b_m \tag{27}$$

where

$$d_{mn} = \tfrac{1}{2}\Phi(n) I_{mn}(\xi_0), \tag{28}$$

with

$$I_{mn}(\xi_0) = \textstyle\int_0^{\xi_0} \tan\frac{t}{2}\, y_m(\cos t)\, y_n(\cos t)\,\mathrm{d}t \tag{29}$$

and

$$b_m = \textstyle\int_0^{\xi_0} \tan\frac{t}{2}\, y_m(\cos t)\,\mathrm{d}t. \tag{30}$$

The solution to the infinite system of equations (27) determines the constants $a_n$ appearing in (26) (Kantarovich and Krylov [1]).

We now calculate the stress $\sigma_{yz}|_{y=0}$. From (6) and (11) via (18) and (19) we obtain

$$\sigma_{yz}|_{y=0} = \mu_0 p_0 \exp(-i\Omega t) - \frac{\pi\mu_0}{L}\sum_{n=1}^{\infty}\{[n-\Phi(n)]D_n \cos n\xi$$
$$\exp(-i\Omega t)\}, \quad 0 < \xi \leqslant \xi_0$$

which is rewritten as

$$\sigma_{yz}|_{y=0} = \exp(-i\Omega t)\left[\mu_0 p_0 - \frac{\pi\mu_0}{L}\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\sum_{n=1}^{\infty} D_n \sin n\xi - \sum_{n=1}^{\infty}\frac{\Phi(n)}{n} D_n \sin n\xi\right\}\right].$$

Ononbstituting $D_n$ from (22) in above and using the relation

$$\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty} y_n(\cos t)\sin n\xi = \frac{\cos\frac{1}{2}\xi H(\xi - t)}{(\cos t - \cos\xi)^{1/2}},$$

$H(\xi - t)$ being the Heaviside unit step function, we obtain

$$\sigma_{yz}|_{y=0} = \exp(-i\Omega t)\left[\mu_0 p_0 - \frac{\pi\mu_0}{L}\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\cos\tfrac{1}{2}\xi\int_0^{\xi_0}\frac{g(t)\,\mathrm{d}t}{(\cos t - \cos\xi)^{1/2}} + \frac{1}{\sqrt{2}}\int_0^{\xi_0} g(t)\left(\sum_{n=1}^{\infty}\frac{\Phi(n)}{n} y_n(\cos t)\sin n\xi\right)\mathrm{d}t\right\}\right], \quad 0 < \xi \leqslant \xi_0 \tag{31}$$

### 4.1. *Stress intensity factor*

Apart from the determination of the distribution of stress near the crack tip, stress intensity factor is also an important physical quantity in view of the fracture criteria. The stress intensity factor (SIF) is defined by the relation

$$K = \lim_{x\to x_0}(2\pi(x - x_0))^{1/2}\sigma_{yz}|_{y=0}. \tag{32}$$

By isolating the singularity and discarding the terms which are bounded as $\xi \to \xi_0$ we can easily obtain from (31)

$$\sigma_{yz}|_{y=0} = \exp(-i\Omega t)\left[\frac{\pi\mu_0}{L} g(\xi_0)\frac{\sin\xi\cos\frac{1}{2}\xi}{\sin\xi_0(\cos\xi_0 - \cos\xi)^{1/2}}\right]$$

whence from (32)

$$K = \exp(-i\Omega t)\left[\frac{\pi\mu_0}{(L\tan\frac{1}{2}\xi_0)^{1/2}} g(\xi_0)\right]. \tag{33}$$

## 5. Numerical results and conclusions

For numerical computation of the SIF near a crack tip we define the nondimensional time $t^* = \pi ct/L$. The SIF in nondimensional form

$$K^*(t^*) = K\sqrt{L}/\pi\mu_0 w_0 \tag{34}$$

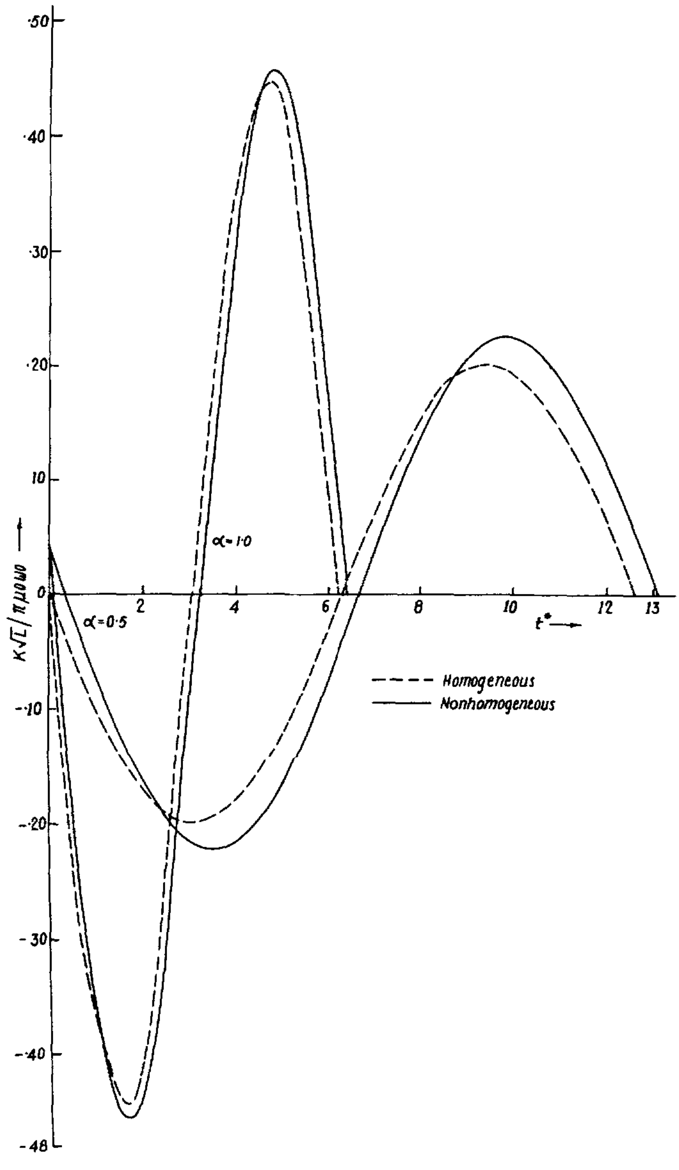


**Figure 1.** Stress intensity factor ($K^*$) as function of time ($t^*$) with the nonhomogeneity parameter $\delta L/2\pi = 0{\cdot}1$.

has been evaluated numerically with $\delta L/2\pi = 0{\cdot}1$ and for values of $\alpha = 0{\cdot}5$ and $\alpha = 1{\cdot}0$.

The variation of $K^*$ with $t^*$ has been shown in figure 1. The effect of nonhomogeneity in the diffracted wave field is found to be more prominent for small values of $\alpha$ than for large values of $\alpha$.

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## References

[1] Kantarovich L V and Krylov V I 1958 *Approximate method for higher analysis* (New York: Interscience Publishers) pp. 20–21
[2] Mal A K 1970 *Int. J. Eng. Sci.* **8** 381
[3] Mal A K 1970 *Int. J. Eng. Sci.* **8** 769
[4] Parton V Z and Morozov E M 1978 *Elastic-plastic fracture Mechanics*—Translated from Russian; (Moscow: Mir Publishers)
[5] Sih G C and Loeber J F 1969 *J. Math. Mech.* **19** 327
[6] Sih G C and Loeber J F 1969 *Q. Appl. Mech.* **27** 193
[7] Singh BM and Dhaliwal R S 1978 *J. Elast.* **8** 313
[8] Sommerfield A 1964 *Optics-Lectures in theoretical physics* (New York: Academic Press) Vol. 4 249
[9] Srivastav R P 1962–63 *Proc. R. Soc. Edinburgh.* **A66** 161–172
[10] Thau SA and Lu T H 1971 *Int. J. Solid Struct.* **7** 731