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Ramanujan and Dirichlet series with Euler products

S S RANGACHARI

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India.

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Abstract. Ramanujan, in his unpublished manuscripts, had written down without proof, explicit linear combinations of cusp forms whose Mellin transforms possess Euler products in the sense of Hecke. All these results are proved here and their connection with the work of Hecke, Rankin and Serre is pointed out.

Keywords. Dirichlet series ; Euler products ; Ramanujan's work.

1. Introduction

In addition to the different results on Dirichlet series with Euler products in his famous paper [7], Ramanujan had stated some more results in the manuscripts recently discovered [1, 8]. We shall discuss in this paper all the Dirichlet series with Euler products found in his published and unpublished work. We divide them for the sake of convenience into three parts although there are some overlaps. We shall remark at the end the connections between the different Euler products.

(i) Euler Products in [7]—they are all Dirichlet series attached to $\eta^k\left(\frac{24z}{k}\right)$, $k|24$ (η: Dedekind's η-function) or Dirichlet series attached to cusp forms occurring in "sums of squares" problem.

(ii) Euler Products in [8]—these are Dirichlet series attached to cusp forms of the type $(\eta(z)\,\eta(Nz))^k$, $k|24$ and $(N+1)|24$ which occur in congruence problems.

(iii) Euler Products in [1]—these are Dirichlet series attached to cusp forms on cycloidal subgroups of the modular group and presumably studied by Ramanujan for congruence properties for $p(n)$. (See appendix in [6]).

The results in (1) were proved by Mordell in [4] and in a more general fashion by Hecke in (35, 36 in [2]) the final result being the proof by Deligne of the Ramanujan conjecture. Some of the results in (2) were proved by Hecke in ((41) in [2]) who was obviously unaware of Ramanujan's statements.

The results in (3) have not been proved so far, although a few of them belong to the category (1) and hence proved by Mordell and Hecke. We shall supply proofs of all the results in (3) and also fill in the gaps left by Ramanujan. For the same, we need the results of Rankin [9].

The most remarkable thing about the results in (3) is the fact that certain linear combinations of Dirichlet series have Euler products, the first published examples of which we encounter only in Hecke, as observed by Birch in [1].

2. Ramanujan's results on Dirichlet series with Euler product

We shall list below all the Euler product developments given by Ramanujan, both published and unpublished. The first list is from his paper [7] and the second list is from his unpublished papers [8] and the third list as given in Birch [1].

(*a*) Euler product development for Mellin transforms of

$$\eta^{a}\left(\frac{24z}{a}\right) \text{ with } a \mid 24 \text{ and for Dirichlet series } \sum_{n=1}^{\infty} e_k(n) \cdot n^{-s}$$

(k = 10, 12, 16) as given in [7].

(*b*) Euler product development for the Mellin transforms of

(1) $\eta(z)\,\eta(7z)$, (2) $\eta(z)\,\eta(11z)$, (3) $\eta(z)\,\eta(23z)$, (4) $(\eta(z)\,\eta(11z))^2$, (5) $(\eta(z)\,\eta(7z))^3$.

More explicitly, they are as follows : (set $x = \exp.(2\pi iz)$)

(1) Let $\sum_{n=1}^{\infty} \phi(n)\, x^{n/3} = x^{1/3} \prod_{n=1}^{\infty} (1 - x^n)(1 - x^{7n})$. Then

$$\sum_{n=1}^{\infty} \phi(n) \cdot n^{-s} = (1 + 7^{-s})^{-1} \prod_{p} (1 + p^{-2s})^{-1} \prod_{q} (1 - q^{-2s})^{-1}$$
$$\prod_{r} (1 + r^{-s})^{-2} \prod_{t} (1 - t^{-s})^{-2}$$

where $p \equiv 2, 8, 11 \pmod{21}$, $q \equiv 5, 17, 20 \pmod{21}$

$r \equiv 1, 4, 16 \pmod{21}$ which are of the form $9A^2 + 7B^2$

$t \equiv 1, 4, 16 \pmod{21}$ which are of the form $A^2 + 63B^2$

(2) Let $\sum_{n=1}^{\infty} \phi(n) \cdot x^{n/2} = x^{1/2} \prod_{n=1}^{\infty} (1 - x^n)(1 - x^{11n})$. Then

$$\sum_{n=1}^{\infty} \phi(n) \cdot n^{-s} = (1 - 11^{-s})^{-1} \prod_{p} (1 - p^{-2s})^{-1} \prod_{q} (1 + q^{-s} + q^{-2s})^{-1}$$
$$\prod_{r} (1 - r^{-s})^{-2}$$

where $p \equiv$ quadratic non-residue mod 11

$q \equiv$ quadratic residue mod 11 of the form $11A^2 + B^2$

$r \equiv$ quadratic residue mod 11 not of the form $11A^2 + B^2$

(3) Let $\sum_{n=1}^{\infty} \phi(n) \cdot x^n = x \prod_{n=1}^{\infty} (1-x^n)(1-x^{23n})$. Then

$$\sum_{n=1}^{\infty} \phi(n) \cdot n^{-s} = \prod_p (1-p^{-2s})^{-1} \prod_q (1+q^{-s}+q^{-2s})^{-1} \prod_r (1-r^{-s})^{-2}$$

where $p \equiv$ quadratic non-residue mod 23

$q \equiv$ quadratic residue mod 23 of the form $23A^2 + B^2$

$r \equiv$ quadratic residue mod 23 not of the form $23A^2 + B^2$

(4) Let $\sum_{n=1}^{\infty} \lambda(n) \cdot x^n = x \prod_{n=1}^{\infty} (1-x^n)^2 (1-x^{11n})^2$. Then

$$\sum_{n=1}^{\infty} \lambda(n) \cdot n^{-s} = (1-11^{-s})^{-1} \prod_{p \neq 11} (1-\lambda(p) \cdot p^{-s} + p^{1-2s})^{-1}$$

(Ramanujan says that " $\lambda(p)$ can be determined. " [8]).

(5) Let $\sum_{n=1}^{\infty} \lambda(n) \cdot x^n = xf^3(-x) f^3(-x^7)$ where $f(-x) = \prod_{n=1}^{\infty} (1-x^n)$

Then $\sum_{n=1}^{\infty} \lambda(n) \cdot n^{-s} = (1+7^{-s})^{-1} \prod_p (1-p^{2-2s})^{-1}$

$$\prod_q (1+2c_q \cdot q^{-s} + q^{2-2s})^{-1}$$

where $q = 7u^2 + v^2$, $c_q = 7u^2 - v^2$ and $q \equiv 1, 2, 4 \pmod 7$ and $p \equiv 3, 5, 6 \pmod 7$.

(*c*) Euler product developments as listed in Birch [1]. With the usual notation for Eisenstein series, writing η instead of f as in [1], the results as given by Birch are as follows :

I. Suppose that A and B are any two integers such that $A^2 + 3B^2 = p$ and $A \equiv 1 \pmod 3$, p being a prime of the form $6k+1$. Let

$$\Sigma a_0(n) q^{n/6} = \eta^4, \ \Sigma a_1(n) \cdot q^{n/6} = \eta^4 P,$$

$$\Sigma a_2(n) \cdot q^{n/6} = \eta^4 Q, \ \Sigma a_3(n) q^{n/6} = \eta^4 R,$$

$$\Sigma a_4(n) q^{n/6} = \eta^4 Q^2 + 288\sqrt{70}\ \eta^{20}, \ \Sigma a_5(n) q^{n/6} = \eta^4 QR$$

$$\Sigma a_7(n) q^{n/6} = \eta^4 Q^2 R + 10080\sqrt{286}\ \eta^{20} R.$$

In all these cases

$$\sum_{n=1}^{\infty} a_K(n) \cdot n^{-s} = \prod_p (1 - a_K(p) \cdot p^{-s} + p^{2K+1-2s})^{-1} \text{ where } p$$

assumes all prime values greater than 3.

If $K = 0, 2, 3, 5$ then $a_K(p) = 0$ for $p \equiv 5\ (6)$,

$$a_K(p) = (A + \sqrt{-3} \cdot B)^{2K+1} + (A - \sqrt{-3}B)^{2K+1}$$

for $p \equiv 1 \pmod 6$. For all values of n, $a_1(n) = na_0(n)$. But $a_4(p)$ and $a_7(p)$ do not seem to have such simple laws.

II. Suppose again that A and B are defined as in I. and let

$$\Sigma a_0(n)\ q^{n/3} = \eta^8,\ \ \Sigma a_1(n)\, q^{n/3} = \eta^8\ p$$

$$\Sigma a_2(n)\ q^{n/3} = \eta^8\, Q + 6\sqrt{10}\,\eta^{16},\ \ \Sigma a_3(n)\, q^{n/3} = \eta^8\, R$$

$$\Sigma a_4(n)\ q^{n/3} = \eta^8\, Q^2 + 6\sqrt{70}\ \eta^{16}\, Q,\ \ \Sigma a_5(n)\ q^{n/3} = \eta^8\, QR + 12\sqrt{55}\,\eta^{16}\ R$$

$$\Sigma a_7(n)\, q^{n/3} = \eta^8\, Q^2 R + 12\sqrt{910}\,\eta^{16}\, QR.$$

In all these cases

$$\sum_{n=1}^{\infty} a_K(n)\cdot n^{-s} = \prod_p (1 - a_K(p)\cdot p^{-s} + p^{2K+3-2s})^{-1}$$

where p assumes all prime values except 3.

If $K = 0$ or 3, then $a_K(p) = 0$ for $p \equiv 2 \pmod 3$,

$$a_K(p) = (A + \sqrt{-3}\cdot B)^{2K+3} + (A - \sqrt{-3}\cdot B)^{2K+3} \text{ for } p \equiv 1\,(3),$$

$$a_1(n) = n\, a_0(n).$$

III. Suppose that C and D are integers such that $C^2 + 4D^2 = p$ where p is of the form $4k + 1$.

$$\Sigma a_0(n)\, q^{n/4} = \eta^6,\ \ \Sigma a_1(n)\, q^{n/4} = \eta^6\, p,\ \ \Sigma a_2(n)\ q^{n/4} = \eta^6\, Q,$$

$$\Sigma a_3(n)\, q^{n/4} = \eta^6\ R + 24i\sqrt{35}\,\eta^{18},\ \Sigma\ a_4(n)\ q^{n/4} = \eta^6\, Q^2,$$

$$\Sigma a_5(n)\ q^{n/4} = \eta^6\, QR + 24i\ \sqrt{1155}\ \eta^{18}\ Q,$$

$$\Sigma a_7(n)\ q^{n/4} = \eta^6\, Q^2 R + 120i\ \sqrt{3003}\ \eta^{18}\ Q^2 \text{ where } i^2 = -1.$$

In all these cases $\sum_{n=1}^{\infty} a_K(n)\cdot n^{-s} = \Pi_1\,\Pi_2$ where

$$\Pi_1 = \prod_p (1 - a_K(p)\cdot p^{-s} - p^{2K+2-2s})^{-1},$$ p assuming primes of all the form $4k - 1$ and

$$\Pi_2 = \prod_p (1 - a_K(p)\cdot p^{-s} + p^{2K+2-2s})^{-1},$$ p assuming all primes of the form $4k + 1$. If $K = 0, 2$ or 4, then $a_K(p) = 0$ for $p \equiv 3 \pmod 4$, $a_K(p) = (C + 2iD)^{2K+2} + (C - 2iD)^{2K+2}$ for $p \equiv 1 \pmod 4$, $a_1(n) = n\, a_0(n)$.

IV. Let $\Sigma a_0(n)\ q^{n/12} = \eta^2$, $\Sigma a_1(n)\ q^{n/12} = \eta^2\, P$

$$\Sigma a_2(n)\, q^{n/12} = \eta^2\, Q + 48\,\eta^{10},\ \Sigma a_3(n)\, q^{n/12} = \eta^2\, R + 360\, i\sqrt{3}\,\eta^{14}$$

$$\Sigma a_4(n)\ q^{n/12} = \eta^2\, Q^2 + 672\,\eta^{10}\, Q,$$

$$\Sigma a_5(n)\, q^{n/12} = \eta^2\, QR + 96\sqrt{1045}\ \eta^{10} R + 216\, i\sqrt{7315}\ \eta^{14}\, Q + 103680\, i\ \sqrt{7}\,\eta^{22}.$$

$$\Sigma a_7(n)\ q^{n/12} = \eta^2\ Q^2 R + 48\sqrt{910\cdot 2911}\ \eta^{10}\, QR + 216\, i\sqrt{5005\cdot 2911}\,\eta^{14}\, Q^2 - 471744\, i\sqrt{22}\,\eta^{22}\, Q.$$

There is a minor difference between Ramanujan's original manuscript and the version appearing in [1] as given above. All the quadratic (real and imaginary) irrationalities and the rational coefficients ($\neq 1$) appear with signs ± 1 in Ramanujan's version.

The following is an additional list of Euler product developments found in [8] which is incomplete except for the first one which is already quoted above.

(1) If $\Sigma\, a(n)\, x^n = x^2 \Pi (1 - x^{3n})^{16}$ and

$$\Sigma\, A(n)\, x^n = x\, \Pi (1 - x^{3n})^8 \cdot \left(1 + 240 \sum_1^\infty \frac{n^3 x^{3n}}{1 - x^{3n}}\right)$$

$$\Sigma \frac{A(n) + 6a_n\sqrt{10}}{n^s} = (1 - 6\sqrt{10} \cdot 2^{-s} + 2^{7-2s})^{-1} \; (1 + 96\sqrt{10} \cdot 5^{-s} + 5^{7-2s})^{-1} \, (1 - 260 \cdot 7^{-s} + 7^{7-2s})^{-1} \times (1 + 1920\sqrt{10} \cdot 11^{-s} + 11^{7-2s})^{-1}$$

(this is the same as $\Sigma a_2(n)\, n^{-s}$ in II).

(2) If $\Sigma\, a(n)\, x^n = x^3 \Pi (1 - x^{4n})^{18}$. Then

$$156\, \Sigma\, a(n) \cdot n^{-s} = (1 - 78 \cdot 3^{-s} + 3^{8-2s})^{-1} \; (1 + 510 \cdot 5^{-s} + 5^{8-2s})^{-1}. (1 + 1404 \cdot 7^{-s} + 7^{8-2s})^{-1} \cdots - (1 + 78 \cdot 3^{-s} + 3^{8-2s})^{-1} (1 + 510 \cdot 5^{-s} + 5^{8-2s})^{-1} (1 + 1404 \cdot 7^{-s} + 7^{8-2s})^{-1} \ldots$$

"Presumably there are analogous results for

$x^5 \Pi (1 - x^{12n})^{10}$, $x^7 \Pi (1 - x^{12n})^{14}$, $x^5 \Pi (1 - x^{6n})^{20}$ and $x^{11} \Pi (1 - x^{12n})^{22}$.'

As stated in the introduction, we shall prove the results in (*c*) and later mention about the results in (*a*) and (*b*) and their connection with the work of Hecke and others. Since we shall be using the results of Rankin in [9], we shall summarize them briefly.

3. Rankin's results

Let $\Gamma(1)$ be the full modular group $SL(2, ZZ)$ and $\Gamma'(1)$ its commutator subgroup. Then $\Gamma(1)/\Gamma'(1)$ is cyclic of order 12 and $\Gamma'(1) \supset \Gamma(12)$ (the principal congruence subgroup of stufe 12). For each real dimension $-k$, there exist 6 possible "multiplier systems" on $\Gamma(1)$ which we denote by $v^{(r)}$, $r \in \mathscr{R} = \{0, 4, 6, 8, 10, 14\}$ defined as follows. If

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{then} \quad v^{(r)}(U) = \exp\, [\pi i (k - r)]/6,$$

$$v^{(r)}(V) = \exp\, [-\pi i (k - r)]/2.$$

We observe that the six multiplier systems agree on $\Gamma'(1)$ and we take $v^{(0)} = u$ and set $M_k = \{\Gamma'(1), k, u\}_0$ = the space of cusp forsm for $\Gamma'(1)$ of weight k and multiplier system u, and $M_k^{(r)} = \{\Gamma(1), k, v^{(r)}\}_0$ = the space of cusp forms for $\Gamma(1)$ of weight k and multiplier system $v^{(r)}$.

Then $M^{(r)}_k \neq 0$ only if $k \geqslant r$ and a basis for $M^{(r)}_k$ is given by $E_{r+12s}\,\eta^{2(k-r)-12s}$ $(0 \leqslant s \leqslant k - r/12)$ except when $k \equiv r \pmod{12}$ and $s = (k-r)/12$. Here E_s denotes the usual Eisenstein series $\epsilon\{\Gamma(1), v, 1\}$. (Later we shall use Ramanujan's notation Q and R for E_4 and E_6).

Now, $M_k = \bigoplus_{r \in \mathcal{R}} M_k^{(r)}$ (direct sum).

$M_k \subset \{\Gamma(12), k, \epsilon\}_0$: the space of cusp forms for the congruence subgroup $\Gamma(12)$ of weight k and character ϵ defined by

$$\epsilon(n) = \left(\sin \frac{\pi n}{2}\right)^k.$$

$$\dim.\quad M_k = \delta_k = \max\left(1,\ -1 - \left[-\frac{k}{2}\right]\right).$$

For each positive divisor t of 12, we denote by $M_k^{(r)}(t)$, the space of cusp forms in whose Fourier expansion only powers $\exp(2\pi imz/12)$ with $(m, 12) = t$ occur. In other words, if $f \in M_k^{(r)}(t)$ and $q = \exp(2\pi iz)$, $f = \sum_{(n,\,12)=1} a_n (q^n)^{12}$. They are of divisor t in the sense of Hecke. We notice that $(k - r, 12) = t$ in case $f \in M_k^{(r)}(t)$ and conversely if $(k - r, 12) = t$ then $f \in M_k^{(r)}(t)$. It follows from Hecke theory that $M_k^{(r)}(t)$ are stable under Hecke operators T_n with $(n, 12) = 1$. We also observe that if $F \in M_k^{(r)}$, $F(z) = \sum_{n=1}^{\infty} \lambda(n)\, q^{nt/12}$ with $\lambda(1) = 1$, is a normalized eigenfunction for the Hecke operators T_n, then the Dirichlet series $\sum_{n=1}^{\infty} \lambda(n)\cdot n^{-s}$ admits an Euler product of the form $\prod_{p \times t} (1 - \lambda(p)\cdot p^{-s} + \epsilon(p)\cdot p^{k-1-2s})^{-1}$ and conversely. A necessary condition for F to be an eigenfunction of $T_p\,(p \times t)$ is given by

$$\lambda(p)^2 = \lambda(p^2) + p^{k-1}\cdot \epsilon(p) \quad (\text{since } \lambda(1) = 1). \tag{1}$$

For the proof of Ramanujan's results in [1] this will be useful.

From the discussion preceding the statements of Ramanujan's results, we see that all Euler products of Ramanujan correspond to cusp forms in $M_k(t)$ for different divisors t of 12 ($t = 2$ in Case I, $t = 4$ in Case II, $t = 3$ in Case III, $t = 1$ in Case IV). The case of $t = 12$ occurs in his results in [7] although not in great detail. The case $t = 6$ occurs only in one example of Ramanujan. [8].

We shall therefore tabulate below the different values of t and corresponding values of k and the type of congruence subgroups and character to which the forms belong. This is followed by tables of bases of $M_k(t)$ $(t \mid 12)$ and $k \leqslant 30$ in case $t = 12$ and $k \leqslant 20$ in other cases, since these are the only cases relevant to Ramanujan's results. In the following $F(z) \in M_k(t)$, $\epsilon(n) = (\sin \pi n/2)^k$.

4. Proof of Ramanujan's results

In this paragraph, we shall prove all statements of § 2, regarding the existence of Euler products. In § 5, we shall take up the results of § 2 concerning the structure of Euler products, since, they overlap with some of the results in [7].

We shall consider the different cases as set out in table 1 (see table 2).

Table 1.

Cases	t	Fourier expansion of $F(z)$	Condition on k	Type
1.	12	$F(z) = \Sigma a_n q^n$	k even, $\neq 14$, $\geqslant 12$	$\{\Gamma(1), k, 1\}_0$
2.	6	$F(z) = \Sigma a_n q^{n/2}$	k even, $\neq 8$, $\geqslant 6$	$\{\Gamma(2), k, 1\}_0$
3.	4	$F(z) = \Sigma a_n q^{n/3}$	k even, $\neq 6$, $\geqslant 4$	$\{\Gamma(3), k, 1\}_0$
4.	3	$F(z) = \Sigma a_n q^{n/4}$	k odd, $\neq 5$, $\geqslant 3$	$\{\Gamma(4), k, \epsilon\}_0$
5.	2	$F(z) = \Sigma a_n q^{n/6}$	k even, $\neq 4$, $\geqslant 2$	$\{\Gamma(6), k, 1\}_0$
6.	1	$F(z) = \Sigma a_n q^{n/12}$	k odd, $\neq 3$, $\geqslant 1$	$\{\Gamma(12), k, \epsilon\}_0$

Table 2.

Cases	t	k	$\{r \in \mathcal{R}\ (k-r, 12) = t\}$	Basis of $M_k(t)$
1.	12	12, 16, 18, 20, 22, 24, 26, 28, 30	$\{0\}, \{4\}, \{6\}, \{8\}, \{10\}, \{0\}, \{14\}, \{4\}, \{6\}$	$\{\Delta\}, \{\Delta Q\}, \{\Delta R\}, \{\Delta Q^2\}, \{\Delta QR\}, \{\Delta E_{12}, \Delta^2\}, \{\Delta Q^2 R\}, \{\Delta^2 Q, \Delta E_{16}\}, \{\Delta^2 R, \Delta E_{18}\}$
2.	6	6, 10, 12, 14, 16, 18, 20	$\{0\}, \{4\}, \{6\}, \{8\}, \{10\}, \{0\}, \{14\}$	$\{\eta^{12}\}, \{\eta^{12} Q\}, \{\eta^{12} R\}, \{\eta^{12} Q^2\}, \{\eta^{12} QR\}, \{\eta^{12} E_{12}, \eta^{36}\}, (\eta^{12} Q^2 R\}$
3.	4	4, 8, 10, 12, 14, 16, 18	$\{0\}, \{0, 4\}, \{6\}, \{4, 8\}\ \{6\ 10\}, \{0\ 8\}, \{10, 14\}$	$\{\eta^8\}, \{\eta^{16}, \eta^8 Q\}, \{\eta^8 R\}, \{\eta^{16} Q, \eta^8 Q^2\}, \{\eta^{16} R, \eta^8 QR\}, \{\eta^{32}, \eta^{16} Q^2, \eta^8 E_{12}\}, \{\eta^{16} QR, \eta^8 Q^2 R\}$
4.	3	3, 7, 9, 11, 13, 15, 17	$\{0\}, \{4\}, \{0, 6\}, \{8\}\ \{4, 10\}, \{0, 6\}, \{8, 14\}$	$\{\eta^6\}, \{\eta^6 Q\}, \{\eta^{18}, \eta^6 R\}, \{\eta^6 Q^2\}, \{\eta^{18} Q, \eta^6 QR\}, \{\eta^{30}, \eta^{18} R, \eta^6 E_{12}\}, \{\eta^{18} Q^2, \eta^6 Q^2 R\}$
5.	2	2, 6. 8, 10, 12, 14, 16	$\{0\}, \{4\}. \{6\}, \{0, 8\}, \{10\}, \{0, 4\}\ \{6, 14\}$	$\{\eta^4\}, \{\eta^4 Q\}, \{\eta^4 R\}, \{\eta^4 Q^2, \eta^{20}\}, \{\eta^4 QR\}, \{\eta^4 E_{12}, \eta^{20} Q, \eta^{28}\}, \{\eta^4 Q^2 R, \eta^{20} R\}.$
6.	1	1, 5, 7, 9, 11, 13, 15	$\{0\}, \{0, 4\}, \{0, 6\}, \{4, 8\}, \{0, 4, 6, 10\}, \{0, 6, 8\}, \{4, 8, 10, 14\}$	$\{\eta^2\}, \{\eta^{10}, \eta^2 Q\}, \{\eta^{14}, \eta^2 R\}, \{\eta^{10} Q, \eta^2 Q^2\}, \{\eta^{22}, \eta^{14} Q, \eta^{10} R, \eta^2 QR\}, \{\eta^2 \Delta, \eta^2 E_{12}, \eta^{14} R, \eta^{10} Q^2\}, \{\eta^{22} Q, \eta^{14} Q^2, \eta^{10} QR, \eta^2 Q^2 R\}$

Case 1: $t = 12,\ F(z) = \sum_{n=1}^{\infty} a_n q^n \in \{\Gamma(1), k, 1\}_0.$

Except in the case of weights 24, 28, 30, the space $M_k(12)$ is 1-dimensional and hence the basis forms are normalized eigenfunctions (since $\lambda(1) = 1$) in all these cases. They all occur in [7]. The case of $k = 24$ does not occur explicitly in Ramanujan, except in his reference to it in his letter to Hardy (Appendix [6]). In this case we see easily that $\triangle E_{12} + \lambda\triangle^2$ is an eigenfunction and the constant λ can be determined. In fact Hecke says [2] $\lambda \in Q(\sqrt{144169})$ as can be verified. λ satisfies the quadratic equation $\lambda^2 + \lambda(2\mu - 1128) + (\mu^2 - 128272\mu + 2048 - 2^{23}) = 0$ (where $\mu = 65520/691$). The discriminant of this equation is $24^2 \times 144169$. In the case of weights 28 and 30, Ramanujan says [8] $\triangle^2 Q$ and $\triangle^2 R$ do not have Euler products, " however they are differences of two such products ". This is obvious from the fact that $\triangle E_{16} + \alpha_1 \triangle^2 Q$ and $\triangle E_{16} + \alpha_2\triangle^2 Q$ are Euler products and similarly $\triangle E_{18} + \beta_1\triangle^2 R$ and $\triangle E_{18} + \beta_2\triangle^2 R$ are Euler products for some constants α_1, α_2, β_1, β_2 which can be determined.

Case 2. $t = 6$. $F(z) = \Sigma\, a_n\, q^{n/2} \in \{\Gamma(2),\ k,\ 1\}_0$.

As in case (1), for all k except 18, the eigenfunctions are given by $\eta^{12}\ E_\nu$ where E_ν is an Eisenstein series of suitable weight. Only in the case of $k = 18$, the eigenfunctions are of the form $\eta^{12} E_{12} \pm \alpha\eta^{36}$ where α can be determined using (1). The only results of Ramanujan in this case are for the weight $k = 6$ in [7] and η^{36} (in the case $k = 18$) in his indirect reference in the letter [6].

Case 3. $t = 4$. $F(z) = \Sigma\, a_n\, q^{n/3} \in \{\Gamma(3),\ k,\ 1\}_0$.

It is easily seen that the above case refers to II of Ramanujan's results stated before. We observe first the Euler product for η^8 was already stated in [7] and proved by Mordell [4]. The function $\eta^8 P$ is not a cusp form, however using the operator $q\, d/dq$, one sees that $a_1(n) = n\, a_0(n)$ as in [7], and this yields the formal Euler product development for $\eta^8 P$ from that of η^8. In the case of $k = 10$, since dim $M_{10}(4) = 1$ and $\eta^8 R$ is normalized, the Dirichlet series of $\eta^8 R$ has an Euler product. In the case of weights $k \geqslant 8$ ($k \neq 10$) any eigenfunction is a linear combination of the forms given below and we need to evaluate the constants and compare them with the values given by Ramanujan. This is done by using the equation (1) for special primes p. The Fourier coefficients $a_0(n)$ can be taken from the table of Newman [5] (in fact $a_0(n) = 0$ if $4 \times n$ and $a_0(4\, m) = p_8(m)$ in Newman's notation). For the computation of Fourier coefficients of η^α, we also use Newman's tables [5]. By explicit computation we can evaluate $a_K(2)$ and $a_K(4)$ for various K. The identity we need is the following:

$$(a_K(2))^2 = a_K(4) + a_K(1) \cdot 2^{k-1}$$

$$= a_K(4) + 2^{2K+3} \text{ since } a_K(1) = 1 \text{ and } k = 2K + 4.$$

(1) $\underline{k = 8}$, $\underline{K = 2}$. $F(z) = \Sigma\, a_2(n)\, q^{n/3} = \eta^8 Q + \lambda_2\, \eta^{16}$ is a normalized eigenfunction for some value of λ_2 since $a_2(1) = 1$ and the term $q^{11/3}$ occurs only in $\eta^8 Q$. Then $a_2(2) = \lambda_2$ and

$$\lambda_2^2 = (a_2(2))^2 = a_2(4) + 2^7 = 360, \text{ i.e., } \lambda_2 = \pm 6\sqrt{10}.$$

(2) $k = 12$, $K = 4$, $F(z) = \Sigma\, a_4(n)\, q^{n/3}$

$$= \eta^8 Q^2 + \lambda_4 \eta^{16} Q \text{ is an eigenfunction.}$$

$$a_4(2)^2 = \lambda_4^2 = a_4(4) + 2^{11}.$$
$$= 472 + 2^{11} = 2520, \quad \lambda_4 = \pm 6\sqrt{70}.$$

(3) $k = 14$, $K = 5$, $F(z) = \Sigma a_5(n)\, q^{n/3}$

$$= \eta^8 QR + \lambda_5 \eta^{16} R \text{ is an eigenfunction}$$

$$a_5(2)^2 = \lambda_5^2 = a_5(4) + 2^{13}$$
$$= -272 + 2^{13} = 7920, \quad \lambda_5 = \pm 12\sqrt{55}.$$

(4) $k = 18$, $K = 7$, $F(z) = \Sigma\, a_7(n)\, q^{n/3}$

$$= \eta^8 Q^2 R + \lambda_7 \eta^{16} QR \text{ is an eigenfunction}$$

$$a_7(2)^2 = \lambda_7^2 = a_7(4) + 2^{17}$$
$$= -32 + 2^{17} = 144 \times 910, \quad \lambda_7 = \pm 12\sqrt{910}.$$

This exhausts Ramanujan's list. As can be seen, Ramanujan does not consider the case $K = 6$ corresponding to $k = 16$ although a reference to η^{32} is made in his letter to Hardy [6]. The eigenfunction is of the form $\eta^{32} + \lambda_6 \eta^{16} Q^2 + \mu_6 \eta^8 E_{12}$ for some constants λ_6 and μ_6. They can be evaluated by the above method.

Case 4. $t = 3$. $F(z) = \Sigma\, a_n\, q^{n/4} \in \{\Gamma(4), k, \epsilon\}_0$.

This case refers to Ramanujan's results in III, stated above. As in II, the Euler product for η^6 was already stated by Ramanujan in [7] and proved by Mordell [4]. The function $\eta^6 P$ is not a cusp form, however the relation $a_1(n) = n\, a_0(n)$ and the formal Euler product for $\eta^6 P$ can be derived as before. In the case of weights 7 and 11, the space is 1-dimensional and since $\eta^6 Q$ and $\eta^6 Q^2$ are normalized, the corresponding Dirichlet series have Euler products. For weights $k = 9$, 13, 15, 17, any eigenfunction is a linear combination of the basis forms as given below and we need to evaluate the constants and compare them with the values given by Ramanujan. This is done by using the identity (1) for the prime $p = 3$. Observe that $a_0(n) = 0$ if n is even or odd and $\not\equiv 1 \pmod 4$. In fact, if $n = 4m + 1$, $a_0(n) = p_6(m)$ in Newman's notation. As before we explicitly evaluate $a_K(3)$ and $a_K(9)$ for various K.

The identity we need is the following:

$$a_K(3)^2 = a_K(9) + \epsilon(3) \cdot 4^{k-1} \cdot a_K(1)$$
$$= a_K(9) - 2^{2K+2} \text{ since } a_K(1) = 1, \epsilon(3) = -1 \text{ and } k = 2K + 3.$$

(1) $k = 9$, $K = 3$. $F(z) = \Sigma a_3(n)\, q^{n/4}$

$$= \eta^6 R + \lambda_3 \eta^{18} \text{ is a normalized eigenfunction for some}$$

value of λ_3, since $a_3(1) = 1$ and the term $q^{1/4}$ occurs only in $\eta^6 R$.

$$a_3(3)^2 = \lambda_3^2 = a_3(9) - 3^8$$
$$= -24^2 \cdot 35, \quad \lambda_3 = \pm 24i\sqrt{35}.$$

(2) $k = 13$, $K = 5$. $F(z) = \Sigma\, a_5(n)\, q^{n/4}$

$$= \eta^6 QR + \lambda_5 \eta^{18} Q \text{ is an eigenfunction.}$$

$$a_5(3)^2 = \lambda_5^2 = a_5(9) - 3^{12}$$
$$= -24^2 \cdot 1155, \quad \lambda_5 = \pm 24i\sqrt{1155}$$

(3) $k = 17$, $K = 7$. $F(z) = \Sigma\, a_7(n)\, q^{n/4}$

$$= \eta^6 Q^2 R + \lambda_7 \eta^{18} Q^2 \text{ is an eigenfunction.}$$

$$a_7(3)^2 = \lambda_7^2 = a_7(9) - 3^{16}$$
$$= -120^2 \cdot 3003, \quad \lambda_7 = \pm 120i\sqrt{3003}.$$

As before, we observe that the case $K = 6$ (corresponding to $k = 13$ is not found in Ramanujan's results although there is an indirect reference to η^{30} in his letter. In this case the eigenfunction is of the form $\eta^6 E_{12} + \lambda_6 \eta^{18} R + \mu_6 \eta^{30}$ and the constants λ_6 and μ_6 can be evaluated as above.

Case 5 : $t = 2$, $F(z) = \Sigma\, a_n q^{n/6} \in \{\Gamma(6), k, 1\}_0$

This case refers to Ramanujan's results in I, stated above. As in other cases, the Euler product for η^4 was already stated by Ramanujan in [7] and proved by Mordell. The function $\eta^4 P$ is not a cusp form and as before we can derive $a_1(n) = na_0(n)$ and the formal Euler product for $\eta^4 P$ from that of η^4. In the case of weights 6, 8, 12 the space $M_k(2)$ is 1-dimensional and since $\eta^4 Q$, $\eta^4 R$, $\eta^4 QR$ are normalized, the corresponding Dirichlet series have Euler products. For $k = 10, 14, 16$ any eigenfunction is a linear combination of the basis forms as given below and we evaluate the constants and compare them with the values given by Ramanujan. This is done by using (1) for the prime $p = 5$. Observe that $a_0(n) = 0$ unless $n \equiv 1 \pmod 6$ and if $n = 6m + 1$, $a_0(n) = p_4(m)$ in Newman's notation. We shall explicitly evaluate $a_K(5)$ and $a_K(25)$ for different K.

The identity we need is the following:

$$a_K(5)^2 = a_K(25) + 5^{k-1} \cdot a_K(1)$$
$$= a_K(25) + 5^{2K+1} \text{ since } a_K(1) = 1 \text{ and } k = 2K + 2.$$

(1) $k = 10$, $K = 4$. $F(z) = \Sigma\, a_4(n)\, q^{n/6}$

$= \eta^4 Q^2 + \lambda_4 \eta^{20}$ is a normalized eigenfunction for some value of λ_4, since $a_4(1) = 1$ and the term $q^{1/6}$ occurs only in $\eta^4 Q^2$.

$$\lambda_4^2 = a_4(5)^2 = a_4(25) + 5^9.$$
$$= 288^2 \cdot 70, \quad \lambda_4 = \pm 288\sqrt{70}.$$

(2) $k = 16$, $K = 7$. $F(z) = \Sigma\, a_7(n)\, q^{n/6}$

$= \eta^4 Q^2 R + \lambda_7 \eta^{20} R$ is a normalized eigenfunction for some value of λ_7 since $a_7(1) = 1$ and the term $q^{1/6}$ occurs only in $\eta^4 Q^2 R$.

$$a_7(5)^2 = \lambda_7^2 = a_7(25) + 5^{15}$$
$$= (10080)^2 \cdot 286, \text{ i.e., } \lambda_7 = \pm 10080\sqrt{286}.$$

As before, the case $K = 6$ (corresponding to $k = 14$) is missing from Ramanujan's results although an indirect reference to η^{28} occurs in his letter [6]. From our table, we can see that the eigenfunction is of the form $\eta^4 E_{12} + \lambda_6 \eta^{20} Q + \mu_6 \eta^{28}$ and the constants λ_6 and μ_6 can be determined as before.

Case 6. $t = 1$. $F(z) = \Sigma\, a_n\, q^{n/12} \in \{\Gamma(12),\, k,\, \epsilon\}_0$.

This case refers to Ramanujan's results in IV quoted above. Although no explicit mention is made about the Euler products, in the context it is obvious that Ramanujan states that $\sum_{n=1}^{\infty} a_K(n) \cdot n^{-s}$ are all Euler products. As before, the Euler product for η^2 already occurs in [7] and proved by Mordell [4]. The function $\eta^2 P$ is not a cusp form; however, from the equation $a_1(n) = n a_0(n)$ (derived as before), the formal Euler product for $\eta^2 P$ follows from that of η^2. In all the other cases, the eigenfunction is a linear combination of the basis forms as given below and we need to evaluate the constants and compare them with the values given by Ramanujan. This is done by using (1) for the primes $p = 5$, 7 and 11 in the different cases.

[Observe that $a_0(n) = 0$ if n is even or odd with $n \not\equiv 1 \pmod{12}$. In fact $a_0(n) = p_2(m)$ if $n = 12m + 1$ in Newman's notation]. The constants $a_K(n)$ are explicitly evaluated for various values of K and n.

(1) $k = 5,\ K = 2$. $F(z) = \Sigma\, a_2(n)\, q^{n/12}$

$= \eta^2 Q + \lambda_2 \eta^{10}$ is a normalized eigenfunction for some value of λ_2, since $a_2(1) = 1$ and the term $q^{1/12}$ occurs only in $\eta^2 Q$.

$$a_2(5)^2 = \lambda_2^2 = a_2(25) + 5^4 \quad (\text{since } \epsilon(5) = 1).$$
$$= 1679 + 5^4 = 2304, \quad \lambda_2 = \pm 48.$$

(2) $k = 7,\ K = 3$. $F(z) = \Sigma\, a_3(n)\, q^{n/12}$

$= \eta^2 R + \lambda_3 \eta^{14}$ is a normalized eigenfunction for some value of λ_3, since $a_3(1) = 1$ and the term $q^{1/12}$ occurs only in $\eta^2 R$.

$$a_3(7)^2 = \lambda_3^2 = a_3(49) - 7^6 \quad (\text{since } \epsilon(7) = -1)$$
$$= -3 \cdot (360)^2 \text{ i.e., } \lambda_3 = \pm 360i\sqrt{3}.$$

(3) $k = 9,\ K = 4$. $F(z) = \Sigma\, a_4(n)\, q^{n/12} = \eta^2 Q^2 + \lambda_4 \eta^{10} Q$

$$a_4(5) = \lambda_4 \quad a_4(25) = 60959$$
$$\lambda_4^2 = a_4(25) + 5^8 = 451584, \text{ i.e., } \lambda_4 = \pm 672.$$

(4) $k = 11,\ K = 5$. $F(z) = \Sigma\, a_5(n)\, q^{n/12}$

$$= \eta^2\, QR + \lambda_5 \eta^{10} R + \mu_5 \eta^{14} Q + \nu_5 \eta^{22}$$

is a normalized eigenfunction for some values λ_5, μ_5, ν_5, since $a_5(1) = 1$ and the term $q^{1/12}$ occurs only in $\eta^2 QR$.

(a) $\lambda_5^2 = a_5(5)^2 = a_5(25) + 5^{10} = 96^2 \times 1045$, i.e., $\lambda_5 = \pm 96\sqrt{1045}$

(b) $\mu_5^2 = a_5(7)^2 = a_5(49) - 7^{10} = -216^2 \cdot 7315$, i.e., $\mu_5 = \pm 216i\sqrt{7315}$

(c) $\nu_5^2 = a_5(11)^2 = a_5(121) - 11^{10} = -7 \cdot (103680)^2$, i.e., $\nu_5 = \pm 103680i\sqrt{7}$.

(4) $k = 15,\ K = 7.$ $F(z) = \Sigma\, a_7(n)\, q^{n/12}$

$$= \eta^2 Q^2 R + \lambda_7\, \eta^{10}\, QR + \mu_7\, \eta^{14}\, Q^2 + \nu_7\, \eta^{22}\, Q$$

is a normalized eigenfunction for some values λ_7, μ_7, ν_7, since $a_7(1) = 1$ and the term $q^{1/12}$ occurs only in $\eta^2 Q^2 R$.

(a) $\lambda_7^2 = a_7(5)^2 = a_7(25) + 5^{14} = 48^2 \times 910 \times 2911;$
$\lambda_7 = \pm\, 48 \sqrt{910 \cdot 2911}$

(b) $\mu_7^2 = a_7(7)^2 = a_7(49) - 7^{14} = -\,216^2 \times 5005 \times 2911;$
$\mu_7 = \pm\, 216 i \sqrt{5005 \cdot 291}$

(c) $\nu_7^2 = a_7(11)^2 = a_7(121) - 11^{14} = -\,2200 \times (471744)^2\,;$
$\nu_7 = \pm\, 4717440 i \sqrt{22}.$

The computations in (3) and (4) were carried out in the TIFR Computer. Notice that all the values for the constants agree with those given by Ramanujan except ν_7, where the difference is a factor of 10. Again as in other cases, $K = 6$ (corresponding to $K = 13$) is missing from Ramanujan's results. We see from our table that the eigenfunction is of the form $\eta^2 E_{12} + \lambda_6 \eta^2 \Delta + \mu_6 \eta^{10} Q^2 + \nu_6\, \eta^{14} R$ for some constants λ_6, μ_6, ν_6 which can be evaluated as above.

5. Explicit form of Euler products of § 2c

In this paragraph, we shall prove the results of § 2c on the explicit form of Euler products.

All the Euler products mentioned in § 2c are Hecke L-series attached to $Q(\sqrt{-3})$ and $Q(\sqrt{-1})$. In fact we shall see that the Euler products in Cases I and II of Ramanujan (whenever the dimension is 1) are Hecke L-series attached to $Q(\sqrt{-3})$ and those in Case III (whenever the dimension is 1) are Hecke L-series attached to $Q(\sqrt{-1})$. These Dirichlet series are in turn Mellin transforms of theta series $\theta_k(\tau, \rho, \theta, Q\sqrt{D})$ introduced by Hecke (23, [2]). We briefly recall their definition and properties.

Let $\varphi(\sqrt{D})$ be an imaginary quadratic field with discriminant D, θ, an integral ideal in $\varphi(\sqrt{D})$, $\rho \in \theta$ and Q, a positive rational integer. Let $A = |N\mathfrak{a}|$. Then for every natural number k, we form the following binary theta series:

$$\theta_k(\tau, \rho, \mathfrak{a}, Q\sqrt{D}) = \sum_{\mu \equiv \rho \,(\mathrm{mod}\ \mathfrak{a}\, Q\sqrt{D})} \mu^{k-1} \exp\left(2\pi i \tau \frac{N\mu}{AQ\,|\,D\,|}\right)$$

Since $\mathfrak{a}$ and Q are fixed throughout, we may denote this for simplicity, $\theta_k(\tau, \rho)$. We have then the following relations.

(1) $\theta_k(\tau, \rho\epsilon) = \epsilon^{k-1} \cdot \theta_k(\tau, \rho)$ for every unit ϵ in $Q(\sqrt{D})$

(2) $\theta_k(\tau + 1, \rho) = \exp\,(2\pi i\,(N\rho/AQ\,|\,D\,|)\ \theta_k(\tau, \rho)$

(3) $\theta_k(-1/\tau, \rho) = \tau^k / Q\sqrt{D} \sum_{\substack{a \bmod \theta\, Q\sqrt{D} \\ a \in \mathfrak{a}}} \exp\,(2\pi i\, tr\,(a\bar{\rho}/AQD)\ \theta_k(\tau, a).$

We shall be concerned only with $D = -3$ and $D = -4$ in this section, and special ρ and $\mathfrak{a}$.

(a) $D = -3$. (i) $\rho = 1$, $\mathfrak{a} = 1$, $Q = 1$

(ii) $\rho = 1$, $\mathfrak{a} = 1$, $Q = 2$.

(i) From properties (1)–(3) listed, we see that in this case $\theta_k(\tau, 1) \in M_k(4$ (case (3) or § 4 or II of results in § 2). The values of k fcr whic dim $M_k(4) = 1$, are $k = 4$ and 10. Hence the cusp forms η^8 and $\eta^8 R$ have for Mellin transforms the Hecke L-series $L(s, \chi_4, Q(\sqrt{-3}))$ and $L(s, \chi_{10}, Q(\sqrt{-3}))$ where $\chi_4((\mu)) = \mu^3$ for $\mu \equiv 1 \pmod{\sqrt{-3}}$ and $\chi_{10}((\mu)) = \mu^9$ for $\mu \equiv 1 \pmod{\sqrt{-3}}$, in agreement with Ramanujan' results in II of § 2.

(ii) In this case we may verify that $\theta_k(\tau, 1, 1, 2\sqrt{-3}) \in M_k(2)$ (case 5 of § 4 or I of results in § 2). The values of k for which dim $M_k(2) = 1$ are $k = 2, 6, 8, 12$ and the cusp forms are given by η^4, $\eta^4 Q$, $\eta^4 R$ and $\eta^4 QR$. Their Mellin transforms are given by $L(s, \chi_k, Q(\sqrt{-3})$ where $\chi_k((\mu)) = \mu^{k-1}$ for $\mu \equiv 1 \pmod{2\sqrt{-3}}$ ($k = 2, 6, 8, 12$) in agreement with Ramanujan's results in I of § 2.

(b) $D = -4$. (i) $\rho = 1$, $\mathfrak{a} = 1$, $Q = 1$ (ii) $\rho = 2$, $\mathfrak{a} = 1$, $Q = 2$.

(i) As in the case of $D = -3$, we deduce that $\theta_k \in M_k(3)$ (Case (4) of § 4 or III of results in § 2). The values of k for which dim $M_k(3) = 1$ are $k = 3, 7, 11$ and the cusp forms are given by η^6, $\eta^6 Q$, $\eta^6 Q^2$. Their Mellin transforms are given by $L(s, \chi_k, Q(\sqrt{-1}))$ where $\chi_k((\mu)) = \mu^{k-1}$ for $\mu \equiv 1 \pmod{\sqrt{-4}}$ and $k = 3, 7, 11$. This is in agreement with Ramanujan's results in III of § 2.

(ii) In this case $\theta_k(\tau, 2, 1, 2\sqrt{-4}) \in M_k(6)$ (Case (2) of § 4) and the values of k for which dim. $M_k(6) = 1$ are $k = 6, 10, 12, 14, 16, 20$ and the cusp forms are given by η^{12}, $\eta^{12} Q$, $\eta^{12} R$, $\eta^{12} Q^2$, $\eta^{12} QR$ and, $\eta^{12} Q^2 R$. Their Mellin transforms are given by $L(s, \chi_k, Q(\sqrt{-4}))$ where $\chi_k((\mu)) = \mu^{k-1}$ for $\mu \equiv 2 \pmod{2\sqrt{-4}}$) and $k = 6, 10, 12, 14, 16, 20$. Although these Euler products do not figure in Ramanujan's results quoted in § 2, the Euler product for η^{12} may be identified with that for $\sum_{n=1}^{\infty} e_{12}(n) \cdot n^{-s}$ (notation as in [7]) as shown by Rankin [9]. The other Euler products $\eta^{12} Q$, $\eta^{12} R$ also occur in Rankin [9].

The Euler products of Case (1) of § 4, corresponding to the full modular group, do not belong to the above category. It has been shown by Hecke that $\Delta(\tau)$ is in fact a theta-series (with spherical harmonics) attached to a quadratic form in 8 variables. (41, [2]).

The Euler product for η^2 (Case (6) of § 4 or IV of § 2) is the Mellin transform of a theta series attached to a real quadratic field. We shall take it up in the next paragraph.

6. Euler products of (a) and (b) of § 2

Regarding (a), except for $\alpha = 1, 2, 3$, all the other cases are included in § 5. The Mellin transforms of $\eta(z)$ and $\eta^3(8z)$ are given by Dirichlet L-series $L(s, \chi)$ where $\chi(n) = (3/n)$ and $(-1/n)$ respectively [4]. The Mellin transform of $\eta^2(12z)$ is the Dirichlet series $L(s, \chi, Q(4\sqrt{12}, i)/Q(i)$ which is also the Mellin transform of $\theta_+(\tau, 1, 1, \sqrt{12})$ (23, [2]).

Proof of (1)-(5). We shall take up (3) first since it is already found in the literature. [10, 11].

(3) $\eta(z)\,\eta(23z)$ is a cusp form (new) for $\Gamma_0(23)$ of nebentype belonging to the character $\epsilon(n) = (23/n)$ and weight -1. From the theorem of Serre–Deligne [11] its Mellin transfoim is an L-series $L(s, \rho)$ attached to a 2-dimensional dihedral representation ρ of $G(\bar{Q}/Q)$. More explicitly, ρ is the 2-dimensional representation induced by the non-trivial cubic character of the Galois group of the Hilbert class field $Q(\sqrt{-23}, \alpha)$ (α satisfies $x^3 - x - 1 = 0$) of $Q(\sqrt{-23})$. It can be verified by using the explicit decomposition of prime ideals of $Q(\sqrt{-23})$ in $Q(\sqrt{-23}, \alpha)$ and the definition of $L(s, \rho)$ that Ramanujan's statement (3) means precisely this.

(2) In this case $\eta(z)\,\eta(11z)$ is a cusp form of weight -1 belonging to $\Gamma_0(44)$ and character $(44/n)$ and as before its Mellin transform is $L(s, \rho)$ where ρ is the 2-dimensional dihedral representation induced by the non-trivial cubic character of Galois group of the Hilbert class field $Q(\sqrt{-11}, \beta)$ (β satisfies $x^3 - 2x^2 + 2x - 2 = 0$) of $Q(\sqrt{-11})$. It can be verified as above that Ramanujan's statement (2) is precisely this.

(1) This case is similar to (2) and (3) but slightly different. The form $\eta(z)\,\eta(7z)$ is a cusp form of weight -1 belonging to $\Gamma_0(63)$ and character $(63/n)$ and by Serre–Deligne [11] its Mellin transform is of the form $L(s, \rho)$ for a 2-dimensional dihedral representation ρ. In this case it is induced by the non-trivial character of the Galois group Gal $(Q(\sqrt{21}, \sqrt{-3})Q/(\sqrt{21}))$. [In the statement of (1), $p \equiv 10, 13, 19 \pmod{21}$ is missing].

(4) The Euler product has been determined by Hecke. [2].

(5) This is also found in Hecke and in fact this is the Hecke L-series $L(s, \chi_3, Q(\sqrt{-7}))$, the type we had in § 5.

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