Proc. Indian Acad. Sci. (Math. Sci.), Vol. 92, No. 1, September 1983, pp. 1–6.

On trade-off solution pairs in a special type of transportation problem

K K ACHARY and V G TIKEKAR

Department of Applied Mathematics, Indian Institute of Science, Bangalore 560012, India

MS received 1 June 1981

Abstract. In this paper, we have developed an algorithm to determine the trade-off solution pairs in a special type of transportation problem considering two objectives, *viz.*, cost and time. It is assumed that the time of transportation from an origin to a destination depends on the quantity transported. This results in a time objective, which is a piecewise constant increasing function. The cost objective function is taken to be linear. A potential physical situation of this model is given and a numerical example is worked out to illustrate the algorithm.

Keywords. Trade-off solution pairs; transportation problem; time objective.

1. Introduction

The time minimizing transportation problem (TMTP), also known as Bottleneck Transportation Problem, was first studied by Hammer [11, 12]. Garfinkel and Rao [9] proposed a threshold algorithm to solve TMTP. Szware [19] refined Hammer's algorithm and also suggested a revision of the proof of a result in Hammer's paper [11]. He also provided a thorough survey of the earlier East European literature on TMTP. Bhatia *et al* [3], Derigs and Zimmermann [6], Finke and Smith [7], Sharma and Swarup [16] provide various algorithms for solving TMTP. Seshan and Tikekar [15] pointed out an error in a step in the algorithm given by Sharma and Swarup [16]. Bhatia [5] has developed a method of solving zero-one time minimization transportation problem. A computational study of this problem was done by Foulds and Gibbons [8]. Isermann [13] presented an algorithm to enumerate all efficient solutions for a linear multiple objective transportation problem.

The fundamental assumption in a TMTP is that the time for transporting a certain amount of goods from a source to a destination is independent of the amount of goods transported. Achary and Seshan [1] replace the assumption by a more general and realistic assumption that the time of transportation over a route depends on the amount of shipment through that route. This assumption leads to a piecewise constant increasing time objective function.

When one considers a transportation problem with two objectives, say, cost and time, one fails to get an optimum solution satisfying both the objectives. In such cases, we see for a sequence of solutions termed as trade-off (efficient, Pareto-optimal, non-dominated) solutions. Methods of obtaining trade-off solutions have been developed by Glickman and Berger [10], Bhatia *et al* [4] and Srinivasan and Thompson [18]. Aneja and Nair [2] developed an algorithm to solve a Bicriteria Transportation Problem having both objective functions linear. Sharma and Swarup [17] provide an algorithm for obtaining trade-off solutions in a multi-dimensional transportation problem.

In this paper, we develop an algorithm for obtaining trade-off solutions in a special type of transportation problem. We consider the usual linear cost objective along with the time objective considered in Achary and Seshan [1]. Section 2 of this paper deals with the mathematical formulation of the problem and provides a potential physical situation in support of the mathematical model. In § 3, some definitions and the methodology of the algorithm are given followed by an algorithm. A numerical example, illustrating the algorithm, is worked out in § 4.

2. The problem

Consider a transportation problem with m sources and n destinations. Let a_i ($i = 1, 2, \ldots, m$) be the availability at source i, b_j ($j = 1, 2, \ldots, n$), the demand at destination j and U_{ij}, the capacity restriction over the route (i, j). The constraints of the problem then are

$$\sum_{j=1}^{n} x_{ij} = a_i \text{ for all } i. \tag{1}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \text{ for all } j. \tag{2}$$

$$0 \leqslant x_{ij} \leqslant U_{ij} \text{ for all } i, j. \tag{3}$$

The cost objective function is given by

$$\min C(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}. \tag{4}$$

Where c_{ij} is the unit cost of transportation over the route (i, j). We consider the time objective function

$$\min T(X) = \max \{t_{ij}(x_{ij})\} \tag{5}$$

where $t_{ij}(x_{ij})$ is defined in (6) below.

For each route (i, j) let the interval $[0, U_{ij}]$ be divided into subintervals as follows:

$$0 = \mu_{ij}^{0} < \mu_{ij}^{1} < \ldots < \mu_{ij}^{r(i,j)} = U_{ij}$$

and let $\quad t_{ij}^{1} < t_{ij}^{2} < \ldots t_{ij}^{r(i,j)}$.

Then

$$t_{ij}(x_{ij}) = \begin{cases} 0 & \text{if} \quad x_{ij} = 0 \\ t_{ij}^{k} & \text{if} \quad \mu_{ij}^{k-1} < x_{ij} \leqslant \mu_{ij}^{k} \\ & \quad (1 \leqslant k \leqslant r(i,j)) \end{cases} \tag{6}$$

It is not necessary that all the intervals $[0, U_{ij}]$ are divided into the same number of subintervals. Without loss of generality we can assume that $U_{ij} \leqslant \min(a_i, b_j)$ for all (i, j). We shall call the problem defined by (1) through (5) as problem P.

Such a problem can arise in a practical situation described below. Let a single mode of transportation be considered for the shipment of goods from any source i to a destination j with the unit cost of transportation c_{ij}. Suppose that there is a capacity limitation m_{ij} for the amount that can be transported from the ith source to the jth destination in a fixed time interval T_{ij}. Let t_{ij} be the time for transporting up to and including m_{ij} units from the ith source to the jth destination. Then $t_{ij}(x_{ij})$ has the form

$$t_{ij}(x_{ij}) = \begin{cases} 0 & \text{if} \quad x_{ij} = 0 \\ kT_{ij} + t_{ij} & \text{if} \quad km_{ij} < x_{ij} \leqslant (k+1)m_{ij}, k \geqslant 0 \end{cases} \tag{7}$$

If $x_{ij} = km_{ij} + y_{ij}$, $0 \leqslant y_{ij} < m_{ij}$, then transportation of m_{ij} units starts at each of the time points $0, T_{ij}, \ldots, (k-1)T_{ij}$ and the transportation of remaining y_{ij} units starts at time kT_{ij}. The cost objective is linear in this case. The function $t_{ij}(x_{ij})$ defined by (6) is a generalisation of $t_{ij}(x_{ij})$ considered in this example.

3. Definitions and methodology

3.1. *Definitions*

3.1a. *Definition* 1: T is said to be a feasible time for the problem P, if there exists a feasible solution X for the problem P, with $T(X) \leqslant T$. Otherwise T is said to be an infeasible time for the problem P.

3.1b. *Definition* 2: Let X be a feasible solution for P which is also cost optimal. Let $T(X) = T$ be the feasible time, and $C(X) = C$, the optimal cost. The pair (C, T) is called a solution pair.

3.1c. *Definition* 3: A solution pair (C, T) is said to be an efficient solution pair (trade-off solution pair, Pareto Optimal solution pair, nondominated solution pair) if there exists no other solution pair (C', T') such that (i) $C' < C$ and $T' \leqslant T$ or (ii) $C' \leqslant C$ and $T' < T$.

Given a time T, its feasibility can be checked as follows:

Define

$$c'_{ij} = \begin{cases} c_{ij} & \text{if} \quad t^k_{ij} \leqslant T \quad \text{for} \quad k \geqslant 1 \\ M \text{ if } t^1_{ij} > T \end{cases} \tag{8}$$

$$U'_{ij} = \begin{cases} \mu^k_{ij} & \text{if} \quad t^k_{ij} \leqslant T < t^{k+1}_{ij}, k \geqslant 1 \\ U_{ij} & \text{if} \quad t^1_{ij} > T. \end{cases} \tag{9}$$

Consider the following cost minimising transportation problem (CMTP):

$$\text{Minimize } Z(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c'_{ij} x^{ij}$$

subject to (1), (2) and

$$0 \leqslant x_{ij} \leqslant U'_{ij} \tag{10}$$

If the problem has a feasible solution with a finite optimal value Z, then the time T is feasible time. Otherwise it is an infeasible time.

3.2. *Methodology of the algorithm*

The algorithm starts with a cost optimal solution. $X_1 = (x_{ij})$ for the transportation problem defined by (1) through (4). The feasible time T_1 corresponding to this solution is obtained using (5) and solution pair (C_1, T_1) is recorded. Next, we determine a time $T' = \max_{i,j,k} \{t_{ij}^k | t_{ij}^k < T_1\}$ and check for its feasibility, using the technique discussed in 3.1. The details of solving the bounded transportation problem are given in Murty [14]. The feasible solution obtained with a finite optimal cost may have a feasible time smaller than T'. In that case, the better value is considered and the solution pair is recorded. This step is repeated until the algorithm establishes the infeasibility of a given time. At each iteration, the optimal solution of the previous iteration is taken as a starting solution for arriving at a feasible solution in the current iteration. The sequence of solution pairs obtained is examined for redundant solutions *i.e.* those solution which have the same optimal cost but different feasible times. Among such solution pairs, only those solution pairs having a smaller (better) feasible time are retained and the final sequence of trade-off solution pairs is recorded.

The algorithm terminates in a finite number of steps, since only a finite number of distinct T values are to be checked for their feasibility.

3.3. *The Algorithm*

STEP 1:

Set $r = 1$. Consider the cost minimising transportation problem in P. Let $X_1 = (x_{ij})$ be an optimal solution and $C(X_1) = C_1$, the optimal value. Find $T_1 = \max \{t_{ij}(x_{ij}) | x_{ij} > 0\}$. Record (C_1, T_1) and go to step 2.

STEP 2:

Set r to $r+1$. Find $T'_r = \max_{i,j,k} \{t_{ij}^k | t_{ij}^k < T_{r-1}\}$. Define c'_{ij} and U'_{ij} as in (8) and (9). Solve CMTP given in (10). If it has a finite optimal value go to step 3. Otherwise go to step 4.

STEP 3:

Let $X_r = (x_{ij})$ be an optimal solution with $C(X_r) = C_r$ and $T(X_r) = T_r$. Record (C_r, T_r) and go to step 2.

STEP 4:

The time T'_r is an infeasible time. Search the sequence of solution pairs for redundant solutions, if any, and eliminate them from the sequence.

4. Numerical example

Consider a numerical example with the data given in tables 1 and 2. The last column in table 1 gives the availabilities (a_i, $i = 1, 2, 3, 4$) and the last row gives the demands (b_j, $j = 1, 2, \ldots 5$). Every (i, j)th cell contains t_{ij}^k's and the corresponding intervals $(\mu_{ij}^{k-1}, \mu_{ij}^k)$. Table 2 gives the cost matrix (c_{ij}). In tables 3–5, the solutions corresponding to the trade-off solution pairs are shown. The upper bounds of the variables are shown,

Table 1. Basic data for the example.

						a_i
	8(0–25) 10(25–40) 12(40–55)	6(0–30) 9(30–60) 12(60–80)	8(0–15) 10(15–30)	9(0–20) 11(20–35)	11(0–25) 13(25–35) 15(35–50)	90
	6(0–20) 9(20–35)	5(0–25) 8(25–35)	6(0–15) 7(15–30)	6(0–20) 8(20–35)	7(0–20) 9(20–35)	35
	8(0–25) 11(25–40) 13(40–55)	7(0–20) 9(20–40) 10(40–60)	8(0–15) 10(15–30)	4(0–20) 6(20–35)	15(0–30) 17(30–50)	60
	6(0–30) 9(30–40) 11(40–55)	5(0–25) 7(25–45) 9(45–65)	5(0–15) 7(15–30)	13(0–20) 15(20–35)	11(0–30) 13(30–50)	65
b_j	55	80	30	35	50	

Table 2. Cost data for the example.

4	3	7	8	6
2	10	11	5	9
7	10	8	5	6
3	8	1	11	2

Table 3. Iteration 1.

(10)	(80)			
(35)				
			(35)	(25)
(10)		(30)		(25)

Table 4. Iteration 2.

	(55)			35 (35)
(20)				(15)
	(25)		(35)	M
(35)		(30)	20	30

Table 5. Iteration 3.

40 (5)	60 60			25 25
(10)				(25)
40 (5)	(20)		(35)	M
(35)		(30)	M	30

wherever necessary, at the right top corner of the corresponding cells. The values of the basic variables at a solution are bracketed.

ITERATION 1:

The cost minimizing problem is solved. The cost optimal solution X_1 is shown in table 3.

$$C(X_1) = C_1 = 785 \quad T(X_1) = T_1 = \max\{t_{ij}(x_{ij}) | x_{ij} > 0\} = 15.$$
$$(C_1, T_1) = (785, 15).$$

ITERATION 2:

$T'_2 = \max\{t^k_{ij} | t^k_{ij} < 15\} = 13$. The modified CMTP is solved. The cost optimal feasible solution is shown in table 4. $C(X_2) = 1110, T(X_2) = 13$ and the efficient pair is (1110, 13).

ITERATION 3:

$T'_3 = \max\{t^k_{ij} | t^k_{ij} < 13\} = 11$. The optimal solution of the modified CMTP is shown in table 5. $(C_3, T_3) = (1505, 11)$.

ITERATION 4:

$T'_4 = \max\{t^k_{ij} | t^k_{ij} < 11\} = 10$. At this iteration we find that 10 is an infeasible time. There are no redundant solutions. The sequence of efficient solution pairs is (785, 15), (1110, 13) (1505, 11).

References

[1] Achary K K and Seshan C R 1981 *Eur. J. Opnl. Res.* Vol. 7 (to appear)
[2] Aneja Y P and Nair K P K 1979 *Mgmt. Sci.* **25** 73
[3] Bhatia H L, Kanti Swarup and Puri M C 1977 *Indian J. Pure Appl. Maths.* **8** 920
[4] Bhatia H L, Kanti Swarup and Puri M C 1976 *Opsearch* **13** 130
[5] Bhatia H L 1979 NZOR **7** 159
[6] Derigs U and Zimmermann U 1979 *Computing* **22** 1
[7] Finke G and Smith P A 1979, *Methods Oper. Res.* **31** 185
[8] Foulds L R and Gibbons P B 1980 NZOR **8** 109
[9] Garfinkel R S and Rao M R 1971 *N.R.L.Q.* **18** 465
[10] Glickman T S and Berger P D 1977 *Oper. Res.* **25** 163
[11] Hammer P L 1969 *N.R.L.Q.* **16** 345
[12] Hammer P L 1971 *N.R.L.Q.* **18** 487
[13] Isermann H 1979 *N.R.L.Q.* **26** 123
[14] Murthy K G 1976 *Linear and Combinatorial Programming* (New York: John Wiley and Sons) Chapter 11
[15] Seshan C R and Tikekar V G 1980, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **89** 101
[16] Sharma J K and Kanti Swarup 1977 *Proc. Indian Acad. Sci.* (*Math. Sci.*) **A86** 513
[17] Sharma J K and Kanti Swarup 1979 *Opsearch* **16** 23
[18] Srinivasan V and Thompson G L 1976 *N.R.L.Q.* **23** 567
[19] Szwarc W 1971, *N.R.L.Q.* **18** 473