Proc. Indian Acad. Sci. (Math. Sci.), Vol. 91, Number 2, July 1982, pp. 123–132.

Couette flow of a non-homogeneous fluid

K N VENKATASIVA MURTHY and K PONNURAJ
Department of Mathematics, Sri Krishnadevaraya University, Anantapur 515003, India

MS received 20 March 1981; revised 30 October 1981

Abstract. The two dimensional Couette flow of a non-homogeneous viscous fluid is studied. The plane boundaries of the channel are maintained at different temperatures. The upper plane moves with a uniform horizontal velocity and the lower plane is at rest. The fluid is subjected to suction and injection at the boundaries. The steady equations are solved by introducing similarity variables which are expanded in series of powers of a small stratification parameter. The non-linear theory predicts that the temperature depends on the distance x from the throat section, an observation which is not predicted by the linear theory. The non-linear effects on velocity and temperature are studied. The rate of heat transfer is discussed.

Keywords. Stratification; heat transfer; suction; injection.

1. Introduction

The flow of a viscous incompressible fluid between two plane boundaries is of fundamental interest in fluid dynamics. This problem is extensively used to find the interplay of various fluid forces like viscous force and buoyancy force. This configuration is used with suction and injection of fluid at the plane boundaries to solve some problems that arise in practice, like the phenomenon of separation [3; 6; 4]. This model is also used to discuss the heat transfer in acquiring the knowledge of several technological devices [1; 2]. The Couette flow is further useful in the theory of hydrodynamic lubrication. In view of these applications to modern technology the investigations on Couette flow remain important.

In this paper a two-dimensional Couette flow of a viscous stratified incompressible fluid, that arises in throat sections of channels is studied. The two plane boundaries of the channel are maintained at different temperatures. The fluid is injected at the upper plane and sucked at the lower plane with different velocities. The upper plane moves with a uniform velocity. The solution for the steady flow is obtained by assuming similarity variables for the velocities. The variables are expanded in suitable series of powers of a small stratification parameter. The formulation is based on the procedure adopted by Verma and Bansal [6] and Venkatasiva Murthy [5]. The velocity, temperature distributions and the rate of heat transfer are derived. They yield certain interesting features. The thickness of the viscous and thermal boundary layers on the planes is determined for fluids of small kinematic viscosity. The non-linearity of the equations brings out certain properties of the temperature distribution which a linear theory cannot predict.

2. Formulation of the problem

The two-dimensional flow of a viscous stratified fluid between two infinite parallel planes $z' = \pm L/2$ is considered with reference to the rectangular cartesian coordinate axes $Ox'z'$. In the basic state the planes are maintained at different temperatures T_1 and T_2. The fluid is injected at the upper plane and sucked at the lower plane with a uniform velocity. The temperature difference between the planes causes an exponential density distribution in the fluid. If (u_e, v_e, w_e) are the velocity components, ρ_e the density and T_e the temperature in the basic state, then

$$u_e = v_e = 0, \quad w_e = -W' (\text{constant}),$$

$$T_e = T_0 + \frac{T_1 - T_2}{2}\left\{\frac{\cosh(\rho_0 c_v L W'/2K) - \exp(-\rho_0 c_v W' z'/K)}{\sinh(\rho_0 c_v L W'/2K)}\right\},$$

$$\rho_e - \rho_0 = -\frac{\epsilon\rho_0}{T_0}(T_e - T_0), \quad T_0 = (T_1 + T_2)/2.$$

The pressure p_e in the basic state satisfies the equation

$$\frac{\partial p_e}{\partial z} = -\rho_e g.$$

T_0 and ρ_0 are the mean values and ϵ is a small constant.

The disturbance in the fluid, over this basic state, is created when (i) the upper plane moves with velocity $\epsilon U'$ and (ii) the normal velocities at the permeables planes $z' = \pm L/2$ are maintained at $-W'(1 \pm \epsilon)$ respectively. The equations of motion for a steady flow are

$$\rho' \vec{q}' \cdot \nabla \vec{q}' = -\nabla \vec{p}' + \mu \nabla^2 \vec{q}' - \rho' g \vec{k},$$

$$\nabla \cdot (\rho' \vec{q}') = 0,$$

$$\rho' c_v \vec{q}' \cdot \nabla T' = K \nabla^2 T' + \phi,$$

$$\rho' - \rho_e = -\epsilon \frac{\rho_0}{T_0}(T' - T_e),$$

where ϕ is the viscous dissipation function and k is the unit vector in the z'-direction. The flow is two-dimensional and the velocity is $\vec{q}' = (u', 0, w')$. The variables ρ', p', T' are the density, pressure and temperature respectively. The constant μ is the coefficient of viscosity. The boundary conditions are:

$$u' = \epsilon U', \quad w' = -W'(1 + \epsilon), \quad T' = T_1, \text{ when } z = L/2;$$

$$u' = 0, \quad w' = -W'(1 - \epsilon), \quad T' = T_2, \text{ when } z' = -L/2;$$

Since the stratification parameter ϵ is small ($\epsilon \ll 1$) for all stable stratifications, we introduce the non-dimensional variables and expansions of variables in series of powers of ϵ as follows:

$$u' = \epsilon V u = \epsilon V (u_0 + \epsilon u_1 + \ldots), \tag{1}$$

$$w' = -W' + \epsilon V w = -W' + \epsilon V (w_0 + \epsilon w_1 + \ldots), \tag{2}$$

$$p' = p_s + \frac{\epsilon \rho_0 g L}{T_0}(T_1 - T_2)\, p, \tag{3}$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \ldots, \tag{4}$$

$$T' = T_s + \epsilon (T_1 - T_2)\theta = T_s + \epsilon (T_1 - T_2)(\theta_0 + \epsilon \theta_1 + \ldots), \tag{5}$$

where

$$V = [(T_1 - T_2)\, gL/T_0]^{1/2}$$

is a constant, which is of the dimension of velocity. We also write $x' = Lx$, $z' = Lz$. With these non-dimensional variables and under the Boussinesq approximation that the influence of density variation with temperature is considered only on the body force term, the equations of motion in the non-dimensional form are

$$\epsilon u \frac{\partial u}{\partial x} + \epsilon w \frac{\partial u}{\partial z} - \delta \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + E \nabla^2 u, \tag{6}$$

$$\epsilon u \frac{\partial w}{\partial x} + \epsilon w \frac{\partial w}{\partial z} - \delta \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + E \nabla^2 w + \epsilon \theta, \tag{7}$$

$$\epsilon u \frac{\partial \theta}{\partial x} + \epsilon w \frac{\partial \theta}{\partial z} - \delta \frac{\partial \theta}{\partial z} + \frac{P\delta \exp(-P\delta z/E)}{2E \sinh (P\delta/2E)}\, w = \frac{E}{P}\nabla^2 \theta + $$

$$+ \epsilon H \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 \right]. \tag{8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{9}$$

where $E = \mu/\rho_0 VL$ is the inverse of a Reynolds number, $P = \mu c_v/K$ is the Prandtl number, $\delta = W'/V$ is suction parameter and $H = 2\mu V/L\rho_0 c_v (T_1 - T_2)$.

In view of the expansions in (1) to (5) we obtain the following differential equations:

$$E \nabla^2 u_0 - \frac{\partial p_0}{\partial x} + \delta \frac{\partial u_0}{\partial z} = 0, \tag{10}$$

$$E \nabla^2 w_0 - \frac{\partial p_0}{\partial z} + \delta \frac{\partial w_0}{\partial z} = 0, \tag{11}$$

$$u_0 \frac{\partial u_0}{\partial x} + w_0 \frac{\partial u_0}{\partial z} - \delta \frac{\partial u_1}{\partial z} = -\frac{\partial p_1}{\partial x} + E \nabla^2 u_1, \tag{12}$$

$$u_0 \frac{\partial w_0}{\partial x} + w_0 \frac{\partial w_0}{\partial z} - \delta \frac{\partial w_1}{\partial z} = -\frac{\partial p_1}{\partial z} + E \nabla^2 w_1 + \theta_0. \tag{13}$$

The energy equation gives

$$\frac{E}{P}\nabla^2\theta_0 + \delta\frac{\partial\theta_0}{\partial z} = \frac{P\delta\exp(-P\delta z/E)}{2E\sinh(P\delta/2E)}\,w_0, \tag{14}$$

$$\frac{E}{P}\nabla^2\theta_1 + \delta\frac{\partial\theta_1}{\partial z} = \frac{P\delta\exp(-P\delta z/E)}{2E\sinh(P\delta/2E)}\,w_1 + u_0\frac{\partial\theta_0}{\partial x} + w_0\frac{\partial\theta_0}{\partial z}$$
$$- H\left\{\left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial w_0}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial w_0}{\partial x} + \frac{\partial u_0}{\partial z}\right)^2\right\} \cdots \tag{15}$$

The continuity equation suggests similarity variables

$$u(x,z) = x\frac{\partial f}{\partial z} + g,\; w(z) = -f(z),$$

and hence

$$u_i(x,z) = x\frac{\partial f_i}{\partial z} + g_i,\; w_i(z) = -f_i(z),\; i = 0, 1, 2, \ldots,$$

where f_i and g_i are unknown functions of z. We eliminate pressure from (6) and (7) to obtain the general form for θ as

$$\theta(x,z) = x^2\theta^{(2)}(z) + x\theta^{(1)}(z) + \theta^{(0)}(z),$$

so that, for $\theta = 0, 1, 2, \ldots,$

$$\theta_i(x,z) = x^2\theta_{i2}(z) + x\theta_{i1}(z) + \theta_{i0}. \tag{16}$$

Equations (6) and (7) then determine pressure as

$$p = x^2 p^{(2)}(z) + xp^{(1)}(z) + p^{(0)}(z) + Ax^2 + Bx + C,$$

where A, B, C are arbitrary constants of integration.
By choosing the origin such that $B = 0$, $A \neq 0$ [6; 5], we obtain the form for p as

$$p = x^2 p^{(2)}(z) + xp^{(1)}(z) + p^{(0)}(z) + Ax^2 + C,$$

and hence the form for $p_i\,(i = 0, 1, 2 \ldots,)$ as

$$p_i = x^2 p_{i2}(z) + xp_{i1}(z) + p_{i0}(z) + A_i x^2 + C_i, \tag{17}$$

where p_{i0}, p_{i1}, p_{i2} are unknown functions and A_i, C_i are arbitrary constants. The boundary conditions are

$$f_0 = \delta,\; f_1 = 0,\; f_0' = f_1' = 0,\; g_0 = U,\; g_1 = 0 \text{ on } z = \tfrac{1}{2}, \tag{18a}$$

$$f_0 = -\delta,\; f_1 = 0,\; f_0' = f_1' = 0,\; g_0 = 0,\; g_1 = 0 \text{ on } z = -\tfrac{1}{2} \tag{18b}$$

$$\theta_{i2} = \theta_{i1} = \theta_{i0} = 0 \text{ at } z = \pm\tfrac{1}{2}, \tag{18c}$$

where $U = U'/V$. It can be shown from (14) and (18c) that

$$\theta_{02} = \theta_{01} = 0. \tag{19}$$

Equations (16), (17) and (19) are substituted in equations (10) to (13) to obtain the following differential equations for f_0, g_0, f_1 and g_1.

$$f_0^{iv} + (\delta/E) f_0''' = 0, \tag{20}$$

$$g_0'' + (\delta/E) g_0' = 0, \tag{21}$$

$$E f_1^{iv} + \delta f_1''' - f_0' f_0'' + f_0 f_0''' = 0, \tag{22}$$

$$E g_1'' + \delta g_1' - g_0 f_0' + f_0 g_0' = 0. \tag{23}$$

The solutions of (20)–(23) subject to boundary conditions (18a, b) are

$$f_0 = A + Bz + Cz^2 + D\exp(2Nz),$$

$$g_0 = F + G\exp(2Nz),$$

$$f_1 = A_1 + A_2 z + A_3 z^2 + A_4 z^3 + A_5 z^4 + A_6 \exp(2Nz) + \\ + z(A_7 + A_8 z + A_9 z^2)\exp(2Nz),$$

$$g_1 = B_1 + B_2 z + B_3 z^2 + (B_4 + B_5 z + B_6 z^2 + B_7 z^3)\exp(2Nz).$$

Also, from (14), (15) and (18c) we obtain

$$\theta_0 = X_0 + X_1 \exp(-P\delta z/E) + \psi(z),$$

$$\psi(z) = \frac{\exp(-P\delta z/E)}{2\sinh(P\delta/2E)}\left[\frac{CP}{3E} z^3 + \left(\frac{BP}{2E} + \frac{C}{\delta}\right) z^2 + \right. \\ \left. + \left(\frac{2CE}{P\delta^2} + \frac{B}{\delta} + \frac{AP}{E}\right) z - \frac{P^2 D\delta \exp(2Nz)}{2NE(2NE - P\delta)}\right],$$

$$\theta_{10} = L_0 + L_1 \exp(-P\delta z/E) + \psi_1(z),$$

$$\theta_{11} = M_0 + M_1 \exp(-P\delta z/E) + Y_0 \exp(2Nz) + Y_1 \exp(4Nz),$$

$$\theta_{12} = C_0 + C_1 \exp(-P\delta z/E) + Y_2 z + Y_3 \exp(2Nz) + Y_4 \exp(4Nz),$$

$$\psi_1(z) = (d_0 z + d_1 z^2 + d_2 z^3) + (d_3 + d_4 z)\exp(2Nz) + d_5 \exp(4Nz) \\ + (d_6 z + d_7 z^2 + d_8 z^3 + d_9 z^4 + d_{10} z^5 + d_{11} z^6)\exp(-P\delta z/E) \\ + (d_{12} + d_{13} z + d_{14} z^2 + d_{15} z^3)\exp\left(2N - \frac{P\delta}{E}\right) z \\ + d_{16}\exp\left(4N - \frac{P\delta}{E}\right) z + d_{17}\exp(-2P\delta z/E),$$

where

$$N = -\delta/2E, \; A = R\delta\left(\cosh N - \frac{N}{2}\sinh N\right),$$

$$B = 2\delta N R\cosh N, \; C = B\tanh N, \; D = -R\delta, \; F = -\frac{U\exp(-N)}{2\sinh N},$$

$$G = U/(2\sinh N), \; R = (N\cosh N - \sinh N)^{-1},$$

$$X_1 = \left[\psi\left(\frac{1}{2}\right) - \psi\left(-\frac{1}{2}\right)\right] \Big/ 2\sinh(P\delta/2E),$$

$$X_0 = -\psi\left(\frac{1}{2}\right) - X_1 \exp(-P\delta/2E),$$

and all other constants appearing in the above solution are lengthy expressions which depend on P, δ, E and H and will run for pages together if presented. Hence they are not presented to save space.

3. Discussion of the results

The velocity and temperature distributions are derived to order ϵ^2. The expressions for the velocities u and w consist of the exponentially decaying terms in z which decay at distances of order (E/δ). When E is small in comparison with δ, the exponential terms represent a boundary layer flow (at the upper plane if $\delta > 0$ and at the lower plane if $\delta < 0$) outside which the horizontal velocity distribution is linear to order ϵ. The differential equations for f_2 and g_2 would depend on θ while those for f_0, g_0, f_1 and g_1 do not depend on the temperature. Hence the temperature produces only changes of order ϵ^2 in the velocities. Since θ_1 contains an exponential term $\exp(-P\delta z/E)$, the temperature variations bring into effect additional exponential modes in the solution representing viscous boundary layers, of amplitude order ϵ^2 and thickness order $E/P\delta$ when $E \ll P\delta$.

The exponential terms in the solution for temperature tend to zero at distances of order E/δ, $E/P\delta$, $E/(P+1)\delta$, $E/(P+2)\delta$. When E is small in comparison with P and δ, these exponential terms represent thermal boundary layers of small thickness near the boundaries. While θ_0 does not depend on x, θ_1 is a function of both x and z. The temperature distribution, to order ϵ^2, is parabolic in x and even though ϵ is small, $\theta_1(x, z)$ can give a non-negligible contribution to the temperature at larger distances x. It may also be noted that this parabolic dependence on x is a non-linear effect of the viscous dissipation.

The velocity and temperature distributions are plotted in figures 1–8. The values of E and δ are chosen to have a greater order than ϵ to make the expansions assumed in the theory valid. Figures 1 and 2 show that as E increases the velocity decreases whereas the velocity increases as δ increases. Figure 3 shows the velocity distribution at various values of x. The continuous lines show the prediction of the linear theory and the dotted lines give the non-linear theory. When E is small in comparison with δ the steep rise in the velocity near the lower plane shows the boundary layer type behaviour of the velocity as observed earlier. It can be seen that the non-linear theory predicts a smaller velocity than the linear theory. Even though the non-linear contribution is small for smaller values of x, it increases with x and hence the linear theory cannot be taken to represent the velocity at moderate values of x, at least in the parts of the channel away from the two plane boundaries.

The velocity for negative values of W' (negative values of δ) is plotted in figure 4. The fluid is now injected at the lower plane and sucked at the upper plane. The velocity now decreases first and then increases to its value at the

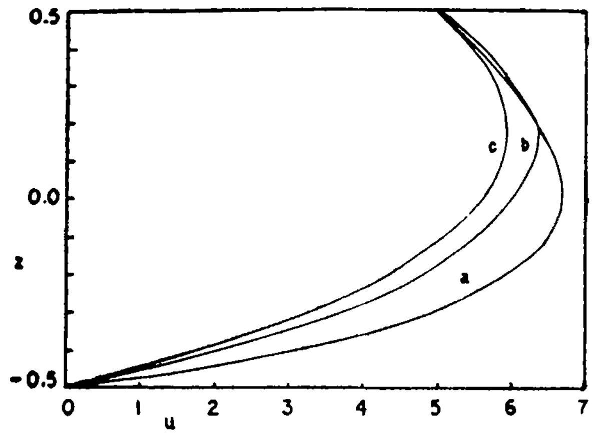

Figure 1. Velocity distribution when $\delta = 0{\cdot}05$, $U = 5$, $x = 20$. (a) $E = 0{\cdot}02$, (b) $E = 0{\cdot}05$, (c) $E = 0{\cdot}1$.

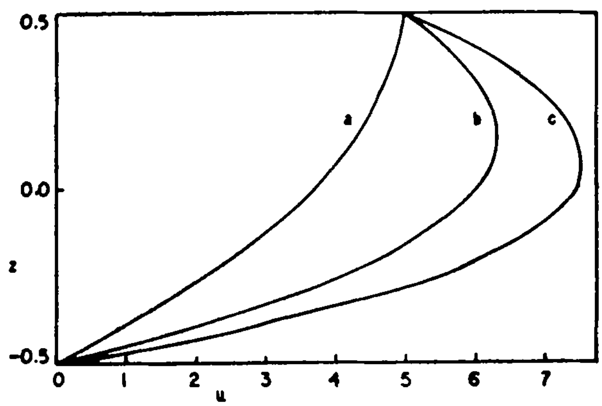

Figure 2. Velocity distribution when $E = 0{\cdot}05$, $x = 20$. (a) $\delta = 0{\cdot}02$, (b) $\delta = 0{\cdot}05$ (c) $\delta = 0{\cdot}07$

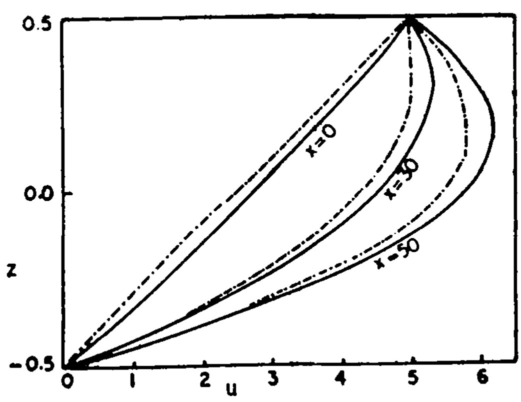

Figure 3. Velocity distribution when $\delta = 0{\cdot}02$, $E = 0{\cdot}05$. ———— linear theory, - · - · - · - non-linear theory.

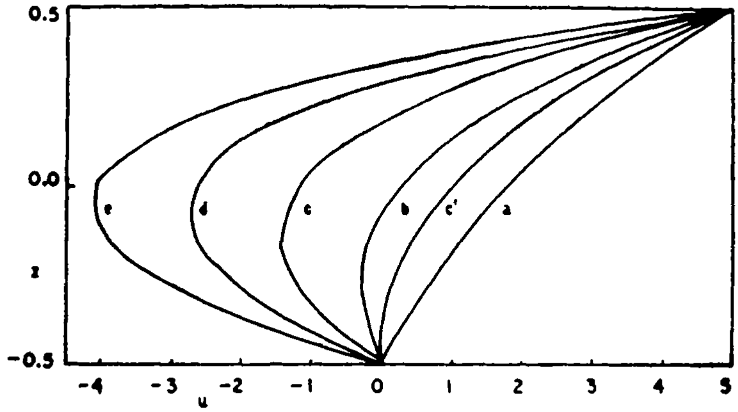

Figure 4. Velocity distribution when δ is negative, $\delta = -0{\cdot}05$. $E = 0{\cdot}05$, (a) $x = 0$, (b) $x = 10$, (c) $x = 20$, (d) $x = 30$, (e) $x = 40$, (c′) $\delta = -0{\cdot}02$, $E = 0{\cdot}05$, $x = 20$.

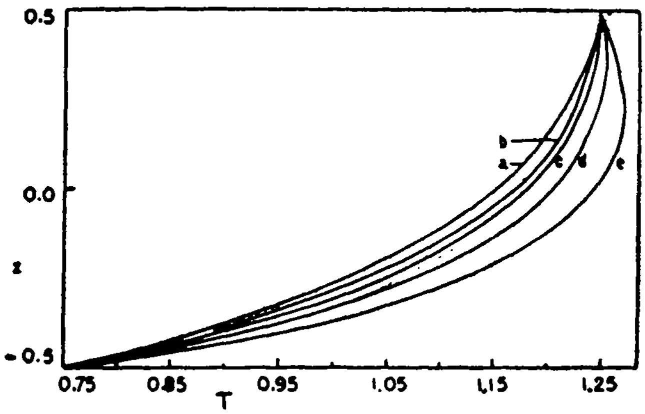

Figure 5. Temperature distribution when $P = 3{\cdot}5$, $\delta = E = 0{\cdot}05$. (a) $x = 10$, (b) $x = 40$, (c) $x = 60$, (d) $x = 80$, (e) $x = 100$.

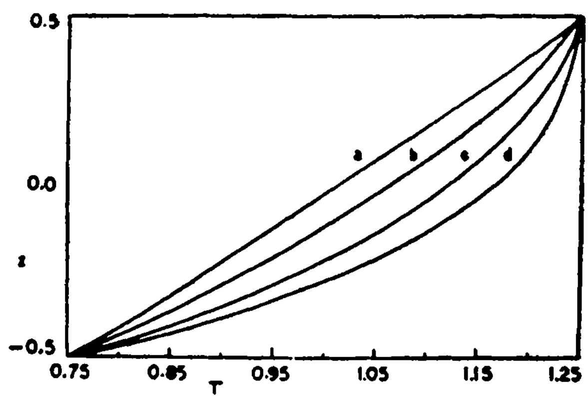

Figure 6. Temperature distribution when $x = 40$, $\delta = E = 0{\cdot}05$. (a) $P = 0{\cdot}5$, (b) $P = 1{\cdot}5$, (c) $P = 2{\cdot}5$, (d) $P = 3{\cdot}5$.

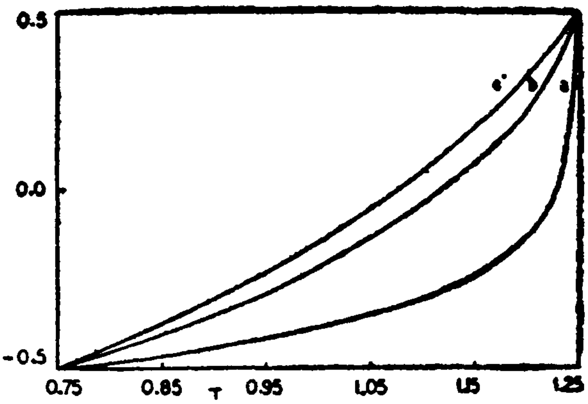

Figure 7. Temperature distribution when $P = 2{\cdot}5$, $\delta = 0{\cdot}05$, $x = 10$. (a) $E = 0{\cdot}02$ (b) $E = 0{\cdot}05$, (c) $E = 0{\cdot}1$.

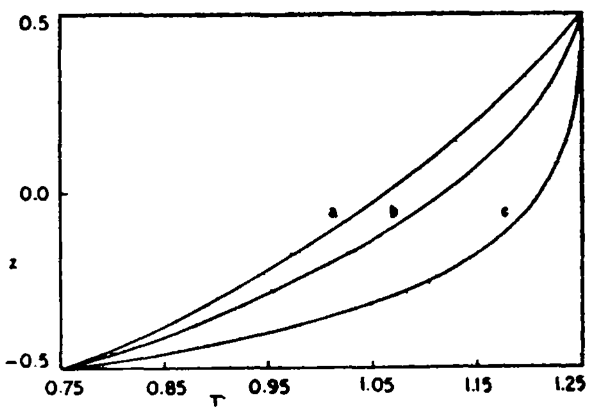

Figure 8. Temperature distribution when $P = 2{\cdot}5$, $E = 0{\cdot}05$, $x = 10$. (a) $\delta = 0{\cdot}02$, (b) $\delta = 0{\cdot}05$, (c) $\delta = 0{\cdot}1$.

upper plane and the velocity curves are comparable to those described by Verma and Bansal [6]. The flow from large values of x to the mouth of the channel is developed near the stationary wall. Thus the dragging action of the faster layers exerted on the fluid particles in the neighbourhood of the stationary wall is insufficient to overcome the influence of the adverse pressure gradient that develops. The velocity decreases (to larger negative values) with x increasing. Also as δ increases in value the velocity increases.

The temperature distribution at various cross-sections of the channel is shown in figure 5. The temperature depends on x parabolically to order ϵ^2. This increases the temperature in the interior beyond its value at the upper plane, as x increases. The important property that the temperature depends on x is brought out by the non-linear theory. The figures 6, 7 and 8 describe the behaviour of the temperature to the changes in the parameters P, δ and E. As P increases there is a rapid change in the temperature at the lower plane as the heat transfer will be more effective for larger P. As E increases the temperature decreases and as δ increases the temperature increases. There is a rapid change in the temperature in a layer near the lower plane and then it remains almost a constant in the upper half of the channel when E is relatively smaller than P and δ.

The rate of heat transfer at the planes is

$$q' = K\left(\frac{\partial T'}{\partial z'}\right)_{z'=\pm L/2}$$

Defining the non-dimensional rate of heat transfer as $q = -q' L/K(T_1 - T_2)$ we obtain the rate of heat transfer at $z = \pm \frac{1}{2}$, to order one, given by

$$q = \frac{P\delta \exp(\mp P\delta/2E)}{2E \sinh(P\delta/2E)}.$$

We will discuss the rate of heat transfer to order one. It is found that perturbation of order ϵ will not altogether change the general behaviour of the rate of heat transfer. The rate of heat transfer at $z = \frac{1}{2}$ is

$$q_1 = \frac{2X}{\exp(2X) - 1}$$

and the rate of heat transfer at the lower plane is

$$q_2 = \frac{2X}{1 - \exp(-2X)}$$

where $X = P\delta/2E$. This shows that as X increases q_1 decreases and q_2 increases. Thus as E decreases or as P increases the rate of heat transfer decreases at the upper plane and increases at the lower plane. Also as $P\delta/2E$ decreases to zero, q_1 and q_2 will tend to unity.

Acknowledgement

The authors are thankful to the referee for suggestions to improve the manuscript. One of the authors, K Ponnuraj is thankful to CSIR for the award of a junior research fellowship.

References

[1] Bleviss Z D 1958 *J. Aeronaut. Sci.* **25** 601

[2] Leadon B M 1957 *Convair Sci. Res. Lab.* RN 13

[3] Sinha K D and Choudhary R C 1965 *Proc. Indian Acad. Sci.* **A61** 308

[4] Terril R M and Sreshta G M 1965 *Z. Angew. Math. Phys.* **16** 470

[5] Venkatasiva Murthy K N 1980 *Proc. Indian Acad. Sci. (Math. Sci.)* **89** 103

[6] Verma P D and Barsal J L 1966 *Proc. Indian Acad. Sci.* **A64** 385