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Approximate analytical solutions for strong shocks with variable energy

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Abstract. Approximate analytical solutions are obtained for self-similar flows behind strong shocks with variable energy deposition or withdrawal at the wavefront in a perfect gas at rest with constant initial density. Numerical solutions are also obtained and the approximate solutions agree with these solutions. The effect of the adiabatic index on the solutions is investigated. The dependence of shock density ratio on the parameter characterizing the energy of the flow is studied. It is observed that the rate of deposition of energy at the wavefront decreases with increase of the parameter that specifies the total energy of the flow.

Keywords. Strong shocks; self-similar flows; variable energy; approximate analytical solutions.

1. Introduction

The study of blast waves with variable energy deposition at the wavefront is of interest in connection with the explosions produced by laser irradiation. With a constant power input, self-similar solutions were obtained for spherical shock [7]. The similarity solutions for blast waves with variable energy have also been studied [2, 3, 6]. A comprehensive parametric study of the solutions for blast waves with variable energy deposition at the wavefront was given by Barenblatt *et al* [1]. However, these studies are confined to obtain only numerical solutions.

In this paper we present approximate analytical solutions for self-similar flows with variable energy behind strong shock waves propagating in a medium at rest. Both deposition and withdrawal of energy at the wavefront are considered. The total energy of the flow behind the shock and the centre of explosition is assumed to be a function of shock radius obeying a power law. The medium considered here is a perfect gas with constant initial density. The basic assumption in obtaining the approximate solutions is that most of the mass of the flow is concentrated behind the shock-wave which is almost true for strong shocks. In particular, for self-similar flows, the method based upon this assumption, proposed by Laumbach and Probstein [4], is found to give simple solutions (see, for example, [5] and its references). Approximate solutions for cylindrical and spherical flows are obtained for the cases with addition and withdrawal of energy at the wavefront. The effect of the adiabatic index on spherical flows with constant power input is also investigated. The dependency of shock density ratio on the parameter characterizing the energy of the flow is studied. To compare with these approximate solutions, numerical solutions are also obtained for the specific values of the parameters characterizing the flows under study which are not investigated earlier. It is observed that there is a fairly good agreement between the numerical and approximate solutions. Also the approximate solutions require much less computational time as compared to numerical solutions.

2. Approximate analytical solutions

An approximate method developed by Laumbach and Probstein [4] is used for obtaining analytical solutions for the problem of self-similar flows of a perfect gas with variable energy deposition at the strong shock-wave propagating in a uniform medium at rest. The equations of continuity, momentum and adiabatic law of the flow in Lagrangian co-ordinates are of the form

$$\rho r^{\alpha} dr = \rho_0 r_0^{\alpha} dr_0, \tag{1}$$

$$\frac{\partial^2 r}{\partial t^2} + \frac{r^{\alpha}}{\rho_0 r_0^{\alpha}} \frac{\partial p}{\partial r_0} = 0, \tag{2}$$

$$p(r_0, t)/p_s(r_0) = [\rho(r_0, t)/\rho_s(r_0)]^{\gamma}, \tag{3}$$

where ρ_0 is the initial density of the perfect gas, p the pressure, ρ the density, t the time, r the Eulerian co-ordinate, r_0 the Lagrangian co-ordinate and γ is the ratio of specific heats. Suffix s represents quantities immediately behind the shock-wave and $\alpha = 0, 1, 2$ for plane, cylindrical and spherical flows respectively. Strong shock conditions can be written in the form

$$u_s = (1-\beta)\dot{r}_s, \tag{4}$$

$$\rho_s = \rho_0/\beta, \tag{5}$$

$$p_s = (1-\beta)\rho_0 \dot{r}_s^2, \tag{6}$$

$$\beta = [\gamma - (1 \pm 2(\gamma^2-1)q)^{1/2}]/(\gamma+1), \quad q = Q/\dot{R}^2, \tag{7}$$

where the dot denotes differentiation with respect to time and Q is the energy per unit mass at the shock front. The negative sign before q represents addition of energy whereas positive sign represents withdrawal. It is assumed that Q varies such that q is always constant so that the problem is self-similar. Then the shock density ratio β is an unknown constant to be determined.

We assume that the total energy of the flow behind the shock varies as

$$E = E_0 r_s^n, \tag{8}$$

where E_0 and n are constants. Equation (2) can be written in the form

$$p(r_0,t) = p_s(r_s) + \int_{r_0}^{r_s} \frac{1}{r^{\alpha}} \partial^2 r/\partial t^2 \, \rho_0 r_0^{\alpha} \mathrm{d}r_0. \tag{9}$$

The integral in (9) is evaluated approximately by replacing the integrand $(1/r^{\alpha})\,(\partial^2 r/\partial t^2)$ by its value at the shock, given in the Appendix. Then we get

$$p(r_0,t) = p_s(r_s) + \frac{\rho_0}{r_s^{\alpha}} \frac{\partial^2 r}{\partial t^2}\bigg|_s (r_s^{\alpha+1} - r_0^{\alpha+1})/(\alpha+1). \tag{10}$$

Using (8), the energy conservation equation in integral form may be written as

$$E_0 r_s^n/\sigma_{\alpha} = \int_0^{r_s} \frac{p}{(\gamma-1)} r^{\alpha} \mathrm{d}r + \int_0^{r_s} 1/2\left(\frac{\partial r}{\partial t}\right)^2 \rho_0 r_0^{\alpha} \mathrm{d}r_0, \tag{11}$$

where $\sigma_{\alpha} = 2\pi\alpha + (\alpha-1)(\alpha-2)$. The integrals in (11) are evaluated approximately by replacing $p(r, t)$ by $p(0, t)$ and $\partial r/\partial t$ by $(\partial r/\partial t)|_s$. Using (10), we finally get a second order differential equation from (11) as

$$\ddot{r}_s + A\dot{r}_s^2/r_s = Br_s^{(n-\alpha-2)}, \tag{12}$$

where

$$A = \frac{(\alpha+1)[2+(\gamma-1)(1-\beta)][\gamma(1-\beta)-\beta]+2\alpha\gamma\beta(1-\beta)}{2[2\beta+\gamma(1-\beta)]},$$

$$B = \frac{(\alpha+1)^2(\gamma-1)[\gamma(1-\beta)-\beta]}{(1-\beta)[2\beta+\gamma(1-\beta)]}\frac{E_0}{\rho_0\sigma_\alpha}.$$

Integration of (12) gives the shock propagation law as

$$r_s = Ct^\delta, \quad \delta = 2/(\alpha+3-n), \tag{13}$$

where

$$C = [2B/(n+2A-\alpha-1)\delta^2]^{\delta/2}.$$

Using (13), we get from (3) and (10) the approximate solutions for pressure and density as

$$p(r_0,t)/p_s(r_s) = F^\gamma(x), \tag{14}$$

$$\rho(r_0,t)/\rho_s(r_s) = F(x)x^b, \tag{15}$$

where

$$F(x) = [1+G(1-x^{\alpha+1})]^{1/\gamma}, \; b = 2(1-\delta)/\gamma\delta, \tag{16a}$$

$$G = [2\beta+\gamma(1+\beta)]\,(1-\delta)+\alpha\gamma\beta(1-\beta)\delta/(\alpha+1)[\beta-\gamma(1-\beta)]\delta, \tag{16b}$$

and $x = r_0/r_s$ is the reduced Lagrangian co-ordinate. From continuity equation, we get a relation between the reduced Eulerian co-ordinate $\lambda = r/r_s$ and x as

$$\lambda^{\alpha+1} = (\alpha+1)\beta\int_0^x \frac{x^{(\alpha-b)}}{F(x)}\,dx. \tag{17}$$

Using the condition at the shock that $\lambda = x = 1$, one gets a relation for obtaining β from (17) as

$$\frac{1}{\beta} = (\alpha+1)\int_0^1 \frac{x^{(\alpha-b)}}{F(x)}\,dx. \tag{18}$$

Differentiation of (17) w.r.t. time t gives approximate solution for velocity as

$$\frac{u(r_0,t)}{u_s(r_s)} = \frac{\beta}{(1-\beta)\lambda^\alpha}\int_0^x \frac{b(1+G)-G(b+(\alpha+1)/\gamma)x^{\alpha+1}}{F^{1+\gamma}(x)}\,x^{(\alpha-b)}dx. \tag{19}$$

For given values of n, α and γ the unknown parameter β is calculated by iterating (18). Then the approximate analytical solutions for pressure, density and velocity are obtained as functions of λ from (14), (15) and (19) respectively with the help of (17).

3. Numerical solutions

For comparison with the approximate solutions studied in § 2, we present here the numerical solutions. Details of the method employed may be found in Wilson and Turcotte [7] and Barenblatt *et al* [1]. We take the similarity transformation as

$$u = \frac{r}{t} V(\lambda), \quad \rho = \rho_0 R(\lambda), \quad p = \rho_0 (r/t)^2 P(\lambda),$$

$$X = R/\gamma P, \quad \lambda = r/r_s. \tag{20}$$

Using transformation (20), the basic equations of the flow behind the shock-wave may be written as

$$\frac{dX}{dV} = \frac{X}{(V-\delta)} \frac{(\gamma-1)[V(V-1)(V-\delta)X - (\alpha+1)(V-K)] + \{[(\alpha+1)(\gamma-1)+2]V-2\}[1-(V-\delta)^2 X]}{V(V-1)(V-\delta)X - (\alpha+1)(V-K)}, \tag{21}$$

$$\frac{d\ln\lambda}{dV} = \frac{1-(V-\delta)^2 X}{V(V-1)(V-\delta)X + (\alpha+1)(K-V)}, \tag{22}$$

$$\frac{d\ln R}{d\ln\lambda} = \frac{(\alpha+1)V}{(\delta - V)} - \frac{V(V-1)(V-\delta)X - (\alpha+1)(V-K)}{(V-\delta)[1-(V-\delta)^2 X]}, \tag{23}$$

where

$$K = 2(1-\delta)/(\alpha+1)\gamma. \tag{24}$$

Equations (20) together with the shock conditions give an expression for the Hugoniot curve as

$$X_s = [\gamma V_s(\delta - V_s)]^{-1}. \tag{25}$$

The integral curve in the X, V phase plane can be obtained by integrating (21). Each of these curves are associated with a saddle point singularity $X = 0$ and $V = K$ which corresponds to the centre of the wave. Using the Runge-Kutta method, (21) is integrated from the singular point with the slope obtained from L'Hospital rule to the point of intersection with the Hugoniot curve given by (25). This point of intersection gives the β value as the shock conditions are satisfied here. Once the integral curve is known, it is straightforward to obtain the dependence of V and R on λ from (22) and (23) by using quadrature. The accuracy of solutions thus obtained can be checked from the mass conservation integral

$$(\alpha+1)\int_0^1 R\lambda^\alpha \, d\lambda = 1. \tag{26}$$

4. Results and discussion

For finding out the approximate solutions given by (14), (15) and (19) for the self-similar flows behind strong shocks propagating in a uniform medium at rest, we first find out the values of β by iterating (18) for different values of n and γ for $\alpha = 1$ and $\alpha = 2$. The β values are also found numerically from (21) for the same cases and are listed in table 1. It is observed that the shock density ratio β increases with increase in n and γ. The approximate solutions and also numerical solutions are plotted in figures 1 to 4 by broken lines and solid lines respectively. One can find a very close agreement between

Table 1. β values vs n for different values of α and γ.*

n	β			
	$\alpha = 2, \gamma = 1{\cdot}4$	$\alpha = 2, \gamma = 5/3$	$\alpha = 1, \gamma = 1{\cdot}4$	$\alpha = 1, \gamma = 5/3$
$-0{\cdot}5$	0·086189 (0·087433)	0·174868 (0·171620)	0·050362 (0·052919)	0·133208 (0·136819)
0·0	0·168439 (0·166661)	0·254260 (0·250000)	0·164140 (0·166666)	0·248125 (0·250000)
0·5	0·258165 (0·255659)	0·338857 (0·334503)	0·296652 (0·298844)	0·370745 (0·373269)
1·0	0·353179 (0·350983)	0·426296 (0·422678)	0·431329 (0·434876)	0·492289 (0·496448)
1·5	0·417732 (0·416032)	0·513874 (0·511409)	0·555953 (0·557539)	0·606247 (0·607381)

*Values in brackets are from numerical solutions.

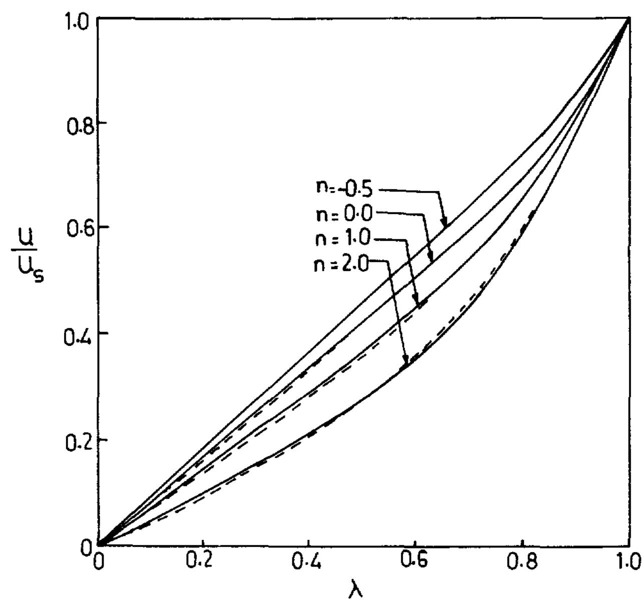

Figure 1. Velocity distribution for $\gamma = 1{\cdot}4$, $\alpha = 2$.

these solutions. The effect of the adiabatic index γ on the approximate solutions is shown in figure 5 for $n = 5/3$ and $\alpha = 2$ which corresponds to the constant power input (Ranga Rao and Ramana [5]). In figure 6 the dependency of non-dimensional energy deposition at the wave-front q on n is shown from which one can conclude that the rate of deposition of energy at the front decreases as n increases.

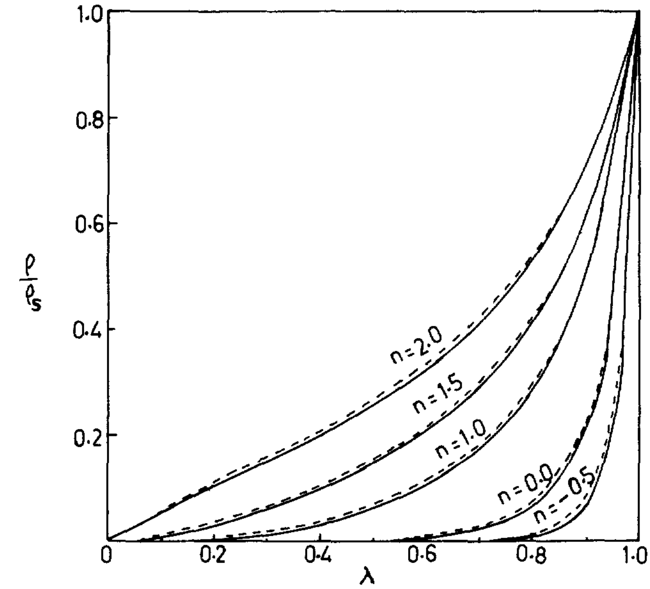

Figure 2. Density distribution for $\gamma = 1{\cdot}4$, $\alpha = 2$.

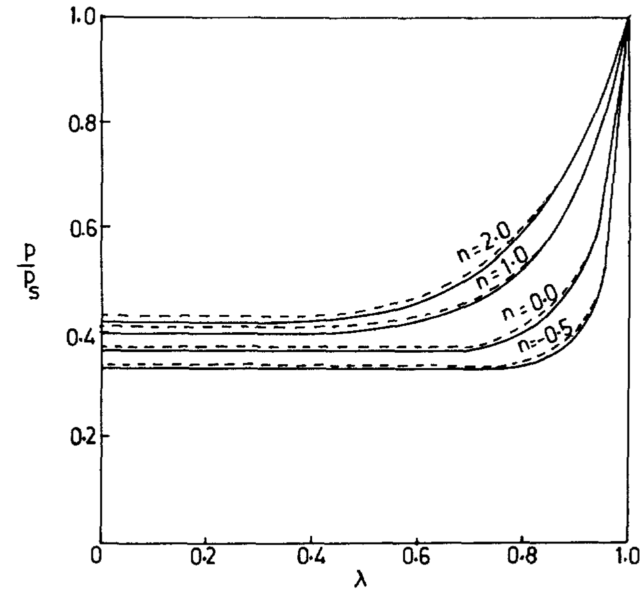

Figure 3. Pressure distribution for $\gamma = 1{\cdot}4$, $\alpha = 2$.

Appendix

Taylor's expansion of the Eulerian co-ordinate $r(r_0, t)$ is written as

$$r(r_0, t) = r_s + \left.\frac{\partial r}{\partial r_0}\right|_s (r_0 - r_s) + 1/2 \left.\frac{\partial^2 r}{\partial r_0^2}\right|_s (r_0 - r_s)^2 + \ldots \tag{A.1}$$

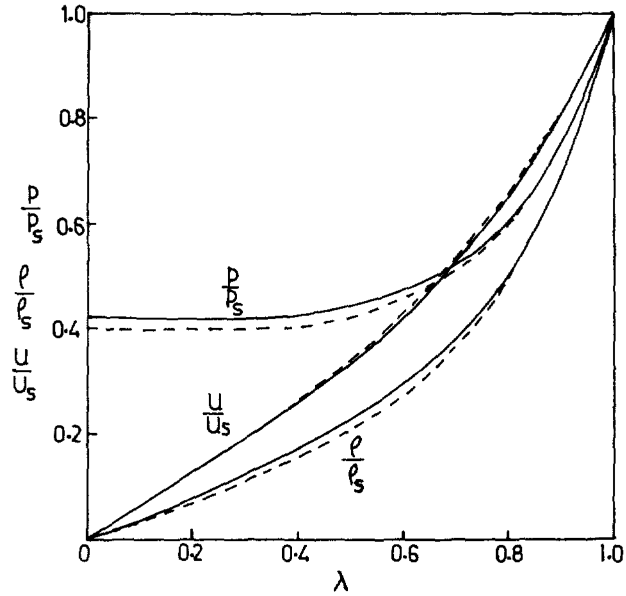

Figure 4. Flow distribution for $\gamma = 1{\cdot}4$, $\alpha = 1$, $n = 1$.

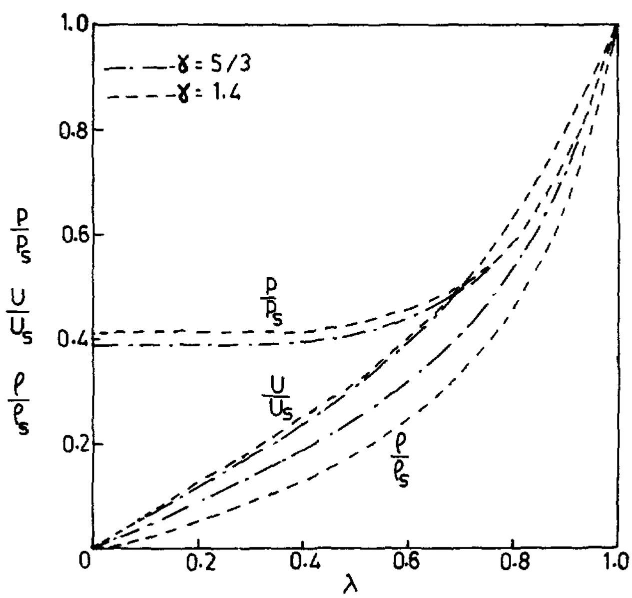

Figure 5. Approximate solutions for $n = 5/3$, $\alpha = 2$.

The equation of continuity (1) and the definition of β yield

$$\left.\frac{\partial r}{\partial r_0}\right|_s = \left.\frac{\rho_0 r_0^\alpha}{\rho r^\alpha}\right|_s = \beta, \tag{A.2}$$

$$\left.\frac{1}{\rho}\frac{\partial \rho}{\partial r_0}\right|_s = \frac{\alpha(1-\beta)}{r_s} - \left.\frac{1}{\beta}\frac{\partial^2 r}{\partial r_0^2}\right|_s. \tag{A.3}$$

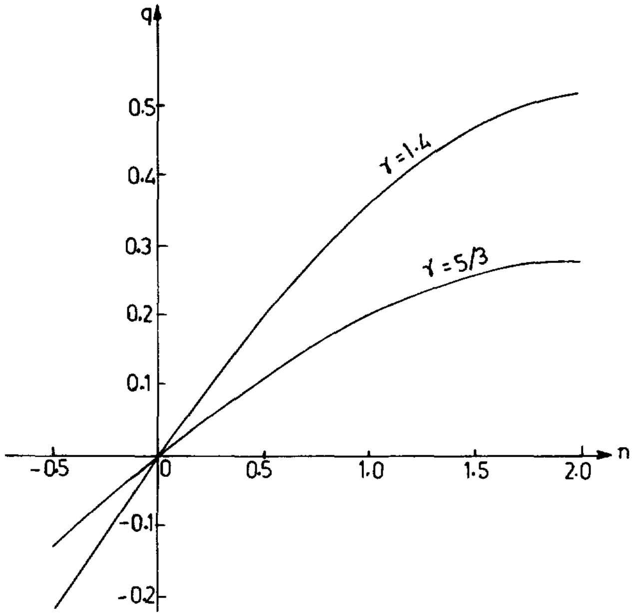

Figure 6. Non-dimensional energy deposition at the wave-front q vs n for $\alpha = 2$.

It follows from equations (A.1) and (A.2) that

$$\left.\frac{\partial r}{\partial t}\right|_s = (1-\beta)\dot{r}_s, \tag{A.4}$$

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_s = (1-\beta)\ddot{r}_s + \left.\frac{\partial^2 r}{\partial r_0^2}\right|_s \dot{r}_s^2. \tag{A.5}$$

Now equation (2) together with the shock condition (6) gives

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_s = -(1-\beta)\dot{r}_s^2\left(\frac{1}{p}\frac{\partial p}{\partial r_0}\right)_s. \tag{A.6}$$

Eliminating $(\partial^2 r/\partial t^2)|_s$ between (A.5) and (A.6) and also making use of (A.3), (5) and (6), we get $(\partial^2 r/\partial r_0^2)|_s$ which after using in (A.5), we get finally

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_s = \frac{(1-\beta)}{\gamma(1-\beta)-\beta}\left[\{\gamma(1-\beta)+2\beta\}\ddot{r}_s + \alpha\gamma\beta(1-\beta)\,(\dot{r}_s^2/r_s)\right]. \tag{A.7}$$

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