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Velocity and temperature distribution in a system consisting of a fluid layer overlying a layer of porous medium

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Abstract. The velocity and temperature distribution in a system consisting of a fluid layer overlying a layer of porous medium is investigated in the presence of buoyancy and surface tension forces. The analysis indicates that buoyancy and surface tension forces are additive and aid the flow by counteracting the effect of permeability. The temperature distribution is sensitive to the variation of the aspect ratio, and conductivities of the media. Further, the inclusion of the viscous dissipation term markedly affects the temperature distribution.

Keywords. Permeable; Navier-Stokes equation; Brinkman model; Boussinesq approximation; slip parameter; surface tension; coupled flow.

1. Introduction

Convection, forced and/or free, in a porous medium saturated with the same fluid is of interest in geophysics, oil recovery techniques, cryogenic industry, space applications, cooling problems and thermal insulation in high temperature reactor technology. The study of flow past and/or through the porous medium has therefore been attracting increasing attention. In recent years there has been a copious literature dealing with the different aspects of the above problem.

Nield [8] investigated the problem of onset of convection in a fluid layer overlying a layer of porous medium and discussed the effect of surface tension in inhibiting the convective instability. Chandrasekhara *et al* [4] investigated the convective instability in a fluid layer overlying a layer of porous medium using Brinkman momentum equation. They have shown that the viscous resistance term is dominant at low values of σ, the porous parameter, and thereby increases the critical Rayleigh-Darcy number. The present investigation is devoted to the study of simple one-dimensional coupled flows in a system consisting of a fluid layer overlying a layer of porous medium to understand the effect of buoyancy and the surface tension forces on the coupled flows when the temperature varies linearly in the direction of flow.

The governing equations of momentum and energy are solved separately for the fluid layer and for the porous medium and are matched at the nominal surface using the continuity conditions of velocity, temperature and heat flux. The solution for the porous medium is obtained by direct and singular perturbation methods. The analysis indicates that the effects of surface tension and buoyancy are additive. It is also observed that the viscous dissipation term is very dominant in affecting the temperature profiles.

2. Formulation of the problem and boundary conditions

The physical model (figure 1) consists of a fluid layer occupying the region $0 \leqslant y \leqslant h_1$, hereafter referred to as zone-I and a porous medium saturated with the same fluid occupying the region $-h_2 \leqslant y \leqslant 0$ hereafter referred to as zone-II. The fluid in zone-I and zone-II is driven by a constant pressure gradient applied in the streamwise direction. Buoyancy and surface tension forces are caused by the entry temperature having a constant temperature gradient in the flow direction. To analyse the problem we assume that Oberbeck-Boussinesq equations are valid in the fluid layer and Brinkman-Oberbeck-Boussinesq equations are valid in the porous medium. The necessity of using Brinkman-Oberbeck-Boussinesq equation in the porous medium instead of Darcy-Oberbeck-Boussinesq equation arises due to the fact that the Darcy equation is valid in regions far away from the boundaries and hence it is applied to large systems where the boundary layer flow does not affect the main body of the fluid. But in a system of finite width the existence of a boundary layer beneath the nominal surface (Beavers and Joseph [1]) due to the presence of slip velocity and the presence of a rigid wall dictate the necessity of boundary layer type of equation of the Brinkman form (Brinkman [2]; Tam [10]; Lundgren [6]; Prabhamani and Rudraiah [9]; Chandrasekhara and Vortmeyer [3]).

To derive the basic equations, we make the following assumptions. (i) The flow is steady and fully developed so that all the physical quantities except pressure and temperature are functions of y only. (ii) The porous medium is homogeneous and isotropic so that its permeability K is constant. (iii) The thickness of the fluid layer is the same as that of the porous layer (*i.e.*, $h_1 = h_2 = h$). The basic equations valid in zone-I and zone-II are:

ZONE-I

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}, \tag{1}$$

$$\partial p/\partial y = -\rho g, \tag{2}$$

$$\rho = \rho_0[1 - \beta(T'_1 - T_A)], \tag{3}$$

$$\rho_0 c_p u_1 \frac{\partial T'_1}{\partial x} = k_1 \frac{\partial^2 T'_1}{\partial y^2} + \mu(\partial u_1/\partial y)^2. \tag{4}$$

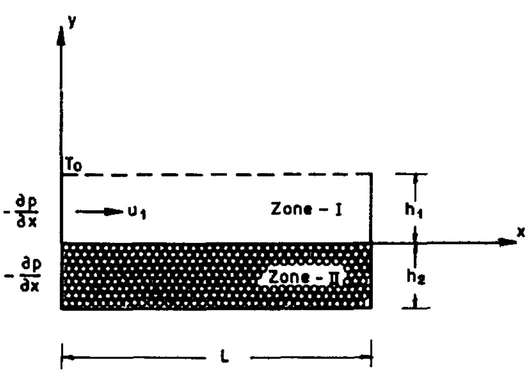

Figure 1. Physical model.

ZONE-II

$$\partial^2 u_2/\partial y^2 - u_2/K = 1/\mu\, \partial p/\partial x, \tag{5}$$

$$\partial p/\partial y = -\rho g, \tag{6}$$

$$\rho_0 c_p u_2 \frac{\partial T_2'}{\partial x} = k_2 \frac{\partial^2 T_2'}{\partial y^2} + \mu\left(\frac{\partial u_2}{\partial y}\right)^2, \tag{7}$$

where u_1, u_2 = velocity in the axial direction; p = pressure; ρ = density; T_1', T_2' = temperature; μ = viscosity; c_p = specific heat at constant pressure; K = permeability; k_1 = thermal conductivity of the fluid; k_2 = average conductivity of the porous medium and is defined by $\varepsilon k_1 + (1-\varepsilon)k_s$; k_s = conductivity of the solid; ε = porosity; ρ_0 = density at $T' = T_A$; β = thermal expansion coefficient; T_A = ambient temperature; g = acceleration due to gravity.

Eliminating the pressure p in (1) using (2) and (3) we get

$$\frac{\partial^3 u_1}{\partial y^3} = \frac{g\beta}{\nu}\frac{\partial T_1'}{\partial x}. \tag{8}$$

Equation (8) describes the fully developed flow only when the right side is independent of x *i.e.*, heat flux in the direction of flow must be constant. To satisfy this condition the temperature T' should be of the form

$$T_1'(xy) = Ax + T_1(y), \tag{9}$$

where $T_1(y)$ is the entrance temperature and A is the axial temperature gradient. Further, we impose the condition that

$$T_1'(0, h) = T_1(h) = T_0. \tag{9a}$$

Similarly the momentum equation in zone-II takes the form

$$\frac{\partial^3 u_2}{\partial y^3} - \frac{1}{K}\frac{\partial u_2}{\partial y} = \frac{g\beta}{\nu}\frac{\partial T_2'}{\partial x}. \tag{10}$$

Introducing the following non-dimensional variables

$$u^* = u/Q, \quad T_1^* = T_1' - T_A/T_0 - T_A, \quad \sigma = h/\sqrt{K},$$

$$\eta = y/h, \quad T_2^* = T_2' - T_A/T_0 - T_A,$$

$$\bar{Q} = -\frac{h^2}{\mu}\frac{\partial p}{\partial x}, \quad x^* = x/L,$$

and removing the asterisks for convenience, the momentum and energy equation in zone-I and zone-II are written in the form:

ZONE-I

$$\partial^3 u_1/\partial \eta^3 = -N_0, \tag{11}$$

$$(\partial^2 T_1/\partial \eta^2) + p_1 \cdot E(\partial u_1/\partial \eta)^2 = P_{e1} a A' u_1. \tag{12}$$

ZONE-II

$$\frac{\partial^3 u_2}{\partial \eta^3} - \sigma^2 \frac{\partial u_2}{\partial \eta} = -N_0, \tag{13}$$

$$(\partial^2 T_2/\partial\eta^2) + P_2 \cdot E(\partial u_2/\partial\eta)^2 = P_{e2} a A' u_2, \tag{14}$$

where $\Delta T = T_0 - T_A$, T_A = ambient temperature; A' = nondimensional temperature gradient, $\partial T/\partial x$; P_{e1} and P_{e2} = Pecklet numbers, $\bar{Q}h/k_1^*$ and $\bar{Q}h/k_2^*$; k_1^* and k_2^* = thermal diffusivity, $(k_1/\rho_0 c_p,\ k_2/\rho_0 c_p)$; P_1 and P_2 = Prandtl numbers, $(\nu\rho c_p/k_1, \nu\rho c_p/k_2)$; E = Eckert number, $(\bar{Q}^2/\Delta T \cdot c_p)$; $N_0 = (g\beta h^3/\bar{Q}\nu)(\Delta T/L)(-\partial T/\partial x)$.

The values of P_{e1}, P_{e2}, P_1 and P_2 depend on the k_1 and k_2 values.

In the above nondimensionalised equations we have chosen $\bar{Q} = -(h^2/\mu)(\partial p/\partial x) = Q\sigma^2$ as the reference velocity, which is the product of Darcy velocity and the square of the porous parameter σ, since both zones are subjected to same horizontal pressure gradient. The length of the porous bed L is taken as the reference length in the X-direction.

3. Boundary conditions

To solve the above equations proper boundary conditions on velocity and temperature have to be specified. The velocity boundary conditions are:

$$\partial u_1/\partial\eta = M_0 \quad \text{at} \quad \eta = 1, \tag{15}$$

$$\partial^2 u_1/\partial\eta^2 = -1 \quad \text{at} \quad \eta = 0, \tag{16}$$

$$\partial u_1/\partial\eta = \alpha\sigma[u_B - (1/\sigma^2)] \quad \text{at} \quad \eta = 0, \tag{17}$$

$$u_2 = u_B \quad \text{at} \quad \eta = 0, \tag{18}$$

$$(\partial^2 u_2/\partial\eta^2) - \sigma u_2 = -1 \quad \text{at} \quad \eta = 0, \tag{19}$$

$$u_2 = 0 \quad \text{at} \quad \eta = -1, \tag{20}$$

where

$M_0 = M/P_{e1}(-\partial T_1/\partial x)$; modified Marangoni number,

$M = -\dfrac{\sigma_T}{\mu}\dfrac{h \cdot \Delta T}{k}$; Marangoni number,

$\Delta T = (T_0 - T_A)$.

Boundary condition (15) represents the continuity of tangential stress at the upper free surface. Equations 1 and (19) express the fact that the flow in zone-I and zone-II is driven by the same pressure gradient. Equations (17) and (18) represent the slip boundary conditions of Beavers and Jospeh [1]. Equation (20) is the no-slip condition.

The temperature boundary conditions are:

$$T_1 = 1 \quad \text{at} \quad \eta = 1, \tag{21}$$

$$\partial T_1/\partial\eta = B(\partial T_2/\partial\eta) \quad \text{at} \quad \eta = 0, \tag{22}$$

$$T_1 = T_2 = T_B \quad \text{at} \quad \eta = 0, \tag{23}$$

$$T_2 = 0 \quad \text{at} \quad \eta = -1, \tag{24}$$

where

$$B = k_2/k_1.$$

Equation (21) is obtained from (9) and it specifies the entry temperature at $\eta = 1$ and $x = 0$. Equation (22) expresses the continuity of heat flux at the nominal surface; and (23) expresses the continuity of temperature at the nominal surface.

3.1. Velocity distribution

In this section analytical expressions for velocity distribution in zone-I and zone-II are obtained and its behaviour is discussed.

The expression for the velocity distribution for zone-I using (11), (15), (16) and (17) is obtained in the form

$$u_1 = -N_0\frac{\eta^3}{6} - \frac{\eta^2}{2} + \left(M_0 + \frac{N_0}{2} + 1\right)\eta + u_B \tag{25}$$

and

$$u_B = \frac{M_0 + (N_0/2) + 1}{\alpha\sigma} + \frac{1}{\sigma^2} \tag{26}$$

where u_B is the slip velocity at the nominal surface.

To understand the influence of slip velocity, buoyancy and surface tension on flow characteristics it is useful to calculate the mass flow rate in the fluid layer. If M_p denotes the nondimensional mass flow rate per unit width, then

$$M_p = \int_0^1 u_1\,d\eta = \frac{5N_0}{24} + \frac{1}{3} + \frac{M_0}{2} + u_B. \tag{27}$$

On the other hand, if the fluid layer is bounded by impermeable walls the mass flow rate M_l is given by

$$M_l = (5N_0 + 8)/24. \tag{28}$$

It may be noted that the net flow M_l in impermeable walls can be stopped by setting $5N_0 = -8$. The ratio of M_p/M_l can be calculated for the conditions of equal pressure gradient and equal thickness of layers from (28) and (29) and has the form

$$\frac{M_p}{M_l} = 1 + \frac{12M_0}{5N_0 + 8} + \frac{[\sigma(M_0 + (N_0/2) + 1) + \alpha]24}{\alpha\sigma^2(5N_0 + 8)} \tag{29}$$

M_p/M_l ratios are evaluated for different values of σ, M_0 and N_0 and is represented in figure 2. It is observed that for low values of σ increase in M_0 the mass flow rate increases by large amounts. For example, $M_0 = N_0 = 1$, $\sigma = 2$ and $\alpha = 0{\cdot}1$, $M_p/M_l = 25{\cdot}4645$, and $M_0 = 5$, $N_0 = 1$, $\sigma = 2$, $\alpha = 0{\cdot}1$, $M_p/M_l = 66{\cdot}0849$. From the above results we can see that surface tension effect is very dominant in increasing the mass flow rate. Hence to obtain greater mass flow rate M_0 should be large. As $\sigma \to \infty$ the ratio M_p/M_l asymptotically approaches the value of,

$$\frac{M_p}{M_l} = 1 + \frac{12M_0}{5N_0 + 8}.$$

The velocity distribution for zone-II is obtained both by direct and singular perturbation methods. The solution obtained by direct method is of the form,

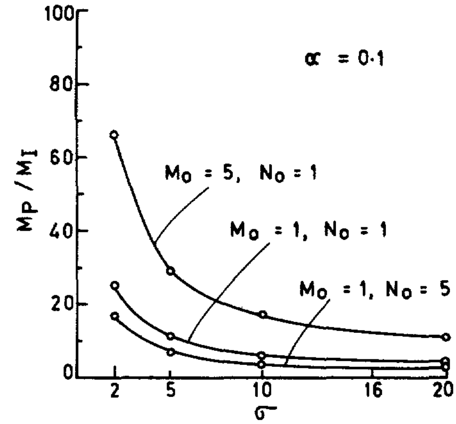

Figure 2. Mass flow rate *vs* porous parameter.

$$u_2 = \frac{1}{\sigma^2}\left[(u_B\sigma^2 - 1)\cosh\sigma\eta + \frac{\sinh\sigma\eta}{\sinh\sigma}\{(u_B\sigma^2 - 1)\cosh\sigma - N_0 + 1\} + N_0\eta + 1\right]. \quad (30)$$

The above solution is valid for arbitrary values of σ. The boundary layer type of solution valid for large values of σ is obtained by singular perturbation method and it is discussed below.

Dividing (13) by σ^2 and writing $1/\sigma^2$ equal to ε^2 we obtain

$$\varepsilon^2\frac{\partial^3 u_2}{\partial\eta^3} - \frac{\partial u_2}{\partial\eta} = -N_0\varepsilon^2. \quad (31)$$

Assuming that, in the limit $\varepsilon \to 0$; $N_0\varepsilon^2$ remains finite, we find that the highest derivative of (31) is lost and the order of the equation reduces from third order to first order. This reduced order equation cannot satisfy all the boundary conditions and hence its solution is not uniformly valid in the entire region of interest. Therefore, we develop inner and outer solutions valid in their respective domins of validity and match them in the overlap domain.

3.2. *Outer solution*

In the limit σ tends to zero (31) reduces to

$$\partial u_2/\partial\eta = N, \quad (32)$$

where $N = N_0\varepsilon^2$.

The solution of (32) is given by

$$u_2 = N\eta + c. \quad (33)$$

Imposing the outer boundary conditions $u_2 = 0$ at $\eta = -1$ we obtain the outer solution in the form

$$u_2(0) = N(\eta + 1). \quad (34)$$

Solution (34) does not satisfy all the boundary conditions and hence is not uniformly valid in the entire flow region. In order to get a uniformly valid solution we develop the inner solution by introducing the stretching variable $\eta^* = \eta/\varepsilon$. Expressing (13) in terms of η^* we have

$$\frac{\partial^3 u_2(i)}{\partial \eta^{*3}} - \frac{\partial u_2(i)}{\partial \eta^*} = -N\varepsilon. \tag{35}$$

Equation (35) in the limit $\varepsilon \to 0$ reduces to the form

$$\frac{\partial^3 u_2(i)}{\partial \eta^{*3}} - \frac{\partial u_2(i)}{\partial \eta^*} = 0. \tag{36}$$

The solution of (36) satisfying the boundary conditions $u_2 = u_B$ at $\eta^* = 0$ is

$$u_2(i) = u_B - D + De^{\eta^*}.$$

In the above solution, the constant D is still undetermined and is obtained by matching the inner and outer solutions in the overlapping domain. The matching condition after Cole (5) and Lagerstrom and Casten [7] is

$$u_2(0) = u_2(i)(-\infty). \tag{37}$$

The inner solution after matching is

$$u_2(i) = N + (u_B - N)e^{\eta^*}. \tag{38}$$

The uniform solution valid in the region of interest is then given by

$$u_2 = \frac{N_0}{\sigma^2}(\eta + 1) + \left(u_B - \frac{N_0}{\sigma^2}\right)e^{\eta\sigma}. \tag{39}$$

Equation (39) satisfies both the boundary conditions at 0 and -1 except for a transcendentally small term $(u_B - N_0/\sigma^2)e^{-\sigma}$. From (39) it is seen that the thickness of the boundary layer is of the order $\eta = 1/\sigma$ and is not provided by the solution (30) obtained by the direct method. The two solutions (30) and (39) are in excellent agreement for $\sigma = 5$.

To illustrate the effects of buoyancy, surface tension and porous parameter σ, on the velocity distribution, expressions (25), (30) and (39) are numerically evaluated for different values of α, σ, N_0 and M_0 and the results are represented in figures 3 and 4. It is observed that for low values of σ, when M_0 and N_0 are kept constant, the magnitude of velocity is large. However, as σ is increased the velocity is greatly reduced in magnitude due to Darcy resistance term and the shape of the profiles appears to be same. Hence, the permeability is the dominant factor in controlling the flow. For a given value of σ increase in M_0 and N_0 the velocity increases. This shows that both buoyancy and surface tension forces are additive and aid the flow. These two forces thereby counteract the effect of permeability. When M_0 and N_0 are negative there is a back flow in both the zones which is not represented in the figure.

3.3. *Temperature distribution*

The expressions for the temperature distribution in zone-I and zone-II are obtained by solving (12) and (14) using the boundary conditions (21)–(24), in the following form:

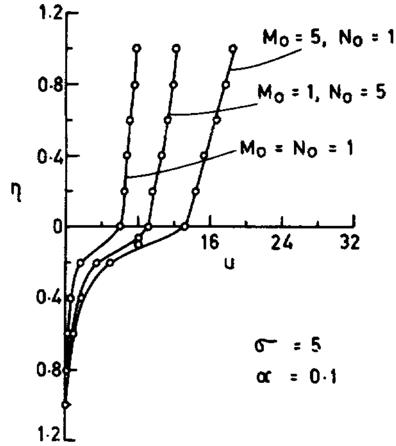

Figure 3. Velocity distribution *vs* η.

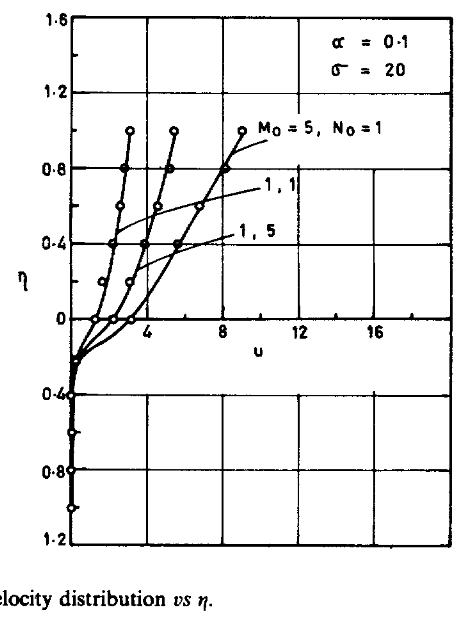

Figure 4. Velocity distribution *vs* η.

Zone-I

$$T_1 = P_{e1} a A' \left(-\frac{N_0 \eta^5}{120} - \frac{\eta^4}{24} + \frac{b_1 \eta^3}{6} + \frac{u_B \eta^2}{2} \right)$$
$$- P_l E \cdot \left(\frac{N_0^2 \eta^6}{120} + \frac{\eta^4}{12} + \frac{b_1^2 \eta^2}{2} + \frac{N_0 \eta^5}{20} - \frac{b_1 \eta^3}{3} - \frac{N_0 \eta^4}{12} b_1 \right) + E\eta + F, \quad (40)$$

where

$$b_1 = (M_0 + (N_0/2) + 1)$$
$$E = (B - BC_1 + BC_2 - BC_3 + BC_4 + BC_7 - BC_8 + BC_5 - BC_6)/(B+1)$$
$$F = (1 - C_1 + C_2 - BC_5 + BC_6 - BC_7 + BC_8 + BC_3 - BC_4)/(B+1).$$

ZONE-II

$$T_2 = P_{e2} aA' \left\{ \frac{N_0}{\sigma^2}\left(\frac{\eta^3}{6} + \frac{\eta^2}{2}\right) + \frac{b_2 e^{\eta\sigma}}{\sigma^2} \right\} - P_2 \cdot E \left(\frac{N_0^2}{\sigma^4}\frac{\eta^2}{2} + \frac{b_2^2 e^{2\eta\sigma}}{4} + \frac{2N_0 b_2 e^{\eta\sigma}}{\sigma^3} \right) + G\eta + H, \tag{41}$$

where

$$b_2 = (u_B - N_0/\sigma^2),$$
$$G = (1 - C_1 + C_2 - C_3 + C_4 - BC_5 + BC_6 + C_7 - C_8)/(B+1),$$
$$H = (1 - C_1 + C_2 - C_3 + C_4 - BC_5 + BC_6 - BC_7 + BC_8)/(B+1).$$

The constants in the above equations are defined below:

$$C_1 = \left(-\frac{N_0}{120} - \frac{1}{24} + \frac{b_1}{6} + \frac{u_B}{2} \right) P_{e1} aA',$$
$$C_2 = \left(\frac{N_0^2}{120} + \frac{1}{12} + \frac{b_1^2}{2} + \frac{N_0}{20} - \frac{b_1}{3} - \frac{N_0 b_1}{12} \right) P_1 \cdot E,$$
$$C_3 = \frac{b_2}{\sigma^2} P_{e2} aA'; C_4 = \left(\frac{b_2^2}{4} + \frac{2N_0 b_2}{\sigma^3} \right) P_2 \cdot E,$$
$$C_5 = \frac{b_2}{\sigma} P_{e2} aA'; C_6 = \left(\frac{b_2 \sigma}{2} + \frac{2N_0 b_2}{\sigma^2} \right) P_2 \cdot E,$$
$$C_7 = \frac{N_0}{3\sigma^2} P_{e2} aA'; C_8 = \frac{N_0}{2\sigma^4} P_2 \cdot E.$$

To evaluate the temperature distribution (presented in figures 5–8) we have considered the specific cases of thermal conductivities in porous media involving glass spheres and water ($B = 1{\cdot}25$, $k_1 = 0{\cdot}0014$, $k_2 = 0{\cdot}00176$) and steel spheres and water ($B = 62$, $k_1 = 0{\cdot}0014$, $k_2 = 0{\cdot}08696$). Figures 5 and 6 represent the temperature distribution for $P.E. = 0$ *i.e.* without dissipation. It is observed that increase in the conductivity value (B) has an appreciable effect on the temperature distribution and the profiles tend to be parabolic in nature. The same trend is observed by keeping B constant and varying the aspect ratio a. Figures 7 and 8 represent the temperature distribution in the presence of viscous dissipation. It is interesting to observe that the temperature profiles are completely reversed and the temperature at the nominal surface increases with increase in M_0, N_0 values. However, the influence of M_0 on the temperature rise is more compared to N_0. In other words, the surface tension is more dominant in increasing the temperature than buoyancy.

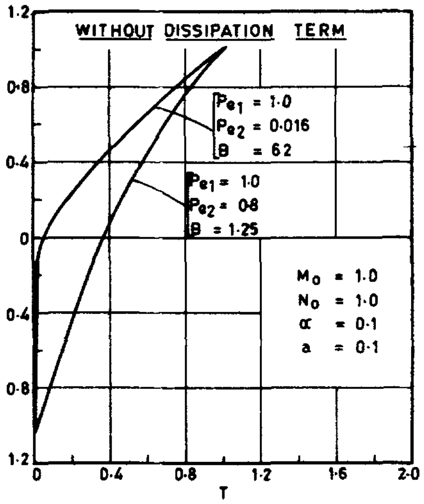

Figure 5. Temperature distribution *vs* η, without dissipation term.

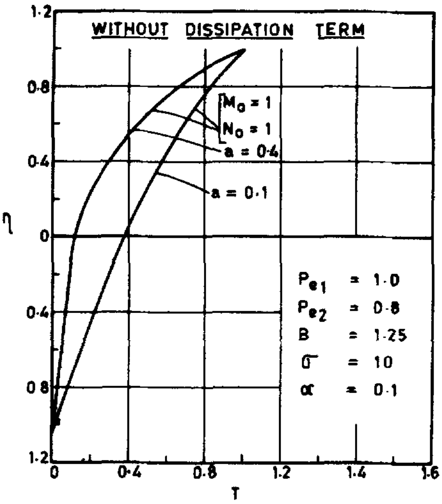

Figure 6. Temperature distribution *vs* η, without dissipation term.

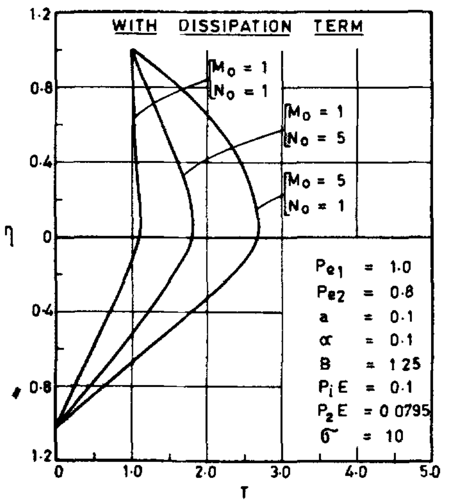

Figure 7. Temperature distribution *vs* η, with dissipation term.

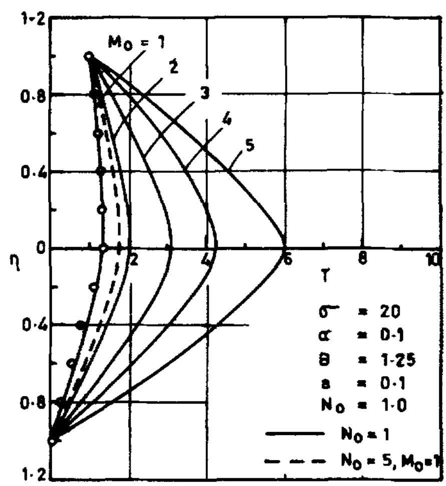

Figure 8. Temperature distribution *vs* η, with dissipation term.

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