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A note on the mean value of L-series

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Abstract. Using Hilbert's inequality, we give a new asymptotic formula (uniform in q and T) for

$$\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi \text{ primitive}}} \int_T^{2T} | L(\tfrac{1}{2} + it, \chi)^4 | \, dt$$

Keywords. L-series; primitive characters; mean value.

1. Introduction

The object of this paper is to give a new asymptotic formula for

$$\sum_{\chi}^{*} \int_T^{2T} | L(\tfrac{1}{2} + it, \chi)^4 | \, dt,$$

uniform in q and T where $\sum_{\chi}^{*}$ denotes summation over primitive characters modulo q.

Before stating our theorem, we introduce our notation.

N denotes the number of primitive characters (mod q). $\sum_{\chi}^{*}$ denotes summation over primitive characters (mod q). We assume $N \geq 1$. It should be noted that $N \geq 1$, if and only if either q is odd or 4 divides q. We write $q = p_1^{l_1} p_2^{l_2} \ldots p_r^{l_r}$, where p_i' s are primes and r = number of distinct prime factors of q.

Also $\quad N = \prod_{i=1}^{r} \theta(p_i^{l_i}),$

where $\quad \theta(p_i^{l_i}) = p_i - 2$ for $l_i = 1$

$$= p_i^{l_i - 2}(p_i - 1)^2 \text{ for } l_i \geq 2,$$

for $i = 1, 2, \ldots, r$.

In what follows, $\phi(n)$, $d(n)$, $\mu(n)$ denote Euler's totient function, divisor function and Möbius function respectively.

We note that

$$\frac{q}{N} \ll \frac{q^2}{\phi^2(q)} \ll (\log\log 3q)^2.$$

We use the fact

$$\sum_{n \leqslant y} d^2(n) \ll y \log^3 y \text{ for } y > 1,$$

quite frequently in the sequel. For convenience, we write $X = qT$. In the sequel, p with or without subscript denotes a prime number. Our theorem heavily depends on lemma 5, which, in its turn, is a consequence of lemmas 3 and 4.

2. Statement of the theorem

Let $L(s, \chi)$ be Dirichlet's L-function to the modulus q. Let $X = qT$ and $N =$ the number of primitive characters (mod q) *and let $\gamma =$ distinct prime factors of q. Then for $T \geq 2$ and for all q (for which $N \geq 1$),*

$$\frac{1}{N} \sum_{\chi}{}^{*} \int_{T}^{2T} | L(\tfrac{1}{2} + it, \chi) |^4 \, dt$$

$$= \frac{1}{2\pi^2} \frac{\prod_{p|q} \left(1 - \frac{1}{p}\right)}{\prod_{p|q} \left(1 - \frac{1}{p^2}\right)} T \log^4 X + O(2^r T \log^3 X (\log\log 3q)^5).$$

Remark. The special case $q = 1$ gives the theorem of Ingham for zeta-function viz.,

$$\int_{T}^{2T} | \zeta(\tfrac{1}{2} + it) |^4 \, dt = \frac{T}{2\pi^2} \log^4 T + O(T \log^3 T).$$

3. Proof of the theorem

For every primitive character χ (mod q), we have functional equation of the type

$$L(s, \chi) = \psi(s, \chi) L(1 - s, \bar{\chi}).$$

We note that

$$(qt)^{\frac{1}{2}-\sigma} \ll \psi(s, \chi) \ll (qt)^{\frac{1}{2}-\sigma}$$

where $s = \sigma + it$, uniformly for all primitive characters χ (mod q) and for σ in a fixed interval.

Lemma 1. (a) : *Let*

$$s = \sigma + it, |\sigma - \tfrac{1}{2}| \leq \frac{1}{100}, \; T \leq t \leq 2T, w = u + iv.$$

Then, for any primitive character $\chi \pmod q$,

$$\begin{aligned} L^2(s, \chi) = & \sum_{n=1}^{\infty} d(n)\, \chi(n)\, e^{-n/X} n^{-s} \\ & + \psi^2(s, \chi) \sum_{n \leq X} d(n)\, \overline{\chi(n)}\, n^{s-1} \\ & - \frac{1}{2\pi i} \int_{u=-\frac{3}{4}} \psi^2(s + w, \chi) \Big(\sum_{n>X} d(n)\, \overline{\chi(n)} n^{w+s-1} \Big) \Gamma(w)\, X^w dw \\ & - \frac{1}{2\pi i} \int_{u=\frac{1}{4}} \psi^2(s + w, \chi) \Big(\sum_{n \leq X} d(n)\, \overline{\chi(n)}\, n^{w+s-1} \Big) \Gamma(w)\, X^w dw \\ & + O(T^{-10}). \end{aligned}$$

Remark. Note that

$$L^2(s, \chi) = \sum_{n=1}^{\infty} \frac{d(n)\, \chi(n)}{n^s} \text{ for } \sigma > 1.$$

This lemma can be proved by starting with the integral

$$\frac{1}{2\pi i} \int_{u=2} L^2(s + w, \chi)\, \Gamma(w)\, X^w\, dw;$$

moving the line of integration to Re $w = u = -3|4$; using the functional equation for $L^2(s + w, \chi)$ there, splitting the series for $L^2(1 - s - w, \bar{\chi})$ and consequently, the integral in an obvious manner and moving the line of integration in one of them back to $u = 1/4$. Also, we note that the error term $O(T^{-10})$ occurs, only when $q = 1$ i.e. in the case of ζ-function, since in that case, before effecting the change of line of integration for

$$\frac{1}{2\pi i} \int_{u=2} \zeta^2(s + w)\, \Gamma(w)\, X^w\, dw$$

we have to break the line of integration suitably, to avoid the pole of zeta function. We rewrite this lemma in a modified form to suit our needs as follows :

Lemma 1. (b): *For* $|\sigma - \tfrac{1}{2}| \leq \frac{1}{100}$ and $T \leq t \leq 2T$,

$$\begin{aligned} \psi(s, \chi)^{-1} L^2(s, \chi) = & \psi(s, \chi)^{-1} \sum_{n \leq X} d(n)\, \chi(n)\, n^{-s} \\ & + \psi(s, \chi) \sum_{n \leq X} d(n)\, \overline{\chi(n)}\, n^{s-1} + \psi(s, \chi)^{-1} \sum_{n \leq X} d(n)\, \chi(n) n^{-s} (e^{-n/X} - 1) \\ & + \psi(s, \chi)^{-1} \sum_{n>X} d(n)\, \chi(n)\, n^{-s} \exp(-n/x) \end{aligned}$$

$$- \frac{\psi(s,\chi)^{-1}}{2\pi i} \int_{u=-\frac{3}{4}} \psi^2(s+w,\chi) \Big(\sum_{n>X} d(n)\, \overline{\chi(n)}\, n^{w+s-1}\Big) \Gamma(w) X^w \, dw$$

$$- \frac{\psi(s,\chi)^{-1}}{2\pi i} \int_{u=\frac{1}{4}} \psi^2(s+w,\chi) \Big(\sum_{n\le X} d(n)\, \overline{\chi(n)}\, n^{w+s-1}\Big) \Gamma(w) X^w \, dw$$

$$+ O(T^{-10}) = \sum_{k=1}^{7} J_k(s,\chi), \text{ say.}$$

We write $J_k(\frac{1}{2} + it, \chi) = J_k(\chi)$ and since $J_1(\chi)$, $J_2(\chi)$ are complex conjugates, we get the following :

Lemma 2. *We have*

$$= \frac{1}{N} \sum_{\chi}^{*} \int_{T}^{2T} \Big| L\Big(\frac{1}{2} + it, \chi\Big)\Big|^4 dt$$

$$= \frac{1}{N} \sum_{\chi}^{*} \int_{T}^{2T} (J_1(\chi) + J_2(\chi))^2 \, dt$$

$$+ O\Big(\frac{1}{N} \sum_{\chi}^{*} \int_{T}^{2T} \sum_{k=3}^{6} |J_k^2(\chi)| \, dt\Big)$$

$$+ O\Big\{\frac{1}{N} \sum_{\chi}^{*} \int_{T}^{2T} (J_1(\chi) + J_2(\chi)) \Big(\sum_{k=3}^{6} J_k(\chi)\Big) dt\Big\}$$

$$= \frac{2}{N} \sum_{\chi}^{*} \int_{T}^{2T} |J_1^2(\chi)| \, dt$$

$$+ O\Big(\frac{1}{N} \sum_{\chi}^{*} \int_{T}^{2T} (J_1^2(\chi) + J_2^2(\chi)) \, dt\Big)$$

$$+ O\Big(\frac{1}{N} \sum_{\chi}^{*} \int_{T}^{2T} \sum_{k=3}^{6} |J_k^2(\chi)| \, dt\Big)$$

$$+ O\Big(\frac{1}{N} \sum_{\chi}^{*} \int_{T}^{2T} (J_1(\chi) + J_2(\chi)) \Big(\sum_{k=3}^{6} J_k(\chi)\Big) dt\Big).$$

Lemma 3. *Let* $\{b(n)\}$ $n = 1, 2, \cdots, R$ *be a sequence of complex numbers and* Q *be any integer. Then*

$$\sum_{\substack{m \equiv n \,(\mathrm{mod}\, Q)}} \sum_{m \ne n} \frac{b(m)\overline{b(n)}}{\log m - \log n}$$

$$= O\Big(\sum_{n\le Q/4} \frac{|b(n)|^2}{\log(Q/n)}\Big) + O\Big(\frac{1}{Q} \sum_{n=1}^{R} |n\, b(n)|^2\Big).$$

Proof. We have

$$\sum_{\substack{m \equiv n \,(\mathrm{mod}\, Q)}} \sum_{m \ne n} \frac{b(m)\overline{b(n)}}{\log m - \log n}$$

$$= \sum_{j=1}^{Q} \Bigg(\sum_{\substack{m', n' \ge 0}} \sum_{m' \ne n'} \frac{b(m'Q + j)\overline{b(n'Q + j)}}{\log(m'Q + j) - \log(n'Q + j)} \Bigg).$$

Now according to equation 1.7 of Montgomery and Vaughan [1], if $\{\lambda_n\}$ $n = 1, 2, \ldots, R$, is a sequence of distinct real numbers and if

$$\delta_n = \min_{m \neq n} |\lambda_n - \lambda_m|,$$

then for any sequence $\{c(n)\}$ $n = 1, 2, \ldots, R$ of complex numbers, we have

$$\left| \sum_{m \neq n} \sum \frac{c(m)\overline{c(n)}}{\lambda_m - \lambda_n} \right| \leq \frac{3\pi}{2} \sum_n |c(n)|^2 \delta_n^{-1}.$$

Using this result, we get

$$\sum_{\substack{m' \neq n' \\ m', n' \geq 0}} \sum \frac{b(m'Q+j)\overline{b(n'Q+j)}}{\log \dfrac{m'Q+j}{n'Q+j}} \ll \sum_{n' \geq 0} |b(n'Q+j)|^2 \delta_{n'}^{-1},$$

where $\delta_{n'} = \log \dfrac{(n'+1)Q+j}{n'Q+j}$,

and for $n' \geq 1$,

$$\delta_{n'} \gg \frac{1}{n'} \text{ for } 1 \leq j \leq Q.$$

Thus
$$\sum_{\substack{m' \neq n' \\ m', n' \geq 0}} \sum \frac{b(m'Q+j)\overline{b(n'Q+j)}}{\log \dfrac{m'Q+j}{n'Q+j}} \ll \frac{|b(j)|^2}{\log \dfrac{Q+j}{j}}$$

$$+ \sum_{n' \geq 1} n' |b(n'Q+j)|^2$$

$$\ll \frac{|b(j)|^2}{\log \dfrac{Q+j}{j}} + \frac{1}{Q} \sum_{n' \geq 1} (n'Q+j)|b(n'Q+j)|^2.$$

Therefore,

$$\sum_{\substack{m \neq n \\ m \equiv n \pmod{Q}}} \sum \frac{b(m)\overline{b(n)}}{\log(m/n)}$$

$$\ll \sum_{j=1}^{Q} \frac{|b(j)|^2}{\log \dfrac{Q+j}{j}} + \frac{1}{Q} \sum_n n|b(n)|^2$$

$$\ll \sum_{j \leq Q/4} \frac{|b(j)|^2}{\dfrac{Q+j}{j}} + \frac{1}{Q} \sum_n n|b^2(n)|$$

Thus
$$\sum_{\substack{m \neq n \\ m \equiv n \pmod{Q}}} \sum \frac{b(m)\overline{b(n)}}{\log(m/n)}$$

$$\ll \sum_{n \leq Q/4} \frac{|b^2(n)|}{\log(Q/n)} + \frac{1}{Q} \sum_n n|b^2(n)|,$$

Lemma 4. *For any sequence* $\{b(n)\}$ *of complex numbers, we have,*

$$\sum^{*}_{\chi \,(\mathrm{mod}\, q)} \sum_{m \neq n} \sum \frac{b(n)\,\overline{b(m)}\,\chi(n)\,\overline{\chi(m)}}{\log(m/n)}$$

$$\ll \sum_{k|q} |\mu(q/k)| \left\{ \phi(k) \sum_{n \leq k/4} \frac{|b^2(n)|}{\log(k/n)} + \frac{\phi(k)}{k} \sum_{n} |b^2(n)| \right\}$$

$$\ll 2^r \left\{ \max_{k|q} \left(k \sum_{n \leq k/4} \frac{|b^2(n)|}{\log(k/n)} \right) + \sum_{n} |b^2(n)| \right\}.$$

Proof. We have,

$$\sum^{*}_{\chi \,(\mathrm{mod}\, q)} \sum_{m \neq n} \sum \frac{b(n)\,b(m)\,\chi(n)\,\overline{\chi(m)}}{\log(m/n)}$$

$$= \sum_{\substack{m \neq n \\ (m,q)=1=(n,q)}} \sum \frac{b(n)\,\overline{b(m)}}{\log m/n} \sum^{*}_{\chi \,(\mathrm{mod}\, q)} \chi(n)\,\overline{\chi(m)}.$$

Now, by virtue of well-known properties of primitive characters,

$$\Sigma^{*}_{\chi \,(\mathrm{mod}\, q)} \chi(n)\,\overline{\chi(m)}$$

$$= \prod_{j=1}^{r} \left(\left(\Sigma^{*}_{\chi \left(\mathrm{mod}\, p_j^{l_j}\right)} \chi(n)\,\overline{\chi(m)}\right) \right)$$

$$= \prod_{j=1}^{r} \left(\left(\Sigma_{t_j=0}^{l_j} \mu(p_j^{l_j - t_j})\, \Sigma_{\chi \left(\mathrm{mod}\, p_j^{t_j}\right)} \chi(n)\,\overline{\chi(m)} \right) \right)$$

$$= \Sigma_{k|q}\, \mu(q/k) \left(\Sigma_{\chi \,(\mathrm{mod}\, k)} \chi(n)\,\overline{\chi(m)} \right).$$

In view of the relation, $\Sigma_{\chi \,(\mathrm{mod}\, Q)} \chi(n)\,\overline{\chi(m)} = \phi(Q)$ if $m \equiv n \,(\mathrm{mod}\, Q)$, $(n, Q) = 1$; and zero, otherwise, for any modulus Q, we get

$$\sum^{*}_{\chi \,(\mathrm{mod}\, q)} \sum_{m \neq n} \sum \frac{b(n)\,\overline{b(m)}\,\chi(n)\,\overline{\chi(m)}}{\log(m/n)}$$

$$= \sum_{k|q} \phi(k)\,\mu(q/k) \left(\sum_{\substack{m \neq n \\ m \equiv n \,(\mathrm{mod}\, k) \\ (n,k)=1}} \sum \frac{b(n)\,\overline{b(m)}}{\log(m/n)} \right).$$

The proof now follows by appealing to the preceding lemma.

Lemma 5. *We have for any sequence* $\{a(n)\}$ *of complex numbers,*

$$\frac{1}{N}\sum_{\chi}^{*}\int_{T}^{T+H}\left|\sum_{n\le X} a(n)\chi(n)n^{-it}\right|^{2}dt$$
$$= H\sum_{\substack{n\le X\\(n,q)=1}} |a^{2}(n)| + O\left(\frac{2^{r}}{N}\max_{k|q} k\sum_{n\le k/4}\frac{|a^{2}(n)|}{\log(k/n)}\right)$$
$$+ O\left(\frac{2^{r}}{N}\sum_{n\le X} n\,|a^{2}(n)|\right).$$

Lemma 6. *We have*

$$\frac{2}{N}\sum_{\chi}^{*}\int_{T}^{2T}|J_{1}^{2}(\chi)|\,dt = 2T\sum_{\substack{n\le X\\(n,q)=1}}\frac{d^{2}(n)}{n}$$
$$+ O\left(2^{r}\frac{q}{N}(T\log^{3}X)(\log\log 3q)\right)$$
$$= \frac{1}{2\pi^{2}}\prod_{p|q}\frac{\left(1-\frac{1}{p}\right)^{4}}{\left(1-\frac{1}{p^{2}}\right)}T(\log X)^{4} + O\left(2^{r}T\log^{3}X(\log\log 3q)^{5}\right).$$

Proof. Using the preceding lemma,

$$\frac{2}{N}\sum_{\chi}^{*}\int_{T}^{2T}|J_{1}^{2}(\chi)|\,dt = 2T\sum_{\substack{n\le X\\(n,q)=1}}\frac{d^{2}(n)}{n}$$
$$+ O\left(2^{r}\frac{q}{N}\max_{k|q}\left(\sum_{n\le k/4}\frac{d^{2}(n)}{n\log k/n}\right)\right) + O\left(\frac{2^{r}}{N}\sum_{n\le X}d^{2}(n)\right).$$

The second error term is easily seen to be

$$\ll 2^{r}\frac{q}{N}T\log^{3}X \ll 2^{r}T\log^{3}X(\log\log 3q)^{2}.$$

We, now, prove

$$\sum_{n\le k/4}\frac{d^{2}(n)}{n\log k/n}$$
$$\ll \log^{3}k(\log\log 3k).$$

This will give the first error term to be

$$\ll 2^{r}\frac{q}{N}\log^{3}q(\log\log 3q) \ll 2^{r}\log^{3}q(\log\log 3q)^{3}$$

writing $K = [\log(k/4)/\log 2]$, we have

$$\sum_{n\le k/4}\frac{d^{2}(n)}{n\log k/n}$$
$$= \sum_{0\le m\le K-1}\left(\sum_{2^{m}\le n<2^{m+1}}\frac{d^{2}(n)}{n\log k/n}\right)$$

$$\leq \sum_{0\leq m\leq K-1} \frac{1}{2^m (\log k/2^{m+1})} \Big(\sum_{n\leq 2^{m+1}} d^2(n)\Big)$$

$$\ll \sum_{0\leq m\leq K} \frac{m^3}{((\log k/\log 2) - m)}$$

$$\ll \log^3 k\,(\log\log 3k).$$

We now prove

$$\sum_{\substack{n\leq X\\(n,q)=1}} \frac{d^2(n)}{n}$$

$$= \frac{1}{4\pi^2} \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^4}{\left(1-\frac{1}{p^2}\right)} \log^4 X + O(\log^3 X(\log\log 3q)^5),$$

consider

$$\frac{\zeta^4(s+1)}{\zeta(2s+2)}\eta(s) = \sum_{(n,q)=1} \frac{d^2(n)}{n^{s+1}},$$

where $\eta(s) = \prod_{p|q} \frac{\left(1-\frac{1}{p^{s+1}}\right)^4}{\left(1-\frac{1}{p^{2s+2}}\right)}.$

We consider

$$\frac{1}{2\pi i}\int_{1+(\log x)^{-1}-iX^3}^{1+(\log x)^{-1}+iX^3} \frac{\zeta^4(s+1)}{\zeta(2s+2)}\eta(s)\frac{X^s}{s}\,ds,$$

which is equal to

$$\sum_{\substack{n\leq X\\(n,q)=1}} \frac{d^2(n)}{n} + O\left(\frac{X}{X^3}\sum_n \frac{d^2(n)}{n^{2+1/\log X}\,|\log(X/n)|}\right).$$

We now move the line of integration (in the above integral) to $\sigma = -1/4$. In doing so, we come across a pole at $s = 0$, the residue at which is

$$\frac{\eta(0)\log^4 X}{4\pi^2} + O(|a_1|\log^3 X),$$

where we write $\eta(s) = \eta(0) + a_1 s + a_2 s^2 + a_3 s^3 + \cdots$

Now $a_1 = \frac{1}{2\pi i}\int \frac{\eta(s)}{s^2}\,ds,$

where we have taken the integral over the circle

$$|s| = \delta, \text{ with } \delta = \frac{1}{\log\log 3q}.$$

Thus $|a_1| \leq 1/\delta \max |\eta(s)|$, the maximum being taken over the circle $|s| = \delta$.

Now $$|\eta(s)| \ll \prod_{p|q} \left(1 + \frac{1}{p^{1-\delta}}\right)^4 \ll \exp\left(4 \sum_{p|q} \frac{1}{p^{1-\delta}}\right).$$

Now, we estimate

$$\sum_{p|q} \frac{1}{p^{1-\delta}};$$

$$\sum_{p|q} \frac{1}{p^{1-\delta}} \leq \sum_{p\leq y} \frac{1}{p^{1-\delta}} + \sum_{\substack{p>y\\ p|q}} \frac{1}{p^{1-\delta}}.$$

Further

$$\sum_{p\leq y} \frac{1}{p^{1-\delta}} \leq y^\delta \sum_{p\leq y} \frac{1}{p}$$

$\leq y^\delta (\log\log y + A_1)$, where A_1 is an absolute constant.

And $$\sum_{\substack{p>y\\ p|q}} \frac{1}{p^{1-\delta}} \leq \sum_{\substack{p>y\\ p|q}} \frac{\log p}{y^{1-\delta}\log y}$$

$$\leq \frac{1}{y^{1-\delta}\log y} \left(\sum_{p|q} \log p\right)$$

$$\leq \frac{\log q}{y^{1-\delta}\log y}.$$

Thus $$\sum_{p|q} \frac{1}{p^{1-\delta}} \leq y^\delta \left\{\log\log y + A_1 + \frac{\log q}{y \log y}\right\}$$

choosing

$$y = \log 3q \text{ and since } \delta = \frac{1}{\log\log 3q}.$$

We get

$$\sum_{p|q} \frac{1}{p^{1-\delta}} \leq \log\log\log 3q + A_2$$

where A_2 is an absolute constant. Thus $|\eta(s)| \ll (\log\log 3q)^5$. Thus the residue at $s = 0$ is equal to

$$\frac{\eta(0)\log^4 X}{4\pi^2} + O(\log^3 X (\log\log 3q)^5).$$

Thus $$\sum_{\substack{n\leq X\\ (n,q)=1}} \frac{d^2(n)}{n} = \frac{1}{4\pi^2}\eta(0)\log^4 X + O(\log^3 X(\log\log 3q)^5)$$

$$+ O\left(\frac{1}{X^2}\sum_n \frac{d^2(n)}{n^{2+1/\log X}\,|\log(X/n)|}\right)$$

$$+ O\left(\int_{-1/4-iX^3}^{-1/4+iX^3} \frac{\zeta^4(s+1)}{\zeta(2s+2)}\eta(s)\frac{X^s}{s}\,ds\right).$$

The second error term is easily $O(1)$. The third error term is

$$\ll X^{-1/4} \prod_{p|q} (1 + p^{-3/4})^4 \int_0^T \frac{|\zeta^4(3/4 + it)|}{|-1/4 + it|} dt.$$

Now $\prod_{p|q} (1 + p^{-3/4})^4$

$$\ll \exp(4 \sum_{p|q} p^{-3/4})$$

$\ll \exp(A_3 (\log q)^{1/4} \log\log\log 3q)$, where A_3 is an absolute constant.

Now $$\int_0^T \frac{|\zeta^4(3/4 + it)|}{|-1/4 + it|} dt$$

$$\ll \sum_{m \geq 0} \int_{T/2^{m+1}}^{T/2^m} \frac{|\zeta^4(3/4 + it)|}{|-1/4 + it|} dt$$

$$\ll \sum_{m \geq 0} \frac{2^{m+1}}{T} \int_{T/2^{m+1}}^{T/2^m} |\zeta^4(\tfrac{3}{4} + it)| dt$$

$\ll (\log T)^5$, by using mean fourth power of zeta function.

Thus the third error term

$$\ll X^{-1/4} \log^5 T \,.\, \exp(A_3 (\log q)^{1/4} \log\log\log 3q) \ll 1.$$

Thus $$\sum_{\substack{n \leq X \\ (n,q)=1}} \frac{d^2(n)}{n} = \frac{\eta(0)}{4\pi^2} \log^4 X + O(\log^3 X (\log\log 3q)^5).$$

This completes the proof of the lemma.

Lemma 7. *We have*

$$\frac{1}{N} \sum_{\chi}{}^{*} \int_T^{2T} |J_k^2(s, \chi)| dt \ll 2^r \frac{q}{N} T (\log X)^3$$

for $3 \leq k \leq 6$.

Proof. To estimate the left hand side, we apply lemma 5 directly for $k = 3, 4$. For $k = 3$, while estimating, we note that $|1 - \exp(-n/X)| \ll n/X$, for $n \leq X$. The cases $k = 4, 6$ we sketch below. The case $k = 5$ can be disposed of, on similar lines.

We have

$$\frac{1}{N} \sum_{\chi}{}^{*} \int_T^{2T} |J_4^2(s, \chi)| dt$$

$$\ll X^{2\sigma-1} \left\{ 2^r \frac{q}{N} T \sum_{n > X} \frac{d^2(n)}{n^{2\sigma}} \exp(-2n/X) \right.$$

$$\left. + \frac{2^r}{N} \sum_{n > X} \frac{d^2(n)}{n^{2\sigma-1}} \exp(-2n/X) \right\}.$$

We deal with the first, viz.,

$$X^{2\sigma-1}\, 2^r \frac{q}{N}\, T \sum_{n>X} \frac{d^2(n)}{n^{2\sigma}} \exp(-2n/X).$$

The second term can be similarly dealt with.

In $\displaystyle\sum_{n>X} \frac{d^2(n)}{n^{2\sigma}} \exp(-2n/X)$,

by summing over intervals $[2^m, 2^{m+1})$ as done before, we get

$$\sum_{n>x} \frac{d^2(n)}{n^{2\sigma}} \exp(-2n/X) \ll \sum_{m\geq M} 2^{m(1-2\sigma)} m^3 \exp(-2^{m+1} X^{-1})$$

$$\ll \sum_{r=0}^{\infty} 2^{(M+r)(1-2\sigma)} (M+r)^3 \exp(-2^{M+r+1} X^{-1}),$$

where $M = [\log X/\log 2]$.

This, in turn, $\ll 2^{M(1-2\sigma)} M^3$.

$$\Big(\sum_{r=0}^{\infty} 2^{r(1-2\sigma)} (1+r/M)^3 \exp(-2^{M+r+1} X^{-1})\Big)$$

$$\ll 2^{M(1-2\sigma)} M^3 \ll X^{1-2\sigma} \log^3 X.$$

Now we sketch the proof that

$$\frac{1}{N} \sum_{\chi}^{*} \int_T^{2T} | J_6^2(s,\chi) |\, dt \ll 2^r \frac{q}{N} T \log^3 X.$$

The l.h.s. $\displaystyle\ll X^{2\sigma-1} \int_T^{2T} dt\, \frac{1}{N} \sum_{\chi}^{*} \cdot \int_{u=1/4} | \psi^4(s+w,\chi) |$

$$\times \Big| \sum_{n\leq X} d(n)\, \overline{\chi(n)}\, n^{w+s-1} \Big|^2 X^{2u} \, | \Gamma(w) | \, | dw |$$

by using Hölder's inequality. Breaking the integral with respect to ω at $|v| \leq \log^2 T$, where $v = \mathrm{Im}\,\omega$, with a small error, we get the above with $u = 1/4$ to be

$$\ll X^{1-2u-2\sigma} \int_{\substack{u=1/4 \\ |v|\leq \log^2 T}} | d\omega | \, | \Gamma(\omega) |$$

$$\times \Big(\int_T^{2T} dt \frac{1}{N} \sum_{\chi}^{*} \Big| \sum_{n\leq X} d(n)\, \overline{\chi(n)}\, n^{\omega+s-1} \Big|^2 \Big).$$

Using lemma 5, this expression

$$\ll X^{1-2u-2\sigma} \Big\{ 2^r \frac{q}{N} \sum_{n\leq X} d^2(n)\, n^{2u+2\sigma-2} + \frac{2^r}{N} \sum_{n\leq X} d^2(n)\, n^{2u+2\sigma-1} \Big\}.$$

The proof now can be completed.

Lemma 8 : *We have*

(a) $\sum_{\chi}^{*} \int_{T}^{2T} | J_1^2(s,\chi) | \, dt \ll 2^r qT \log^3 X$ for $\sigma \leq \frac{1}{2} - 10^{-8}$ and

(b) $\sum_{\chi} \int_{T}^{2T} | J_2^2(s,\chi) | \, dt \ll 2^r qT \log^3 X$ for $\sigma \geq \frac{1}{2} + 10^{-8}$.

Lemma 9. *Let* $s = \sigma + it$ with $t = T$ or $2T$. *Then*

(a) $\int_{\frac{1}{2}-10^{-8}}^{\frac{1}{2}} d\sigma \sum_{\chi}^{*} | J_1^2(s,\chi) | \ll 2^r \cdot qT \log^3 X \cdot (\log\log 3q)$

(b) $\int_{\frac{1}{2}}^{\frac{1}{2}+10^{-8}} d\sigma \sum_{\chi}^{*} | J_2^2(s,\chi) | \ll 2^r qT \cdot \log^3 X \cdot (\log\log 3q)$

(*c*) $\sum_{\chi}^{*} | J_k^2(s,\chi) | \ll 2^r \cdot qT \log^3 X$

for $\frac{1}{2} - 10^{-8} \leq \sigma \leq \frac{1}{2} + 10^{-8}$ and $3 \leq k \leq 6$.

(d) $\int_{\frac{1}{2}-10^{-8}}^{\frac{1}{2}} d\sigma \sum_{\chi}^{*} | J_1(s,\chi) J_k(s,\chi) | \ll 2^r \cdot qT \log^3 X (\log\log 3q)$

for $3 \leq k \leq 6$.

(e) $\int_{\frac{1}{2}}^{\frac{1}{2}+10^{-8}} d\sigma \sum_{\chi}^{*} | J_2(s,\chi) J_k(s,\chi) | \ll 2^r \cdot qT \cdot \log^3 X \cdot (\log\log 3q)$

for $3 \leq k \leq 6$.

Proof: We prove (a). The proof of (b) follows on the same lines. Similarly (c) can be proved. First, using Hölder's inequality with respect to integration (in σ) and then using (a), (b) and (c) the proof of (d) and (e) follows.

Now $\sum_{\chi}^{*} | J_1^2(s,\chi) |$

$$\ll (qT)^{2\sigma-1} \sum_{\chi}^{*} \Big| \sum_{n<X} \frac{d(n)\chi(n)}{n^s} \Big|^2$$

$$\ll (qT)^{2\sigma-1} \sum_{m,n<X} \sum \frac{d(m)\,d(n)}{m^{\sigma+it} \cdot n^{\sigma-it}} \Big(\sum_{\chi}^{*} \chi(m) \overline{\chi(n)} \Big)$$

$$\ll \sum_{k|q} | \mu(q/k) | \phi(k) X^{2\sigma-1} \sum_{m \equiv n \,(\mathrm{mod}\, k)} \sum \frac{d(m)\,d(n)}{(mn)^\sigma}$$

$$\ll \sum_{k|q} | \mu(q/k) | \phi(k) X^{2\sigma-1} \sum_{j=1}^{k} \left\{ \frac{d(j)}{j^\sigma} + \sum_{1 \leq n' \leq (X-j)/k} \frac{d(n'k+j)}{(n'k+j)^\sigma} \right\}^2$$

$$\ll \sum_{k|q} | \mu(q/k) | \phi(k) X^{2\sigma-1} \sum_{j=1}^{k} \left\{ \frac{d^2(j)}{j^{2\sigma}} + \Big(\sum_{n' \geq 1} \frac{d(n'k+j)}{(n'k+j)^\sigma} \Big)^2 \right\}$$

Thus $\int_{\frac{1}{2}-10^{-8}}^{\frac{1}{2}} d\sigma \sum_{\chi}^{*} | J_1^2(s,\chi) |$

$$\ll \sum_{k|q} \frac{\phi(k) | \mu(q/k) |}{X} \left\{ \sum_{j=1}^{k} d^2(j) \int_{\frac{1}{2}-10^{-8}}^{\frac{1}{2}} \Big(\frac{X}{j} \Big)^{2\sigma} d\sigma \right.$$

$$+\sum_{j=1}^{k}\sum_{n'\geq 1}\left(\left(\frac{X}{n'k+j}\right)^{1/2} d(n'k+j)\right)^{2}\Bigg\}$$

$$\ll \sum_{k|q} \phi(k)\,|\,\mu(q/k)\,|\left\{\sum_{j=1}^{k}\frac{d^2(j)}{j\log(X/j)}+\sum_{j=1}^{k}\left(\sum_{n'\geq 1}\frac{d^2(n'k+j)}{(n'k+j)^{1/2}}\right)^{2}\right\}.$$

It can be shown easily that

$$\sum_{j=1}^{k}\frac{d^2(j)}{j\log(x/j)} \ll \log^3 k \cdot \log\log 3k$$

and using Hölder's inequality,

$$\sum_{j=1}^{k}\left(\sum_{n'\geq 1}\frac{d(n'k+j)}{(n'k+j)^{1/2}}\right)^{2} \ll \left(\sum_{1\leq n'\leq X/k}\frac{1}{(n'k)^{1/2}}\right)$$

$$\times\left(\sum_{j}\sum_{1\leq n'\leq (X-j)/k}\frac{d^2(n'k+j)}{(n'k+j)^{1/2}}\right) \ll (X^{1/2}k)\sum_{n\leq X}\frac{d^2(n)}{n^{1/2}} \ll X/k\log^3 x.$$

The proof can now be completed.

Lemma 10. *We have*

(a) $\sum_{\chi}^{*} \int_{T}^{2T} J_i^2(\chi)\,dt \ll 2^r qT\log^3 X \cdot (\log\log 3q)$

for $i = 1, 2$.

(b) $\sum_{\chi}^{*} \int_{T}^{2T} J_i(\chi) J_k(\chi)\,dt \ll 2^r qT\log^3 X \cdot (\log\log 3q)$

for $i = 1, 2$ and $3 \leq k \leq 6$.

Proof: We prove only

$$\sum_{\chi}^{*} \int_{T}^{2T} J_1^2(\chi)\,dt \ll 2^r qT\log^3 X \cdot (\log\log 3q).$$

To prove this, we move the line of integration from $\sigma = \frac{1}{2}$ to $\sigma = \frac{1}{2} - 10^{-8}$. Appealing to lemma 9 (a) and lemma 8 (a), the proof can be completed.

This completes the proof of the theorem.

Reference

[1] Montgomery H L and Vaughan R C 1974 *J. London Math. Soc.* **8** 73–82

Note added in proof

Using

$$\int_{T}^{T+H} \Big|\sum_{n} a(n)\chi(n) n^{-it}\Big|^2 dt$$

$$= H \sum_{\substack{n\\(n,q)=1}} |\,a^2(n)\,| + O\Big(\sum_{n} n\,|\,a^2(n)\,|\Big)$$

(which can be trivially proved using Hilbert's inequality), our above proof can be trivially modified to give the result that 'for a primitive character $\chi \pmod q$ and for $T \geq 2$,

$$\int_T^{2T} \left| L\left(\frac{1}{2} + it, \chi\right)\right|^4 dt = 2T \sum_{\substack{n \leq qT \\ (n, q)=1}} \frac{d^2(n)}{n} + O(qT \log^3 qT)\text{'}.$$