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On a generalization of the class of functions with bounded Mocanu variation

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Abstract. The object of this paper is to generalise the well-known class of functions analytic in the unit disc having bounded Mocanu variation. Certain properties of this more general class are investigated using convolution techniques.

Keywords. Convolution; Mocanu variation; bounded argument rotation bounded boundary rotation; hypergeometric function.

Introduction

Let M denote the class of functions $f(z)$ analytic in the unit disc $E(|z|<1)$ with $f(0)=f'(0)-1=0$. We represent by $(f*g)(z)$ the 'convolution' or the 'Hadmard product' of two functions belonging to M. Let $K_a(z)$ denote the function $z(1-z)^{-a}$ where $a \in \mathbf{C}$ with $\operatorname{Re} a>0$. Clearly $K_a \in M$.

In this paper we introduce and study a new class of functions MV $[\alpha, k, a]$ defined as follows:

Let a, $\alpha \in \mathbf{R}$, $a>0, \alpha \geq 0$. We say $f \in \mathrm{MV}[\alpha, k, a]$ if and only if $f \in M$;

$$\frac{(K_{a+1}*f)(z)}{z} \neq 0; \ \frac{(K_a*f)(z)}{z} \neq 0 \ \text{ for } z \in E \text{ and}$$

$$\int_0^{2\pi} \left| \operatorname{Re}\left\{\alpha(a+1)\left[\frac{(K_{a+2}*f)(re^{i\theta})}{(K_{a+1}*f)(re^{i\theta})}-\frac{a}{a+1}\right]\right.\right.$$
$$\left.\left. +(1-\alpha)a\left[\frac{(K_{a+1}*f)(re^{i\theta})}{(K_a*f)(re^{i\theta})}-\frac{a-1}{a}\right]\right\}\right| d\theta \leq k\pi$$

where $z=re^{i\theta}$ and $k\geq 2$. (1)

It is easy to observe that when $a=1$, MV $[\alpha, k, a]$ coincides with the well-known class MV $[\alpha, k]$ consisting of functions having a bounded Mocanu variation introduced by Coonce and Ziegler [1]. Similarly we see that MV $[1,2,1] \equiv K$

the class of convex functions; MV [0, 2, 1] $\equiv S^*$ the class of starlike functions; MV [α, 2, 1] $\equiv S_\alpha$ the class of α-convex functions introduced by Mocanu [4]; MV [0, k, 1] $\equiv U_k$ the class of normalized functions having bounded argument rotation (Tammi [6]) and MV [1, k, 1] $\equiv V_k$ the class of normalized functions having bounded boundary rotation (Löwner [2]).

2. Geometric interpretation and preliminary remarks

A geometric interpretation of (1) can be obtained and the class MV [α, k, a] can be defined in a manner analogous to MV [α, k]. Let

$$J(\alpha, f, a) \equiv \alpha (a+1)\left[\frac{(K_{a+2} * f)(z)}{(K_{a+1} * f)(z)} - \frac{a}{a+1}\right] + (1-\alpha)\, a\left[\frac{(K_{a+1} * f)(z)}{(K_a * f)(z)} - \frac{a-1}{a}\right]. \quad (2)$$

If $z = re^{i\theta}$ and χ is the angle defined by

$$\chi = (1-\alpha)\arg (K_a * f)(z) + \alpha \arg (iK_{a+1} * f)(z),$$

using the relation

$$z(K_a * f)'(z) = a(K_{a+1} * f)(z) - (a-1)(K_a * f)(z), \quad (3)$$

it can be shown that $\partial\chi/\partial\theta = \operatorname{Re}\{J(\alpha, f, a)\}$. Thus $f \in$ MV [α, k, a] if and only if the angle χ defined above has a bounded variation as $\arg z = \theta$ increases continuously on $|z| = r$. When $a = 1$, χ is the Mocanu angle given by $\chi = (1-\alpha) \arg f(z) + \alpha \arg (izf'(z))$.

Next we proceed to give an integral representation to members in the class MV [α, k, a]. We require the following results.

Let M_k denote the class of real valued functions $m(t)$ of bounded variation on $[-\pi, \pi]$ which satisfy the conditions

$$\int_{-\pi}^{\pi} dm(t) = 2, \quad \int_{-\pi}^{\pi} |dm(t)| \leq k.$$

The class P_k introduced by Pinchuk [5] consists of analytic functions

$$p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \text{ in } E \text{ satisfying for every}$$

$$r < 1 \, (z = re^{i\theta} \in E), \int_0^{2\pi} |\operatorname{Re}\, p(z)|\, d\theta \leq k\pi, \, k \geq 2.$$

Clearly P_2 is the class P of functions with positive real part in E. It is known [5] that functions in the class P_k have the representation,

$$f(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, dm(t), \quad m(t) \in M_k, \quad (4)$$

and functions in the class U_k can be written in the form

$$f(z) = z \exp \int_{-\pi}^{\pi} -\log(1 - ze^{-it})\, dm(t),\ m(t) \in M_k. \tag{5}$$

3. Theorems and their proofs

Theorem 1. $f \in \mathrm{MV}\,[\alpha, k, a]$ if and only if

$$\alpha z \frac{(K_{a+1} * f)'(z)}{(K_{a+1} * f)(z)} + (1 - \alpha) \frac{z(K_a * f)'(z)}{(K_a * f)(z)} \in P_k. \tag{6}$$

Proof. Using (3) it is easy to check that $J(\alpha, f, a)$ given by (2) is also equal to the left side expression given by (6). Hence the theorem follows from (1) and definition of P_k.

Corollary. Putting $\alpha = 0$ we see $f \in \mathrm{MV}\,[0, k, a]$ if and only if $(K_a * f) \in U_k$ and similarly $f \in \mathrm{MV}\,[1, k, a]$ if and only if $(K_{a+1} * f) \in U_k$. It follows therefore that $f \in \mathrm{MV}\,[1, k, a]$ if and only if $(K_a^{-1} * K_{a+1}) * f \in \mathrm{MV}\,[0, k, a]$. For $a = 1$ we obtain the well-known result [5] $f \in V_k$ iff $zf' \in U_k$.

Theorem 2. $f(z)$ is in $\mathrm{MV}\,(\alpha, k, a)$ if and only if there is an $F(z)$ in U_k such that for a suitable branch

$$(K_a * f)(z) = \left[\frac{a}{\alpha} z^{-(a-1)/\alpha} \int_0^z t^{-1+(a-1)/2\alpha} F(t)^{(a+1)/2\alpha}\, dt\right]^\alpha, \quad z \in E. \tag{7}$$

Proof. Using (6) we obtain

$$\alpha \frac{(K_{a+1} * f)'(z)}{(K_a * f)(z)} + (1 - \alpha) \frac{(K_a * f)'(z)}{(K_a * f)(z)} - \frac{1}{z} = \frac{p(z) - 1}{z}$$

where $p(z) \in P_k$.

Integration and simplification making use of (4) and (5) yield

$$(K_{a+1} * f)(z) \{(K_a * f)(z)\}^{1/\alpha - 1} z^{(a-1)/2\alpha} = F(z)^{(a+1)/2\alpha} \tag{8}$$

where $F(z) \in U_k$. This on further simplification becomes

$$\frac{d}{dz} \left\{(K_a * f)(z)^{1/\alpha} z^{(a-1)/\alpha}\right\} = \frac{a}{\alpha} z^{(a-1)/2\alpha - 1} F(z)^{(a+1)/2\alpha}, \tag{9}$$

from which we derive (7). The converse can be verified easily.

Corollary. Since $G(z) \in V_k$ if and only if $zG'(z) \in U_k$ we can write (7) also in the form

$$(K_a * f)(z) = \left[\frac{a}{\alpha} z^{-(a-1)/\alpha} \int_0^z G'(t)^{(a+1)/2\alpha} t^{a/\alpha - 1}\, dt\right]^\alpha$$

where $G(t) \in V_k$. (10)

Remark. Putting $a = 1$ we obtain the representation for functions with bounded Mocanu variation (Coonce and Ziegler [1]). When $a = 1$, $k = 2$ we obtain the representation for α-convex functions (Mocanu [4]). When $\alpha = 1$, $a = 1$, the functions satisfying (7) are all known to be in V_k.

The representation formula (7) enables us to obtain a distortion theorem concerning functions in MV $[\alpha, k, a]$. We require the following lemma.

Lemma. If $F(z)$ is in U_k and $|z| = r$ then

$$r(1-r)^{(k-2)/2}(1+r)^{-(k+2)/2} \leq |F(z)| \leq r(1+r)^{(k-2)/2}(1-r)^{-(k+2)/2}. \tag{11}$$

The proof of the lemma can be found in [1].

Theorem 3. If $f(z)$ is in MV $[\alpha, k, a]$ and $M(r) = \max |(K_a * f)\, re^{i\theta}|$ $(0 \leq \theta \leq 2\pi)$, then

$$M(r) = 0\,[(1-r)^{-(k+2)(a+1)-4\alpha/4}] \text{ for } 0 \leq \alpha < \frac{(k+2)(a+1)}{4}, \tag{12}$$

$$M(r) = 0\,[\log(1-r)^{-1}]^{\alpha} \text{ for } \alpha = \frac{(k+2)(a+1)}{4} \tag{13}$$

$$M(r) \leq \left[\frac{a}{\alpha} 2^{(k-2)(a+1)/4\alpha} r^{-(a-1)/a} \frac{\Gamma(a/\alpha)\,\Gamma(1-(k+2)(a+1)/4\alpha)}{\Gamma(a/\alpha + 1 - (k-2)(a+1)/4\alpha)}\right]^{\alpha},$$

$$\text{for } \alpha > \frac{(k+2)(a+1)}{4}. \tag{14}$$

Proof. By Theorem 2, there exists $F(z)$ in U_k such that

$$(K_a * f)(z) = \left[\frac{a}{\alpha} z^{-(a-1)/a} \int_0^z t^{-1+(a-1)/2a} F(t)^{a+1/2a}\, dt\right]^{\alpha},$$

and applying (11) to the above we obtain

$$|(K_a * f)(re^{i\theta})|^{1/\alpha} \leq \frac{a}{\alpha} r^{-(a-1)/a} \int_0^r \rho^{a/\alpha - 1}(1-\rho)^{-(k+2)(a+1)/4\alpha}$$

$$\times (1+\rho)^{(k-2)(a+1)/4\alpha}\, d\rho$$

$$\leq \frac{a}{\alpha} r^{-(a-1)/a} 2^{(k-2)(a+1)/4\alpha} \int_0^r \rho^{a/\alpha-1}(1-\rho)^{-(k+2)(a+1)/4\alpha}\, d\rho.$$

Changing the variable by putting $\rho = ru$ we obtain

$$|(K_a * f)(re^{i\theta})|^{1/\alpha} \leq \frac{a}{\alpha} r^{1/a} 2^{(k-2)(a+1)/4\alpha} \int_0^1 u^{a/\alpha-1}(1-ru)^{-(k+2)(a+1)/4\alpha}\, du. \tag{15}$$

using hypergeometric functions given by

$$G(l, m, n; z) = \frac{\Gamma(n)}{\Gamma(l)\Gamma(m)} \sum_{p=0}^{\infty} \frac{\Gamma(l+p)\Gamma(m+p)}{\Gamma(n+p)} \frac{z^p}{p!}$$

$$= \frac{\Gamma(n)}{\Gamma(l)\Gamma(n-1)} \int_0^1 u^{l-1}(1-u)^{n-l-1}(1-zu)^{-m}\, du, \tag{16}$$

where $\operatorname{Re} l > 0$ and $\operatorname{Re}(n-l) > 0$ we can write

$$|(K_a * f)(re^{i\theta})|^{1/\alpha} \le r^{1/\alpha}\, 2^{(k-2)(a+1)/4\alpha}\, G\left(\frac{a}{\alpha}, \frac{(k+2)(a+1)}{4\alpha}, 1+\frac{a}{\alpha}; r\right). \tag{17}$$

Hence $$M(r) \le r\, 2^{\{\frac{1}{4}[(k-2)(a+1)]}\left[G\left(\frac{a}{\alpha}, \frac{1}{4\alpha}[(k+2)(a+1)], 1+\frac{a}{\alpha}; r\right)\right]^{\alpha} \tag{18}$$

If we now take $0 < \alpha < \frac{1}{4}[(k+2)(a+1)]$ then

$$\lim_{r\to 1-0} \frac{G\left(\frac{a}{\alpha}, \frac{1}{4\alpha}[(k+2)(a+1)], 1+\frac{a}{\alpha}; r\right)}{(1-r)^{1-\frac{1}{4\alpha}[(k+2)(a+1)]}}$$

$$= \frac{4a}{(k+2)(a+1)-4\alpha} \quad \text{([7] p. 299).}$$

Combining this with (18) we get

$$M(r) = O\{r(1-r)^{-\frac{1}{4}[(k+2)(a+1)-4\alpha]} = O\{(1-r)^{-\frac{1}{4}[(k+2)(a+1)-4\alpha]}\}$$

$$\text{as } r \to 1-0$$

when $\alpha = \frac{1}{4}[(k+2)(a+1)]$ we obtain from ([7], p. 299)

$$\lim_{r\to 1-0} \frac{G\left(\frac{a}{\alpha}, 1, 1+\frac{a}{\alpha}; r\right)}{\log[(1-r)^{-1}]} = a/\alpha,$$

and combining with (18) we have

$$M(r) = O\{r[\log(1-r)^{-1}]^{\alpha}\} = O[\log(1-r)^{-1}]^{\alpha} \text{ as } r \to 1-0.$$

In the case $\alpha > [(k+2)(a+1)]/4$ we have

$$\int_0^1 \rho^{\frac{a}{\alpha}-1}(1-\rho)^{-\frac{1}{4\alpha}[(k+2)(a+1)]}\, d\rho = \frac{\Gamma(a/\alpha)\,\Gamma\left(1-\frac{1}{4\alpha}[(k+2)(a+1)]\right)}{\Gamma\left(\frac{a}{\alpha}+1-\frac{1}{4\alpha}[(k+2)(a+1)]\right)}.$$

Hence we obtain

$$M(r) \leq \left[\frac{a}{\alpha} \, 2^{\frac{1}{4\alpha}[(k-2)\,(a+1)]} \, r^{\frac{-(a-1)}{\alpha}} \, \frac{\Gamma(a/\alpha)\, \Gamma\left(1 - \frac{1}{4\alpha}[(k+2)\,(a+1)]\right)}{\Gamma\left(\frac{a}{\alpha} + 1 - \frac{1}{4\alpha}[(k+2)(a+1)]\right)} \right]^{\alpha}$$

which completes the proof of the theorem.

The function $f(z)$ defined by

$$(K_a * f)(z) = \frac{a}{\alpha} z^{-(a-1)\alpha} \int_0^z t^{a/\alpha - 1} (1-t)^{-\frac{1}{4\alpha}[(k+2)\,(a+1)]} (1+t)^{\frac{1}{4\alpha}[(k-2)\,(a+1)]} \, dt$$

shows that the orders given in (12) and (13) are the best possible.

Remark. When $a = 1$ we obtain the results in [1] and when $a = 1$, $k = 2$ the results in [3].

It is worth noting that when $a = 1$, MV $[\alpha, k, a] \subset S$, the class of univalent functions iff $k \leqslant 2 + 2\alpha$ and MV $[\alpha, k, a] \subset$ MV $[\beta, k, a]$ if $\alpha \geq \beta \geq 0$; [1]. These results are under investigation by the author for other values of $a \in \mathbf{R}$.

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