Proc. Indian Acad. Sci. (Math. Sci.), Vol. 91, Number 2, July 1982, pp. 155-165.

The wall jet flow of a conducting gas over a permeable surface in the presence of a variable transverse magnetic field

J L BANSAL, M L GUPTA and S S TAK
Department of Mathematics, University of Jodhpur, Jodhpur, India

MS received 8 May 1981; revised 10 February 1982

Abstract. The effects of the magnetic field, Mach number and the permeability parameter on the wall jet flow (radial or plane) of an electrically conducting gas spreading over a permeable surface have been investigated. Taking the Prandtl number of the fluid as unity and assuming a linear relationship between viscosity and temperature, it is found that similar solutions for the velocity distribution exist for a specified distribution of the normal velocity along the wall and the corresponding distribution of the transverse magnetic field. Previous non-magnetic flow results have been improved by adopting a new and simple transformation of variables.

Keywords. Boundary layer theory; magnetohydrodynamics; jet flow.

1. Introduction

The non-magnetic wall jet in a gas medium has been studied by means of similar solutions [7, 4, 9, 5, 2, 3]. Fox and Steiger [5] studied the fluid dynamic characteristics of a wall jet of an electrically non-conducting gas spreading out over a permeable plane surface in otherwise stationary ambient surroundings by posing an eigenvalue problem on the velocity field. The problem was solved numerically for various values of suction/injection parameter. They also obtained a particular solution of the temperature distribution assuming the Prandtl number of the fluid as unity. Bansal [1] reconsidered the work of Fox and Steiger [5] and extended it to the study of thermal boundary layer for arbitrary values of the Prandtl number. He found that similar solutions exist both for radial and plane wall jets, for a specified distribution of normal velocity along the wall.

In the present paper we have studied the effects of the magnetic field, Mach number and the permeability parameter on the wall jet (radial or plane) of an electrically conducting gas spreading over a permeable plane wall. Keeping the works of Fox and Steiger [5] and Bansal [1] in view, we have obtained the similar solution for the velocity field for various values of the suction/injection parameter α and magnetic parameter m. A linear relationship between viscosity and temperature has been assumed and the Prandtl number of the fluid is taken as unity. It is found that the similar solution exists for a specified distribution of the normal velocity along the wall and the corresponding distribution of the transverse magnetic field. The improved values of the results of Fox and Steiger have been obtained as a particular case.

One of the important conclusions of the present study is that the coefficient of skin-friction is independent of the Mach number but decreases with the increase in the value of the magnetic parameter m. It is noted that the effect of injection is to reduce the skin friction coefficient at all stations downstream as compared to the value for impermeable wall, whereas in the case of suction it first decreases and then increases monotonically downstream both for radial and plane jets.

2. Formulation of the problem

Let an electrically conducting gas at a temperature T_∞ be discharged through a small orifice (slit or circular) spreading out over a permeable plane surface in the presence of a variable transverse magnetic field and mix with an ambient gas being initially at rest having a temperature T_∞, so that the thermal boundary layer is formed only due to viscous dissipation and Joule heating.

Taking the origin in the orifice and the coordinate axes x and y along and normal to the plane wall respectively, the equation of state, continuity and the boundary layer approximations to the momentum and the energy equations are:

State :

$$p = \rho RT, \tag{1}$$

Continuity :

$$\frac{\partial}{\partial x}(\rho x^t u) + \frac{\partial}{\partial y}(\rho x^t v) = 0, \tag{2}$$

Momentum :

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \sigma_e B^2 u, \tag{3}$$

Energy :

$$\rho u \frac{\partial \bar{T}}{\partial x} + \rho v \frac{\partial \bar{T}}{\partial y} = \frac{1}{\sigma}\frac{\partial}{\partial y}\left(\mu \frac{\partial \bar{T}}{\partial y}\right) + \frac{\mu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma_e B^2}{c_p} u^2, \tag{4}$$

where

$$\bar{T} = T - T_\infty, \ \sigma = \frac{\mu c_p}{k} \text{ (Prandtl number)}, \tag{5}$$

The boundary conditions are as follows :

$$y = 0: \ u = 0, \ v = v_w(x); \ \bar{T} = 0 \ \text{(isothermal wall)},$$

$$\text{or} \quad \frac{\partial \bar{T}}{\partial y} = 0 \ \text{(adiabatic wall)}, \tag{6}$$

$$y = \infty : \frac{\partial u}{\partial y} = 0,\ \frac{\partial u}{\partial x} \to -\frac{\sigma_e B^2}{\rho},$$

$$\frac{\partial \bar{T}}{\partial y} = 0,\quad \frac{\bar{\partial} T}{\partial x} \to \frac{\sigma_e B^2}{\rho c_p} u; \tag{7}$$

where for injection $v_w(x) > 0$ and for suction $v_w(x) < 0$.

By the boundary layer approximation the pressure is uniform everywhere and hence the equation of state (1) implies that

$$\rho T = \text{constant} = \rho_\infty T_\infty. \tag{8}$$

3. Analysis

Introducing the stream function ψ, such that

$$\rho x^t u = \frac{\partial \psi}{\partial y} \text{ and } \rho x^t v = -\frac{\partial \psi}{\partial x}. \tag{9}$$

The solution of the coupled partial differential equations of § 2 is facilitated by taking a linear relationship between viscosity μ and temperature T, i.e.,

$$\frac{\mu}{\mu_\infty} = \frac{T}{T_\infty}. \tag{10}$$

Hence,

$$\rho\mu = \text{constant} = \rho_\infty \mu_\infty. \tag{11}$$

Following Levy–Lees [8], let us transform the independent variables (x, y) to (ξ, η), such that

$$\xi = \int_0^x \mu_\infty \rho_\infty u_r x^{2t} \mathrm{d}x, \tag{12}$$

$$\eta = \frac{\rho_\infty u_r x^t}{\sqrt{2\xi}} \int_0^y \frac{\rho}{\rho_\infty} \mathrm{d}y, \tag{13}$$

where the subscript r represents some reference value, yet to be prescribed, of the velocity component u in the boundary layer.

Taking the following forms of stream function and the temperature difference $\bar{T}$:

$$\psi = (2\xi)^{1/2} f_m(\eta), \tag{14}$$

$$\bar{T} = -\frac{u_r^2}{2c_p} h_m(\eta), \tag{15}$$

the momentum and energy equations reduce to

$$f_m''' + f_m f_m'' - 2\alpha f_m'^2 - 2\beta f_m' = 0, \tag{16}$$

and

$$h_m'' + \sigma (f_m h_m' - 4\alpha f_m' h_m) = 2\sigma (f_m''^2 + 2\beta f_m'^2), \tag{17}$$

respectively, with the boundary conditions

$$\eta = 0:\ f_w = f_w(0)\ \text{(say)},\ f'_w = 0;\ \ h_w = 0\ \text{(isothermal wall)},$$
$$\text{or } h'_w = 0\ \text{(adiabatic wall)}, \tag{18}$$
$$\eta \to \infty:\ f'_w \to -\frac{\beta}{\alpha};\ h_w \to \frac{\beta^2}{\alpha^2}$$

where a prime denotes differentiation with respect to η and α, β are given by

$$\alpha = \frac{\xi}{u_e}\frac{\mathrm{d}u_e}{\mathrm{d}\xi}\ \text{(wall permeability parameter)}, \tag{19}$$

and

$$\beta = \left(\frac{\xi}{\rho_\infty \mu_\infty x^{2i} u_e^2}\right)\frac{\sigma_e B^2}{\rho}. \tag{20}$$

The requirement of similar solution is that α and β must be independent of x. Taking α as constant, from (19) and (12) it follows that

$$\frac{u_e}{u_{e_0}} = \left(\frac{x}{x_0}\right)^{\alpha(2i+1)/1-\alpha},\ \alpha \neq 1. \tag{21}$$

Hence,

$$v_w(x) = -A(1-\alpha)^{-1/2} f_w(0)\left(\frac{x}{x_0}\right)^n, \tag{22}$$

where

$$A = \mu_w\left\{\frac{(2i+1)\,u_{e_0}}{2\rho_\infty \mu_\infty x_0}\right\}^{1/2} \text{ and } n = \frac{2\alpha(i+1)-1}{2(1-\alpha)}. \tag{23}$$

Therefore, $f_w(0) < 0$ (injection) and $f_w(0) > 0$ (suction). For β to be independent of x, B is assumed to be of the following form:

$$B = B_0\, x^{(2\alpha+2i\alpha-1)/2(1-\alpha)},\ \alpha \neq 1. \tag{24}$$

Hence,

$$\beta = \frac{m(1-\alpha)\rho_\infty}{\rho}, \tag{25}$$

where

$$m = \frac{x_0^{\alpha(2i+1)/(1-\alpha)}}{(2i+1)\,u_{e_0}}\frac{\sigma_e B_0^2}{\rho_\infty}\ \text{(magnetic interaction parameter)}. \tag{26}$$

A useful simplification is made by taking the Prandtl number of the conducting gas as unity. In such a case, it may easily be seen that

$$h_w = f'^2_w,\ \sigma = 1, \tag{27}$$

is a solution of the energy equation (17) both for isothermal and adiabatic walls. It is, therefore, required to solve the momentum equation only.

Hence,

$$\frac{T}{T_w} = 1 - \frac{u_e^2}{2c_p T_w} f'^2_w, \tag{28}$$

and

$$\frac{\rho_\infty}{\rho} = 1 - \frac{1}{2}(\gamma - 1) Ma^2 f_m'^2, \tag{29}$$

where the relationship

$$u_e^2 = (\gamma - 1) Ma^2 c_p T_\infty, \tag{30}$$

has been used. In general, Mach number is variable and shall violate the similarity requirements. However, in the present solution it has been taken as constant, an assumption which is usually made in the study of compressible boundary layers [10].

Therefore, the momentum equation (16), in view of (25) and (29), becomes

$$f_m''' + f_m f_m'' - 2a f_m'^2 - 2m(1-a)\left\{1 - \frac{(\gamma-1)}{2} Ma^2 f_m'^2\right\} f_m' = 0, \tag{31}$$

with the boundary conditions

$$\begin{aligned} \eta = 0 &: f_m = f_m(0), f_m' = 0, \\ \eta = \infty &: \left[\frac{m(1-a)}{a}\left\{1 - \frac{(\gamma-1)}{2} Ma^2 f_m'^2\right\} + f_m'\right] \to 0. \end{aligned} \tag{32}$$

In adiditon, the two-point boundary value problem posed by (32) implies the existence of an eigenvalue problem that is, for each value of a only one solution $f_m(\eta)$ with the correct asymptotic behaviour as $\eta \to \infty$ can be found. Therefore, each case of suction or injection is related to the eigenvalue a and requires a specific form of the reference velocity u_e and the normal velocity $v_w(x)$ along the wall for a given magnetic field. This means that a and $f_m(0)$ cannot be prescribed simultaneously. In fact one determines the other and it will be easy to prescribe a and determine $f_m(0)$, with the help of Lagrange's three-point interpolation formula, such that the condition at infinity is achieved.

Before we presctibe a, let us determine its range which is obtained by the following two expressions:

$$f_0(0) f_0''(0) = -2(a+1)\int_0^\infty f_0 f_0'^2 \, d\eta, \tag{33}$$

$$f_0''(0) = -(2a+1)\int_0^\infty f_0'^2 \, d\eta. \tag{34}$$

Equation (33) is obtained by multiplying (31) with f_0 and then integrating with respect to η between the limits 0 to ∞ and putting $m = 0$; whereas (34) is obtained by direct integration of (31) between the same limits and taking $m = 0$.

(i) For impermeable wall, since $f_0(0) = 0$, relation (33) gives $a = -1$ and $\int_0^\infty f_0 f_0'^2 \, d\eta = \text{constant}$, which is the same integral condition as was obtained by Akatnow and Glauert [6].

(ii) For $f_0''(0) = 0$ (blow-off condition), it is concluded from (34) that

$$a = -\tfrac{1}{2} \text{ and } \int_0^\infty f_0'^2 \, d\eta = \text{constant},$$

Further, (33) gives

$$\int_0^\infty f_0 f_0'^2 \, d\eta = 0.$$

Since $f_0'^2$ is always positive, it is implied in this case that f_0 must be asymmetric function varying from $f_0 = f_0(0) < 0$ to $f_0 = f_0(\infty) = 1$ as η varies from 0 to ∞. Hence, $\alpha = \frac{1}{2}$ implies injection at the permeable wall.

(iii) For $f_0''(0) < 0$ (region of back flow), since $\int_0^\infty f_0'^2 \, d\eta$ is always positive, it is concluded from (34) that $\alpha > -\frac{1}{2}$. As the boundary layer approximations breakdown in the region of back-flow, we shall limit ourselves to the value of $\alpha < -\frac{1}{2}$.

(iv) For $f_0''(0) > 0$, from (i), (ii) and (iii) it may be concluded that

$$-1 < \alpha \leqslant -\tfrac{1}{2} \text{ for } f_0(0) < 0 \text{ (injection)},$$

$$\alpha = -1 \text{ for } f_0(0) = 0 \text{ (impermeable wall)},$$

$$-\infty < \alpha < -1 \text{ for } f_0(0) > 0 \text{ (suction)}.$$

Finally, the range of α is

$$-\infty < \alpha \leqslant -\tfrac{1}{2}. \tag{35}$$

4. Transformation of variables

To simplify the analysis let us make the following transformation of variables :

$$f_m(\eta) = \{a\alpha(am-1)\}^{-1/2} F_m(\zeta) \tag{36}$$

and

$$\zeta = a^{-1/2}\{\alpha(am-1)\}^{1/2}\eta. \tag{37}$$

Hence,

$$f_m'(\eta) = a^{-1} F_m'(\zeta), \tag{38}$$

$$f_m''(\eta) = a^{-3/2}\{\alpha(am-1)\}^{1/2} F_m''(\zeta), \tag{39}$$

and (31) reduces to

$$\alpha(am-1) F_m''' + F_m F_m'' - 2\alpha F_m'^2$$
$$- 2am(1-\alpha)\left\{1 - \frac{(\gamma-1)Ma^2}{2a^2} F_m'^2\right\} F_m' = 0, \tag{40}$$

with the boundary conditions

$$\zeta = 0 : F_m = \{a\alpha(am-1)\}^{1/2} f_m(0), \; F_m' = 0, \; F_m'' = 1 \quad m > 0$$

$$\zeta = \infty : F_0(\infty) = (-a\alpha)^{1/2} ;$$

$$\left[\frac{am(1-\alpha)}{\alpha}\left\{1 - \frac{(\gamma-1)Ma^2}{2a^2} F_m'^2\right\} + F_m'\right] \to 0, \tag{41}$$

where a prime denotes differentiation with respect to ζ.

Here we have taken $F_m''(0) = 1$, which is permissible without loss of generality, due to the presence of a free constant a in (39).

4.1. *Determination of a*

The value of a, which is independent of m, can be easily obtained by taking $m = 0$ in (40) and (41).

The corresponding equation was solved by Fox and Steiger [5] by a very tedious method. A simple and accurate method, which we suggest, is the following: Prescribe a value of α and interpolate the value of $F_0(0)$ such that $F_0'(\infty) = 0$. This will give us a corresponding value of $F_0(\infty)$, which in turn will determine a, $f_0(0)$ and $f_0''(0)$ as follows:

$$a^{1/2} = (-\alpha)^{-1/2} F_0(\infty), \tag{42}$$

$$f_0(0) = F_0(0)/F_0(\infty), \tag{43}$$

$$f_0''(0) = \alpha^2/F_0^3(\infty). \tag{44}$$

The calculated values of a, $f_0(0)$ and $f_0''(0)$ for various values of α are given in table 1. One may see that this method gives more accurate results.

Now, we may think of the integration of equation (40) in the following manner: Prescribe the values of γ, Ma and m $(am < 1)$. For a chosen value of α, take the corresponding value of a from table 1 and interpolate $F_m(0)$ so that the given condition at infinity is achieved. The value of $f_m(0)$ is, now, given by

$$f_m(0) = \{a\alpha(am-1)\}^{-1/2} F_m(0), \tag{45}$$

and

$$f_m''(0) = a^{-3/2}\{\alpha(am-1)\}^{1/2}. \tag{46}$$

It may be noted here that $f_m''(0)$ is independent of Mach number; however, $f_m(0)$ will very much depend on it.

Table 1. Solution of the equation (40), $m = 0$ (initial values).

	α	a	$f_0(0)$ Present method	$f_0(0)$ Fox and Steiger [5]	$f_0''(0)$ Present method	$f_0''(0)$ Fox and Steiger [5]
Injection	−0·55	5·50806	−0·76139	−0·749	0·0948	0·061
	−0·60	3·87370	−0·61563	−0·591	0·10160	0·106
	−0·70	3·02650	−0·38746	−0·379	0·15890	0·159
	−0·75	2·88975	−0·29793	−0·298	0·1763	0·186
Impermeable wall	−1·0	2·72575	0·0	0·0	0·2222	0·222
Suction	−1·25	2·83350	0·17982	0·193	0·2344	0·235
	−1·5	3·03178	0·30500	0·305	0·23200	0·232
	−2·0	3·55905	0·46538	0·472	0·2106	0·208
	−5·0	7·24786	0·77604	0·776	0·1146	0·117
	−10·0	13·55271	0·88592	0·892	0·06338	0·0688
	−50·0	57·36662	0·9771	0·976	0·01627	0·0235

5. Transformation to the physical plane

Inverting the variables, we find that

$$y = \frac{\sqrt{2a\xi}}{\rho_w u_e x^i \sqrt{a(am-1)}}\left[\zeta - \frac{(\gamma-1)Ma^2}{2a^2}\int_0^{\zeta} F_m'^2 \, d\zeta\right] \tag{47}$$

and

$$\xi = \rho_\infty \mu_\infty u_e (1-a)\frac{x^{2i+1}}{(2i+1)}. \tag{48}$$

Thus we see that the effect of the index i (i.e., the nature of the orifice) is not felt in the transformed plane, it comes into play only when we switch over to the physical plane.

6. Coefficient of skin-friction

The shearing stress τ_w is given by

$$\tau_w = \left(\mu\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu_w \rho_w u_e^2 x^i}{\sqrt{2\xi}} f_m''(0). \tag{49}$$

Hence, the coefficient of skin-friction is obtained as

$$C_{f_m}(m,a) = \frac{\tau_w}{\frac{1}{2}\rho_\infty u_e^2}$$

$$= \left\{\frac{2(2i+1)\mu_\infty}{x_0 u_{e_0}\rho_\infty}\right\}^{1/2}\left(\frac{x}{x_0}\right)^{-(2i\alpha+1)/2(1-\alpha)}(1-a)^{-1/2} f_m''(0). \tag{50}$$

For impermeable wall ($a = -1$) and in the absence of magnetic field ($m = 0$), we have [6] :

$$[f_0''(0)]_{a=-1} = \frac{2}{9}. \tag{51}$$

Hence,

$$\frac{C_{f_m}(m,a)}{c_{f_0}(0,-1)} = \frac{9}{\sqrt{2}}\left(\frac{x}{x_0}\right)^{-(2i+1)(1+a)/4(1-a)}(1-a)^{-1/2} f_m''(0). \tag{52}$$

7. Numerical discussion

The velocity distribution, in the absence of magnetic field (i.e., $m = 0$) and for various values of the permeability parameter a, is plotted against the transformed similarity variable ζ in figure 1. It is noted that the effect of fluid injection is to increase the velocity in a considerable neighbourhood of the wall and to decrease it thereafter, resulting in the decrease of the width of the jet. A reverse phenomenon happens in the case of suction.

Taking $\gamma = 1\cdot4$ and $m = 0\cdot15$, the velocity distribution in a hydromagnetic plane wall jet spreading over a permeable surface is plotted against the transformed similarity variable ζ in figure 2, for two values of Mach number 0 and 2

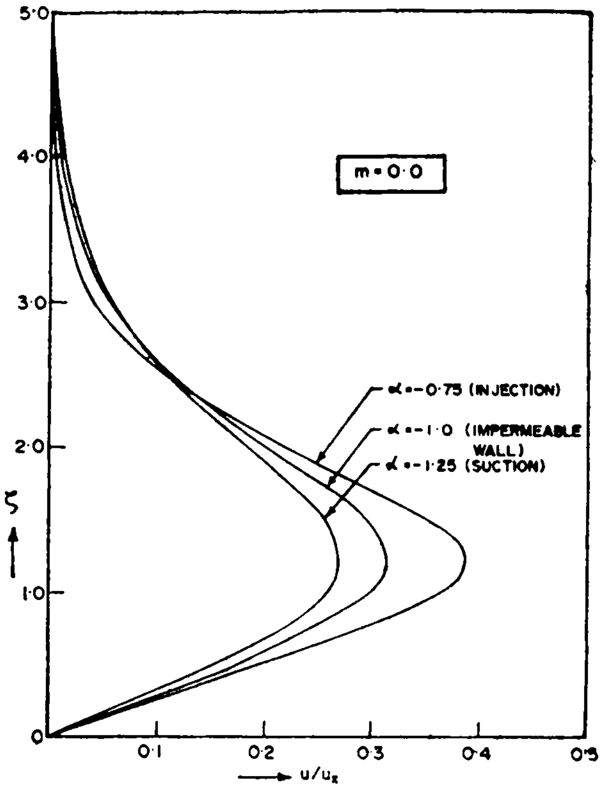

Figure 1. Velocity distribution in a wall jet of a gas spreading over a permeable surface for various values of α.

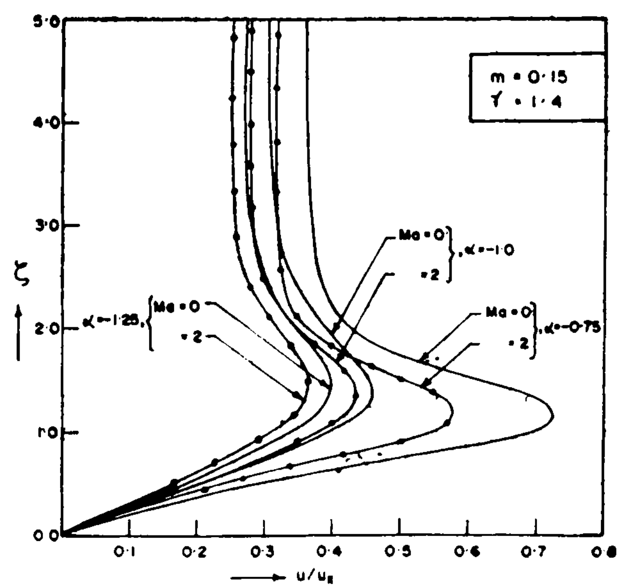

Figure 2. Velocity distribution in a hydromagnetic wall jet of a conducting gas spreading over a permeable surface for various values of $\alpha \cdot m = 0{\cdot}15$ and $Ma = 0, 2$.

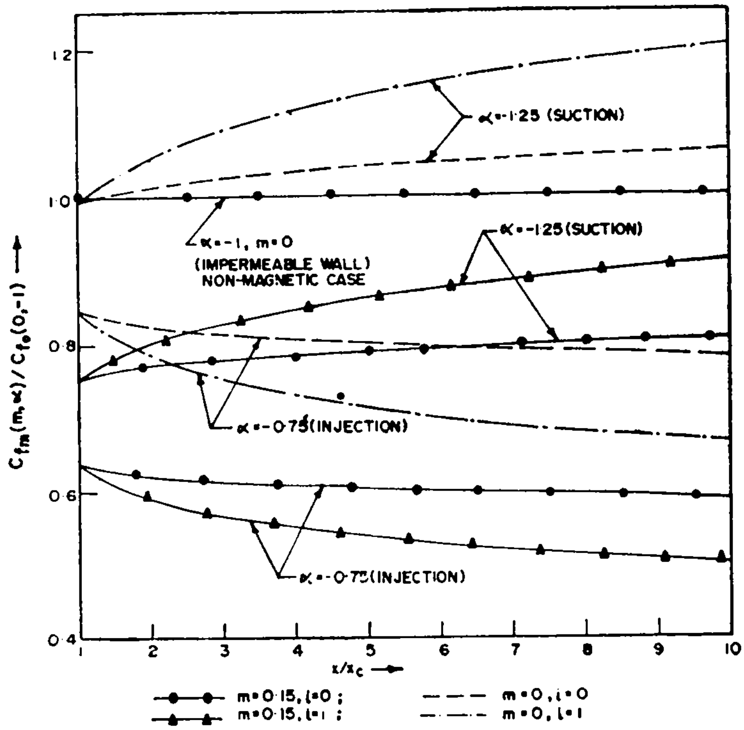

Figure 3. Variation of relative coefficient of skin-friction in a hydromagnetic wall jet of a conducting gas spreading over a permeable surface for various values of α and m and for $i = 0, 1$.

and for three values of the wall permeability parameter $\alpha = -0{\cdot}75$, -1 and $-1{\cdot}25$. It is found that the effect of the Mach number is to decrease the velocity at all points and in all cases of the permeability of the wall. As far as the question of the permeability of the wall is concerned, its effect is of the same nature as we observed in figure 1. The effect of the magnetic field can be seen by combining figures 1 and 2 and it is concluded that the magnetic field contributes towards the increase of the width of the jet.

One of the important studies of the present analysis is the variation of the relative coefficient of skin-friction, which is plotted in figure 3 against the dimensionless distance x/x_c down the stream. It is found to be independent of Mach number, an outcome of the linear relationship between viscosity and temperature, but decreases with the increase in the value of the magnetic parameter m. It is noted that the effect of gas injection is to reduce the skin-friction coefficient at all stations downstream as compared to the value for impermeable wall, whereas in the case of suction it first decreases and then increases monotonically downstream both for radial and plane jets.

Nomenclature

$B(x)$ strength of variable transverse magnetic field

c_p specific heat at constant pressure

$C_{fm}(m, a)$	coefficient of skin-friction
f, F	velocity distribution functions introduced in (14) and (36) respectively
h	temperature distribution function
i	0 or 1 for plane or radial wall jet respectively
m	magnetic interaction parameter defined by (26)
Ma	Mach number
p	pressure
T	temperature in the boundary layer
u, v	velocity components along and normal to the plane wall respectively
x, y	coordinates along and normal to the plane wall

Greek symbols

α	suction/injection parameter
β	defined by (20)
γ	ratio of the two specific heats c_p and c_v
ζ	similarity variable defined by (37)
ξ, η	Levy-Lees variables defined in (12) and (13)
k	thermal conductivity
μ	coefficient of viscosity
ρ	density
σ	Prandtl number
σ_e	electrical conductivity
τ_w	shearing stress at the wall
ψ	stream function

Subscripts

c	refers to values at a reference station in the boundary layer
o, m	refers to non-magnetic and magnetic cases respectively
w	refers to values at the wall
x	denotes reference value in the boundary layer
∞	refers to values at the edge of the boundary layer.

References

[1] Bansal J L 1981 A new approach to some hydrodynamic and hydromagnetic boundary layer flow problems. D.Sc. Thesis, Univ. of Jodhpur, Jodhpur

[2] Bansal J L and Tak S S 1978 *ZAMP* **29** 742–756

[3] Bansal J L and Tak S S 1979 *Indian J. Pure Appl. Math.* **10** (12) 1469–1483

[4] Bloom M H and Steiger M H 1958 *Proc. Third U.S. Nat. Congr. Appl. Mech.* p. 717 Brown University

[5] Fox H and Steiger M H 1963 *J. Fluid Mech.* **15** 4

[6] Glauert M B 1956 *J. Fluid Mech.* **1** (6) 625–643

[7] Glauert M B 1957 Boundary Layer Research Symp. Freiburg/Br.

[8] Lees L 1956 *Jet. Prop.* **26** 259–269

[9] Riley N 1958 *J. Fluid Mech.* **4** (6) 615–628

[10] White Frank M 1974 *Viscous fluid flow* (USA ; McGraw-Hill) pp. 589–592