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Flow between torsionally oscillating noncoaxial cylinders

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Abstract. The flow of an incompressible viscous fluid between two torsionally oscillating noncoaxial cylinders has been investigated. Closed form solutions for symmetric and first order asymmetric flow are obtained for the cases when the gap between the cylinders is finite. Solutions of the governing equations under the geometrical restriction of narrow gap are also presented. These solutions coincide with the solutions of the finite gap by incorporating in them the condition of narrow gap. The components of the force acting on the inner cylinder are calculated.

Keywords. Noncoaxial system; torsional oscillations; incompressible viscous fluid; closed form solution; symmetric flow.

1. Introduction

Two-dimensional flows between moving nearly coaxial cylinders are important in the design of control mechanisms for aircrafts and rockets and rheometers. Several authors have studied the viscous flow between two noncoaxial cylinders rotating about their axes which are parallel and situated slightly apart. Wood [8] in his study of the viscous flow between two noncoaxial rotating cylinders has obtained the exact solution of the governing Navier-Stokes equations assuming small eccentricity and discussed the asymptotic behaviour of these solutions for large Reynolds number. The conformal transformation technique has been used by Segel [5] to study the unsteady flow between noncoaxial cylinders when the outer cylinder is kept fixed and the inner cylinder rotates or vibrates about a slightly eccentric point. Separation in the flow between eccentric rotating cylinders has been investigated by Kamal [3]. Kulinski and Ostrach [4] also studied the flow between rotating cylinders with axes slightly apart in connection with their study of the flow phenomena in journal bearings. Flow between rotating eccentric cylinders has been considered by Abbot and Walters [1] in understanding some rheometrical flows. The steady flow between two noncoaxial rotating cylinders has been studied by Urban [6] by a simple approach using polar coordinate system. The principle of transfer of boundary conditions discussed by van Dyke [7] has been used to handle the conditions at the outer boundary which is not a coordinate curve.

In this paper, the primary oscillatory flow of a viscous incompressible fluid contained between two infinite cylinders with parallel axes situated slightly apart has been investigated. The solutions are obtained for the cases when the gap between the cylinders is finite and narrow. Components of the force acting on the inner cylinder are calculated. Further, it is observed from the profiles of transverse velocity distributions that the outer boundary condition is more clearly satisfied for large frequency parameter.

2. Formulation

Let O_1 and O_2 be the centres of the inner and outer cylinders with radii R_1 and R_2 respectively. O_1 is taken as the origin of the cylindrical polar coordinate system (r, θ, Z) with the Z-axis along the axis of the inner cylinder (figure 1). The distance $O_1 O_2$ is the eccentricity e of the system. We define the non-dimensional parameters

$$\varepsilon = e/d, \ (0 < \varepsilon < 1), \text{ the eccentricity ratio,}$$

$$\delta = \frac{d}{R_0}(0 < \delta < 2), \text{ the gap ratio,}$$

where $d = R_2 - R_1$ and $R_0 = (R_1 + R_2)/2$. Using the cosine rule for the triangle O_1 O_2 P, the radial distance h which is the variable gap between the cylinders is obtained as

$$h(\theta) = -R_1 + \varepsilon d \cos\theta + R_2\left(1 - \frac{4\varepsilon^2\delta^2}{(2+\delta)^2}\sin^2\theta\right)^{1/2}. \tag{1}$$

Consider the flow problem when the inner and outer cylinders execute rotatory oscillations with $\Omega_1 \exp(i\sigma t)$ and $\Omega_2 \exp(i\sigma t)$ respectively, where σ is the frequency of oscillations and Ω_1 and Ω_2 are the angular velocities of rotation. Assuming that there is no flow in the axial direction since the cylinders are infinite, the velocity and the pressure fields are functions of r, θ, and t only. The governing equations of the primary flow are

$$u_t = -\frac{1}{\rho'}p_r + \nu\left[u_{rr} + \frac{1}{r}u_r - \frac{u}{r^2} + \frac{1}{r^2}u_{\theta\theta} - \frac{2}{r^2}v_\theta\right] \tag{2}$$

$$v_t = -\frac{1}{r\rho'}p_\theta + \nu\left(v_{rr} + \frac{1}{r}v_r - \frac{v}{r^2} + \frac{1}{r^2}v_{\theta\theta} + \frac{2}{r^2}u_\theta\right). \tag{3}$$

and the continuity equation

$$u + ru_r + v_\theta = 0, \tag{4}$$

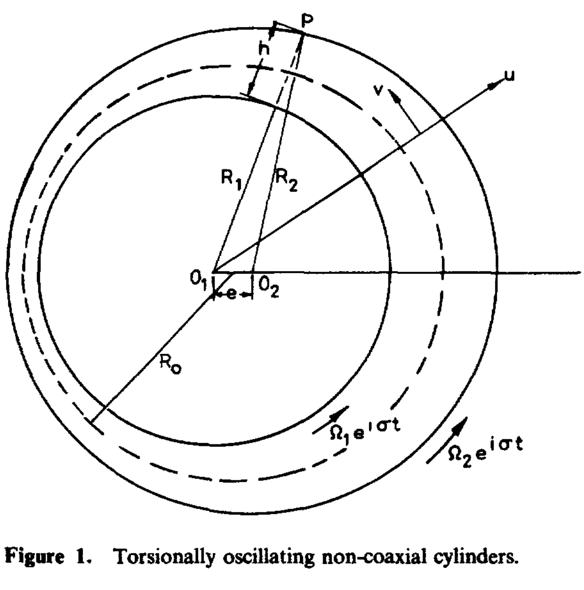

Figure 1. Torsionally oscillating non-coaxial cylinders.

where u is the radial velocity, v is the transverse velocity ρ' is the density, ν is the kinematic viscosity and p is the pressure. The subscripts denote the partial derivatives with respect to the corresponding variables and this notation is used throughout the paper. In a primary flow the nonlinear terms are completely omitted in the equations of motion, as their inclusion leads to a solution with second harmonic.

The boundary conditions are

$$\left.\begin{aligned} u &= 0 \\ v &= R_1 \Omega_1 \exp(i\sigma t) \end{aligned}\right\} \quad \text{on } r = R_1, \tag{5}$$

$$\left.\begin{aligned} u &= -\Omega_2 \varepsilon d \sin\theta \exp(i\sigma t), \\ v &= R_2 \Omega_2 \{1 - (\varepsilon d/R_2)^2 \sin^2\theta\}^{1/2} \exp(i\sigma t) \end{aligned}\right\} \quad \text{on } r = R_1 + h. \tag{6}$$

The boundary conditions (6) are derived making use of the fact that $r = R_1 + h$ is not a coordinate curve. Eliminating the pressure from (2) and (3) we get

$$u_t - r v_{tr} - v_t = \left[u_{rr\theta} - \frac{1}{r} u_{r\theta} + \frac{1}{r^2} u_\theta + \frac{1}{r^2} u_{\theta\theta\theta} \right.$$

$$\left. - \frac{1}{r^2} v_{\theta\theta} - 2 v_{rr} - r v_{rrr} - \frac{v}{r^2} + \frac{1}{r} v_r - \frac{1}{r} v_{\theta\theta r} \right]. \tag{7}$$

In order to obtain the solution we expand u and v in powers of the small parameter ε in the form

$$u(r,\theta) \exp(i\sigma t) = \{u_0(r) + \varepsilon u_1(r,\theta) + \varepsilon^2 u_2(r,\theta) + \ldots\} \exp(i\sigma t), \tag{8}$$

$$v(r,\theta) \exp(i\sigma t) = \{v_0(r) + \varepsilon v_1(r,\theta) + \varepsilon^2 v_2(r,\theta) + \ldots\} \exp(i\sigma t). \tag{9}$$

Using (8) and (9) in (7) and (4) and equating different powers of ε on either side, we obtain various order equations which are solved to obtain the solutions.

3. Solution for finite gap

The symmetric flow due to torsionally oscillating coaxial cylinders is given by the zeroth order equation which is given in the non-dimensional form by

$$R^2 V_{0RR} + R V_{0R} + (-i\alpha^2 R^2 - 1) V_0 = 0, \tag{10}$$

where

$$\alpha^2 = \frac{\sigma}{\nu} R_1^2.$$

In writing (10) the following non-dimensional variables (capital letters) $UR_1\Omega_1 = u$, $VR_1\Omega_1 = v, R_1 R = r, T = \sigma t$, have been used. The corresponding boundary conditions are

$$V_0 = 1, \text{ at } R = 1,$$

$$V_0 = \mu/\eta \text{ at } R = 1/\eta, \tag{11}$$

where

$$\eta = R_1/R_2 \text{ and } \mu = \Omega_2/\Omega_1. \tag{12}$$

The solution of (10) satisfying (11) is

$$V_0 = \frac{1}{D}[Y_1(\rho_2)J_1(\rho) - J_1(\rho_2)Y_1(\rho) + \mu/\eta\{J_1(\rho_1)Y_1(\rho) - Y_1(\rho_1)J_1(\rho)\}], \tag{13}$$

where

$$\lambda = \exp\left(-i\frac{\pi}{4}\right)\alpha, \quad \rho_1 = \lambda, \quad \rho = \lambda R, \quad \rho_2 = \lambda/\eta$$

and

$$D = J_1(\rho_1)Y_1(\rho_2) - J_1(\rho_2)Y_1(\rho_1), \tag{14}$$

J_n, Y_n being Bessel functions of order n. It is easily seen that the expression for V_0 given in (13) coincides with the one given by Urban [6], for steady flow, in the limit $\lambda \to 0$.

The equation governing the asymmetric flow is

$$i\alpha^2(U_{1\theta} - RV_{1R} - V_1) = \left[U_{1RR\theta} \times \frac{1}{R^2}U_{1\theta\theta\theta} + \frac{2}{R^2}U_{1\theta} + \frac{1}{R}V_{1R} - 2V_{1RR} - RV_{1RRR} - \frac{1}{R}V_{1\theta\theta} - \frac{V_1}{R^2}\right]. \tag{15}$$

In order to obtain the first order boundary conditions, we adopt the following procedure. In view of (1) it is noted that the perturbation parameter ε appears implicitly in the radial coordinate in the expressions for u_1 and v_1 as well as explicitly. Hence it is not possible to equate the various powers of ε directly. This difficulty is overcome by expanding u_1 and v_1 in Taylor's series about r equal to R_2, which gives the explicit dependence of u_1 and v_1 on ε. This method is known as 'the transfer of boundary conditions', see van Dyke [7]. Now, the boundary conditions (5) and (6) reduce at first order in ε to

$$\left.\begin{aligned} U_1 &= 0 \\ V_1 &= 0 \end{aligned}\right\} \quad \text{on } R = 1, \tag{16}$$

$$\left.\begin{aligned} U_1 &= (-\mu/R_1)d\sin\theta \\ V_1 &= (-K_0 d/R_1)\cos\theta \end{aligned}\right\} \quad \text{on } R = 1/\eta, \tag{17}$$

where

$$K_0 = (V_{0R})_R = 1/\eta. \tag{18}$$

Equation (15) admits a solution of the form

$$U_1 = \tilde{U}(R)i\exp(i\theta), \; V_1 = -(R\tilde{U})_R\exp(i\theta). \tag{19}$$

Substituting (19) in (15) we get

$$\tilde{U}_{RRRR} + \frac{6}{R}\tilde{U}_{RRR} + \frac{3}{R^2}\tilde{U}_{RR} - \frac{3}{R^3}\tilde{U}_R - i\alpha^2(\tilde{U}_{RR} + \frac{3}{R}\tilde{U}_R) = 0. \tag{20}$$

The corresponding boundary conditions are

$$\left.\begin{aligned} \tilde{U} &= 0, \\ \tilde{U}_R &= 0 \end{aligned}\right\} \quad \text{at } R = 1, \tag{21}$$

$$\left.\begin{aligned} \tilde{U} &= \mu d/R_1 \\ \tilde{U}_R &= \eta d(K_0-\mu)/R_1 \end{aligned}\right\} \quad \text{at } R = 1/\eta. \tag{22}$$

The solution of (20) satisfying (21) and (22) is

$$\tilde{U} = AI_1(\rho) + BI_2(\rho), \tag{23}$$

where

$$A = \frac{d}{R_1\Delta}\{\mu I_4(\rho_2) - K_0 I_2(\rho_2)\},$$

$$B = \frac{d}{R_1\Delta}\{K_0 I_1(\rho_2) - \mu I_3(\rho_2)\},$$

$$I_1(\rho) = \{J_0(\rho_1) - J_0(\rho) - J_2(\rho) + \rho_1^2 J_2(\rho_1)/\rho^2],$$

$$I_2(\rho) = [Y_0(\rho_1) - Y_0(\rho) - Y_2(\rho) + \rho_1^2 Y_2(\rho_1)/\rho^2],$$

$$I_3(\rho) = [J_0(\rho_1) - J_0(\rho) + J_2(\rho) - \rho_1^2 J_2(\rho_1)/\rho^2],$$

$$I_4(\rho) = [Y_0(\rho_1) - Y_0(\rho) + Y_2(\rho) - \rho_1^2 Y_2(\rho_1)/\rho^2]$$

and

$$\Delta = I_1(\rho_2)I_4(\rho_2) - I_3(\rho_2)I_2(\rho_2).$$

The radial and transverse velocities are given by

$$U_1 = \frac{d}{R_1\Delta}[\mu\{I_4(\rho_2)I_1(\rho) - I_3(\rho_2)I_2(\rho)\} + K_0\{I_1(\rho_2)I_2(\rho) - I_2(\rho_2)I_1(\rho)\}]\, i\exp(i\theta), \tag{24}$$

$$V_1 = \frac{d}{R_1\Delta}[\mu\{I_3(\rho_2)I_4(\rho) - I_4(\rho_2)I_3(\rho)\} - K_0\{I_1(\rho_2)I_4(\rho) - I_2(\rho_2)I_3(\rho)\}]\exp(i\theta). \tag{25}$$

It is not possible to derive the asymmetric steady flow results discussed by Urban [6] from our results (24) and (25) as the nonlinear terms are neglected completely in obtaining them. Segel [5] has obtained results similar to (24) and (25) in the study of the oscillatory flow between two circular cylinders where the inner cylinder remains fixed and the centre of the outer cylinder vibrates causing the asymmetry.

4. Solution for narrow gap

In this section some useful and simple results are derived by considering an extra restriction namely, the gap between the cylinders is very small in addition to the small eccentricity. First the geometrical condition of narrow gap is introduced into the governing equations and boundary conditions from which the solutions are presented. We introduce a non-dimensional independent variable X by

$$r = R_0(1 + \delta X), \tag{26}$$

where δ, the gap ratio defined earlier is very small. The new independent variable has the range $-\frac{1}{2} \leqslant X \leqslant \frac{1}{2}$ and any point in the fluid domain is prescribed by X and θ.

Introducing (26) in (10) one obtains

$$i\beta^2\delta^2 V_0 = \left[V_{0XX} + \frac{\delta}{(1+\delta X)} V_{0X} - \frac{\delta^2 V_0}{(1+\delta X)^2} \right], \tag{27}$$

where

$$\beta^2 = R_0^2 \frac{\sigma}{\nu}. \tag{28}$$

Here two interesting cases arise:

(i) $\beta^2\delta^2 \to 0$ as $\delta \to 0$ implying that the frequency parameter β is small,

(ii) $\beta^2\delta^2 = (\sigma/\nu)d^2 = i\gamma^2$, (finite) as $\delta \to 0$ which implies that β is large enough to make $\beta\delta$ finite in the limit $\delta \to 0$. Taking the limit as $\delta \to 0$ in (27) in case (i) we get

$$V_{0XX} = 0. \tag{29}$$

The corresponding boundary conditions are

$$V_0 = 1 \text{ at } x = -\tfrac{1}{2},$$

$$V_0 = \mu \text{ at } x = \tfrac{1}{2}. \tag{30}$$

The solution of (29) satisfying (30) is

$$V_0 = \left[\frac{1+\mu}{2} - (1-\mu)X \right]. \tag{31}$$

Taking the limit as $\delta \to 0$ in (27) in case (ii) one obtains

$$V_{0XX} + \gamma^2 V_0 = 0. \tag{32}$$

The solution of (32) satisfying (30) is

$$V_0 = \frac{1}{\sin\gamma} [\mu \sin\gamma(x+\tfrac{1}{2}) - \sin\gamma(x-\tfrac{1}{2})]. \tag{33}$$

It is easily seen that (33) reduces to (31) as $\gamma \to 0$. We observe that the solution given in (33) for the narrow gap is also derivable from the finite gap solution (13). This is achieved in the following way: Replace the Bessel function in (13) by their asymptotic series for large frequency parameter, change the independent variable R to X as defined in (26) and then take the limit as $\delta \to 0$, keeping $\beta\delta$ finite.

Introducing the narrow gap approximation in (19)–(21) for case (i) leads to

$$\tilde{U}_{XXXX} = 0, \tag{34}$$

$$\tilde{U}(X) = 0, \quad \tilde{U}_X = 0 \text{ at } x = -\tfrac{1}{2}, \tag{35}$$

$$\tilde{U}(X) = \mu \quad \tilde{U}_X = (\mu - 1) \text{ at } x = \tfrac{1}{2}. \tag{36}$$

The solution for (34) satisfying (35) and (36) is

$$\tilde{U} = \frac{3\mu+1}{8} + \frac{5\mu+1}{4} X + \frac{\mu-1}{2} X^2 - (\mu+1)X^3. \tag{37}$$

The first order radial and transverse velocities are obtained as

$$U_1(X,\theta) = \left[\frac{3\mu+1}{8} + \frac{5\mu+1}{4}X + \frac{\mu-1}{2}X^2(\mu+1)X^3\right] i\exp(i\theta), \tag{38}$$

$$V_1(X,\theta) = -\left[\frac{5\mu+1}{4} + (\mu-1)X - 3(\mu+1)X^2\right]\exp(i\theta). \tag{39}$$

By taking the real parts of the results given in (31), (38) and (39) we get the results of the steady flow given by Urban [6]. In our problem these are to be multiplied with $\exp(iT)$ as given in (8) and (9) and then the real parts are to be taken. Thus in case (i), the conclusions drawn by Urban [6] holds good in our problem also except for the time dependence. The governing equations and boundary conditions in case (ii) become

$$\tilde{U}_{XXXX} + \gamma^2 \tilde{U}_{XX} = 0, \tag{40}$$

$$\tilde{U} = 0, \quad \tilde{U}_X = 0 \text{ at } x = -\tfrac{1}{2}, \tag{41}$$

$$\tilde{U} = \mu, \quad \tilde{U}_X = \frac{\gamma}{\sin\gamma}(\mu\cos\gamma - 1) \text{ at } x = \tfrac{1}{2}. \tag{42}$$

The solution of (40) satisfying (41) and (42) is obtained and the velocities are given by

$$U_1(X,\theta) = [C_1 + C_2X + C_3\sin\gamma(X+\tfrac{1}{2}) + C_4\sin\gamma(X-\tfrac{1}{2})]i\exp(i\theta), \tag{43}$$

$$V_1(X,\theta) = -[C_2 + C_3\gamma\cos\gamma(X+\tfrac{1}{2}) + C_4\gamma\cos\gamma(X-\tfrac{1}{2})]\exp(i\theta). \tag{44}$$

where

$$C_1 = d_1/d^*,\ C_2 = d_2/d^*,\ C_3 = d_3/d^*,\ C_4 = d_4/d^*,$$

$$d_1 = \gamma/2\,[(1-\mu)(\gamma+\gamma\cos\gamma - 2\sin\gamma) - \gamma\mu\cos^2\gamma],$$

$$d_2 = \gamma^2[(1+\mu)(\cos\gamma - 1)],$$

$$d_3 = \gamma/\sin\gamma\,[(1+\mu)\sin\gamma + \gamma\cos\gamma(\mu\cos\gamma - 1) - 2\mu\sin\gamma\cos\gamma],$$

$$d_4 = \gamma/\sin\gamma\,[\mu\sin\gamma - \sin\gamma - \gamma(\mu\cos\gamma - 1)],$$

$$d^* = \gamma/\sin\gamma\,(2 - 2\cos\gamma - \gamma\sin\gamma).$$

The solutions given in (43) and (44) can be obtained from (24) and (25) following the same procedure explained for zeroth order velocity.

5. Discussion

The physical quantities of interest in this problem are the forces acting on the cylinders. If X' and Y' denote the components of force on the inner cylinder in the x and y directions respectively, then we have for a cylinder of length h, Abbot *et al* [2],

$$X' = R_1 h \int_0^{2\pi} (P_{rr}\cos\theta - P_{r\theta}\sin\theta)\,d\theta, \tag{45}$$

$$Y' = R_1 h \int_0^{2\pi} (P_{rr}\sin\theta + P_{r\theta}\cos\theta)\,d\theta, \tag{46}$$

where P_{rr} and $P_{r\theta}$ are the components of stress tensor. Substituting the expressions for P_{rr} and $P_{r\theta}$ in terms of non-dimensional velocities, we get

$$X' = \mathrm{i}\pi R_1 h\nu\rho'\Omega_1 \varepsilon \left\{ R\frac{\mathrm{d}^2\tilde{U}}{\mathrm{d}R^2} + 3\frac{\mathrm{d}\tilde{U}}{\mathrm{d}R} \right\}_{R=1} \exp(iT), \tag{47}$$

$$Y' = iX. \tag{48}$$

where the real parts on the right sides of (47) and (48) only are relevant in a physical problem. Using $\tilde{U}$ of finite gap given by (23) in (47), the force in the x-direction is

$$X' = 2\pi i\nu\rho' R_1 h\Omega_1 \varepsilon\lambda[AJ_1(\lambda) + BY_1(\lambda)]\exp(iT). \tag{49}$$

From (47) using the narrow gap expression for $\tilde{U}$ in case (ii), we get

$$X' = i\pi R_1 \nu\rho' h\Omega_1 \varepsilon/\delta[d_4\gamma^2 \sin\gamma/d^*]\exp(iT). \tag{50}$$

Employing in (47) the narrow gap expression for $\tilde{U}$ given in (37) for case (i) and collecting the real parts we get

$$X' = -2\pi R_1 \nu\rho' h\Omega_1 \varepsilon/\delta(2\mu + 1)\sin\sigma t, \tag{51}$$

$$Y' = -2\pi R_1 \nu\rho' h\Omega_1 \varepsilon/\delta(2\mu + 1)\cos\sigma t. \tag{52}$$

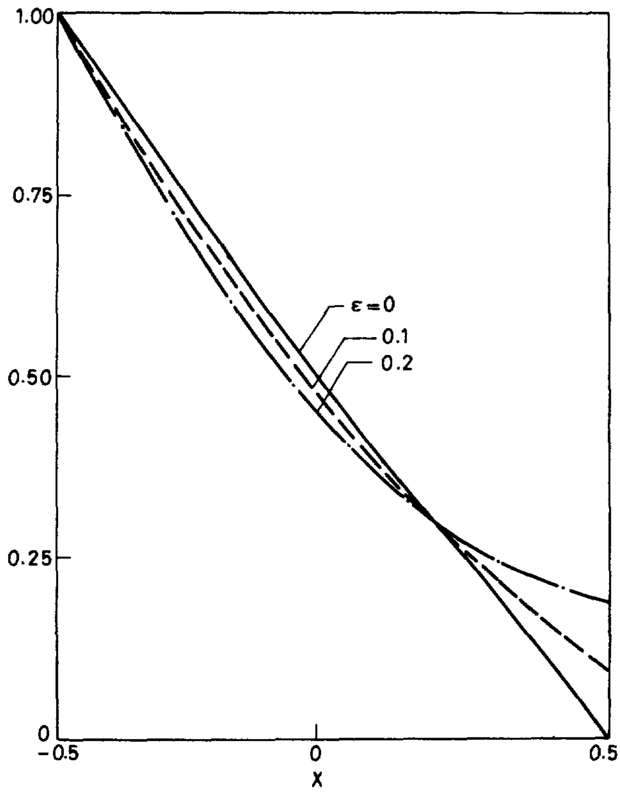

Figure 2. Transverse velocity profiles for $\gamma = 1 - i$, $\theta = 0°$ and $T = 0$.

From (51) and (52) we observe that in the limit $\sigma \to 0$, Y' only exists. This means that for steady, rotating non-coaxial cylinders, the force on the inner cylinder is always perpendicular to the line of centres.

The small gap transverse velocity plays a significant role in a stability analysis and has been discussed by Urban [6] for the steady problem. As our results in case (i) at time $t = 0$ are the same as the results of Urban [6], we discuss only the narrow gap transverse velocity profiles in case (ii). Figures 2 and 3 depict the transverse velocity profiles when the outer cylinder is at rest ($\mu = 0$) for various values of eccentricity ratio ε and the frequency parameter γ with values $1 - i$ and $(1{\cdot}5) - (1{\cdot}5i)$ respectively. The symmetric velocity corresponding to $\varepsilon = 0$ is not a straight line, as seen from (33) and the figures, unlike in case (ii). It is seen that the outer boundary condition is not well satisfied when we include the first order solution. The error in the transverse velocity at the outer boundary is positive, increases with the increase of the eccentricity ratio ε. Comparing figures 2 and 3 we see that the outer boundary condition is more closely satisfied when the value of the frequency parameter is large. The transverse velocity profiles for $\theta = 180°$ and the other parameters remaining the same as in the previous case are shown in figures 4 and 5. Here the error in the transverse velocity at the outer boundary is negative. The error at the outer boundary is found to decrease with increase of the frequency parameter at both the locations $\theta = 0°$ and $\theta = 180°$. Thus for given ε small,

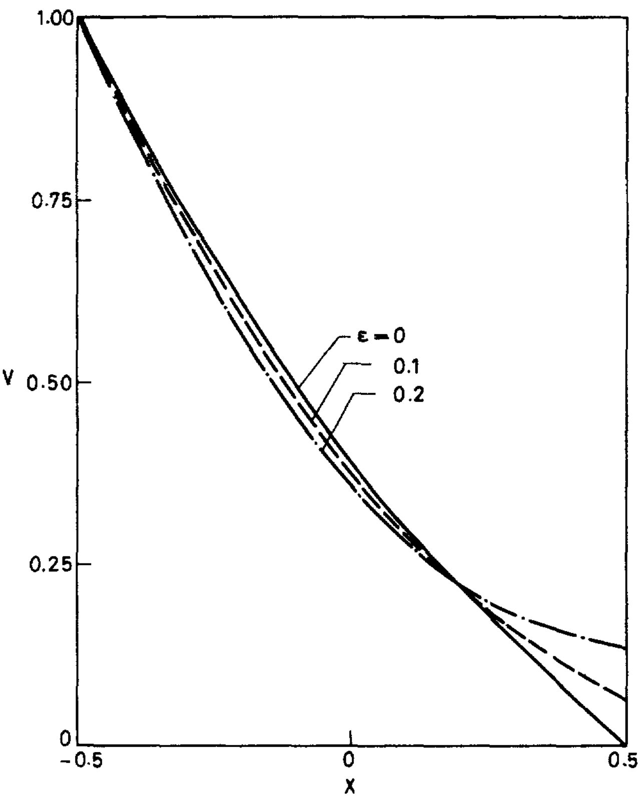

Figure 3. Transverse velocity profiles for $\gamma = 1{\cdot}5 - (1{\cdot}5)i$, $\theta = 0°$ and $T = 0$.

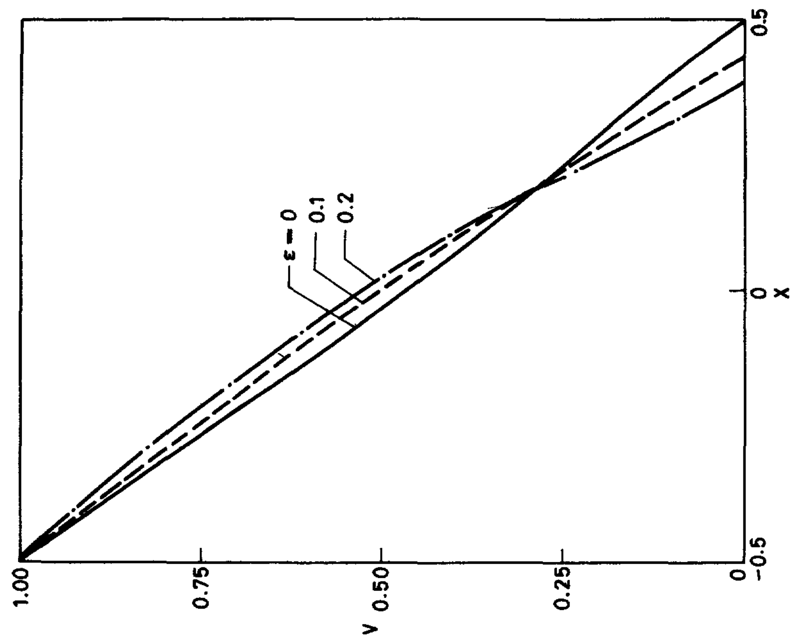

Figure 4. Transverse velocity profiles for $\gamma = 1 - i$, $\theta = 180°$ and $T = 0$.

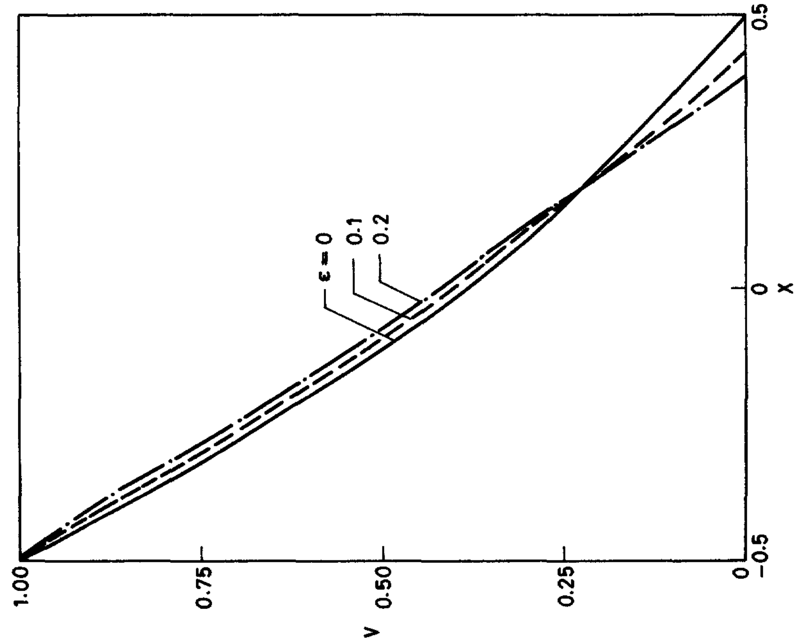

Figure 5. Transverse velocity profiles for $\gamma = 1{\cdot}5 - i(1{\cdot}5)$, $\theta = 180°$ and $T = 0$.

the boundary condition at the outer boundary is satisfied more closely for sufficiently large frequency parameter.

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