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# On the mean square value of Hurwitz zeta function

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**Abstract.** R Balasubramanian has shown that

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2 dt = T\log\frac{T}{2\pi} + (2\gamma-1)T + O(T^{\theta+\epsilon})$$

with $\theta = \frac{1}{3}$. In this paper we develop a hybrid analogue for the mean square value of the Hurwitz zeta function $\zeta(s, a)$ and show that (i) new asymptotic terms arise in the expression for $\zeta(s, a)$ which are not present in the above expression for the ordinary zeta function and (ii) the corresponding error term is given by

$$O(T^{5/12}\log^2 T) + O\left(\frac{\log T}{\|2a\|}\right)$$

for $0 < a < 1$.

**Keywords.** Hybrid analogue for L-series; Balasubramanian's theorem.

## 1. Introduction

In a recent paper, Balasubramanian [1] proved the following theorem.

*Theorem* 1 :

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt = T\log(T/2\pi) + (2\gamma-1)T + O(T^{\theta+\epsilon}),$$

with $\theta = \frac{1}{3}$.

The above theorem with $\theta = 5/12$ is a deep theorem of Titchmarsh [7] and Balasubramanian was the first to make a major improvement on it. Later Good [2] gave a different proof of part of Balasubramanian's result. Heath-Brown [3], by a different set of more powerful ideas, proved Theorem 1 and a very important result on the fourth power mean.

In an attempt to get a hybrid analogue, we are led to the consideration of the mean square of the Hurwitz zeta function on the critical line. So we try to get a result similar to Theorem 1, for Hurwitz zeta function $\zeta(s, a)$, defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty}\left(\frac{1}{(n+a)^s} - \int_n^{n+1}\frac{du}{(u+a)^s}\right) - \frac{a^{1-s}}{1-s},$$

for $0 < a < 1$ and $\operatorname{Re} s > 0$.

Using the approximate functional equation for $\zeta(s, a)$, Rane [5] has already proved

*Theorem* 2 :

$$\int_1^T |\zeta(\tfrac{1}{2}+it, a)|^2\, dt = A(a, T) + O(a^{-1/2}\, T^{1/2} \log T)$$

where $A(a, T) = T \log (T/2\pi) + T[C(a) + \gamma - 1] - 1/a$

and $C(a) = \lim_{m\to\infty} \left[ \sum_{n=0}^{m} \frac{1}{n+a} - \log m \right].$

We shall use a much more delicate analysis introduced by Titchmarsh in [7] and prove

*Theorem* 3 : If $0 < a < 1$ and $A(a, T)$ *is as in Theorem* 2, *then for* $a \neq \frac{1}{2}$

$$\int_0^T |\zeta(\tfrac{1}{2}+it, a)|^2\, dt$$

$$= A(a, T) + A_1(a)\, T^{1/2} + O(T^{5/12} \log^2 T) + O\left(\frac{\log T}{\| a \|}\right),$$

where $$A_1(a) = 2\pi \left[ -2H(a) + \int_{1-a}^{1} \frac{\cos \pi (u^2 - u - \frac{1}{4})}{\cos \pi u}\, du \right.$$

$$\left. + 2 \sum_{r=1}^{\infty} \int_r^{\infty} \cos \pi (u^2 - \tfrac{1}{4})\, du \right] < -(\sqrt{2\pi})\, H(a).$$

with $H(a) = \int_{(1-a)/2}^{(1+a)/2} G^2(u)\, du + \int_{a/2}^{1-a/2} G^2(u)\, du$

and $G(u) = \cos 2\pi (u^2 - u - \frac{1}{16}) / \cos 2\pi u,$

*and for a real number a,*

$\| a \|$ = *distance between a and the integer nearest to a.*

For $a = \frac{1}{2}$,

$$\int_1^T |\zeta(\tfrac{1}{2}+it, \tfrac{1}{2})|^2\, dt = A(\tfrac{1}{2}, T) + A_2\, T^{1/2} \log T + A_3 T^{1/2}$$
$$+ O(T^{5/12} \log^2 T),$$

where $A_2 = -1/4\sqrt{2\pi},$

and $A_3 = \frac{1}{\sqrt{2\pi}} [\frac{1}{4} \log 2\pi - 4 \log 2 - 2\gamma - 4\pi H(\tfrac{1}{2}) + \tfrac{1}{2} + A_0],$

where $A_0 = 8\sqrt{2} \sum_{r=1}^{\infty} \int_r^{\infty} \frac{\sin \pi (u^2 + u + \frac{1}{4})}{(2u+1)^2}\, du.$

*Remarks* : (1) It is surprising that the term $A_1(a)$ is different from zero for $0 < a < 1$, $a \neq \frac{1}{2}$ and $A_2 \neq 0$ for $a = \frac{1}{2}$.

(2) In this paper, we apply mainly the methods of Titchmarsh [7]. It is obvious from the proof of Lemma 20 in this paper why Balasubramanian's [1] multiple average integral method is not applicable in our case.

(3) The results of this paper, as well as those of Rane [5] have been announced by Ramachandra in [4].

## 2. Some preliminary results and a few definitions

*Lemma P1* :

$$\int_{2\pi a^2}^{\infty} \exp\left[ i\left(t \log \frac{t}{2\pi ab} - t - \frac{\pi}{4}\right)\right] dt$$

$$= 2\pi a \exp\left[2\pi i a^2\left(\log\frac{a}{b} - \frac{1}{2}\log^2\frac{a}{b}\right)\right]$$

$$\times \int_{a \log a/b}^{\infty} \exp\left[ i\pi\left(\theta^2 - \frac{1}{4}\right)\right] d\theta + O\left(\frac{1}{(a-b)^4}\right),$$

for $0 < b < a < Ab$, *$A, a, b$ being real numbers.*

Titchmarsh proves the above lemma in [7] with $a$ and $b$ as positive integers It is not difficult to see that practically the same proof works in our case.

*Lemma P2* : *For $0 \leqslant r$ and $0 < \alpha \leqslant 1$, and $r$ an integer*

$$\int_{r+\alpha}^{\infty} \exp\left[i\pi\left(\theta^2 - \tfrac{1}{4}\right)\right] d\theta$$

$$= -\frac{(-1)^r \exp\left[i\pi\left(\alpha^2 + 2r\alpha - \frac{1}{4}\right)\right]}{2\pi i (r+\alpha)} + \frac{(-1)^r \exp\left[i\pi\left(\alpha^2 + 2r\alpha - \frac{1}{4}\right)\right]}{4\pi^2 (r+\alpha)^3}$$

$$+ O\left(\frac{1}{(r+\alpha)^4}\right).$$

*Proof* : Follows from integration by parts.

*Lemma P3* : *If $0 < \alpha \leqslant 1$ and $0 \leqslant \sigma \leqslant 1, \sigma = \operatorname{Re} s$,*

$$k_1 = \left[\sqrt{\frac{t}{2\pi}} - \alpha\right], \quad k = \left[\sqrt{\frac{t}{2\pi}}\right], \quad \beta = k - k_1, \ s = \sigma + it, \textit{ then}$$

$$\zeta(\sigma + it, \alpha)$$

$$= \sum_{0 \leqslant n \leqslant k_1} \frac{1}{(n+\alpha)^s} + \left(\frac{2\pi}{t}\right)^{it} \exp\left[ i\left(t + \frac{\pi}{4}\right)\right]$$

$$\times \sum_{1 \leqslant m \leqslant k} \left[\frac{\exp(-2\pi i m\alpha)}{m^{\frac{1}{2}-it}}\right] + g_3(t) + O(t^{-3/4}), \tag{1}$$

where $$g_3(t) = (2\pi/t)^{1/4} i^{\beta} (-1)^k \exp\left[-\frac{7}{8}\pi i + \frac{\pi i \alpha^2}{2} + i\frac{t}{2}\log\frac{2\pi}{t}\right.$$

$$\left. + \pi i (2\beta y - 2\alpha y - \alpha\beta)\right]\psi(t),$$

with

$$\psi(t) = \frac{\cos\left[t - \sqrt{2\pi t}(2k + 1 + \alpha - \beta) + \pi\left\{\frac{(\alpha-\beta)^2}{2} + (\alpha-\beta)(2k+1) - \frac{1}{8}\right\}\right]}{\cos[\sqrt{2\pi t} - (\alpha - \beta)\pi]}$$

and $y = (t/2\pi)^{1/2}$.

*Proof.* This is the approximate functional equation for $\zeta(s, \alpha)$, and as in Rane's work [5] it is obtained in the same way as the approximate functional equation for $\zeta(s)$ in Titchmarsh [6], equation (4.17.1).

*Remark* : If we denote the four terms on the R.H.S. of equation (1) by $g_i(\alpha, t)$, $i = 1, 2, 3, 4$, then it is easy to see that $g_3(\alpha, t) = O(t^{-1/4})$.
For the proofs of the following two lemmas, we refer to Titchmarsh [7].

*Lemma P4 : If $f(x)$ is real and in $C^2$ over $a \leqslant x \leqslant b$, and $f''(x) \geqslant r > 0$ (or $\leqslant -r < 0$) in $a < x < b$ and $|f'(b) - f'(a)| = (b - a)R$, then if $n$ denotes integer,*

$$\sum_{a \leqslant n \leqslant b} \exp[2\pi i f(n)] = O\left\{\frac{(b-a)R}{\sqrt{r}}\right\} + O(1/\sqrt{r}).$$

*Lemma P5 : If $f(x)$ is real and is in $C^3$ over $a \leqslant x \leqslant b$, and $f'''(x) \geqslant r_2 > 0$ (or $\leqslant -r_2 < 0$) in $a < x < b$, then if $|f''(b) - f''(a)| = (b - a)R_2$,*

$$\sum_{a \leqslant n \leqslant b} \exp[2\pi i f(n)] = O\left\{\frac{(b-a)R_2^{1/3}}{r_2^{1/6}}\right\} + O\left\{\frac{(b-a)^{1/2}R_2^{1/6}}{r_2^{1/3}}\right\}.$$

We now give the notation to be used in the proof :

$K = [(T/2\pi)^{1/2}]$, $K_1 = [(T/2\pi)^{1/2} - \alpha]$,

where for a real number $a$, $[a]$ = the largest integer not greater than $a$.

$Q_i$, $i = 1, 2, 3, 4$, will be integers with $Q_i = K + O(1)$, $Q_i \leqslant K$.

$$B_1 = 2\,\mathrm{Re} \sum_{n=0}^{K_1} \frac{1}{(n+\alpha)^{1/2}} \sum_{n+1 \leqslant q}^{Q_1} (-1)^{q-1} \int_{2\pi q^2}^{2\pi(q+\alpha)^2} i\left(\frac{2\pi}{t}\right)^{1/4}$$

$$\times \exp\left[it\log(n+\alpha) + \frac{it}{2}\log\frac{2\pi}{t} + \frac{it}{2} + i\pi(2y - 2\alpha y - \alpha) + \frac{i\pi}{8} + \frac{i\pi\alpha^2}{2}\right]\psi(t)\,dt,$$

$$B_2 = 2\,\mathrm{Re} \sum_{n=0}^{K_1} \frac{1}{(n+\alpha)^{1/2}} \sum_{n \leqslant q}^{Q_2} (-1)^q \int_{2\pi(q+\alpha)^2}^{2\pi(q+1)^2} \left(\frac{2\pi}{t}\right)^{1/4}$$

$$\times \exp\left[it\log(n+\alpha) + \frac{it}{2}\log\frac{2\pi}{t} + \frac{it}{2} + i\pi(-2\alpha y)\right.$$

$$+\frac{i\pi}{8}+\frac{i\pi\alpha^2}{2}\Big]\psi(t)\,dt,$$

$$B_3 = 2\,\mathrm{Re}\sum_{1\leqslant n}^{K}\frac{1}{n^{1/2}}\sum_{n\leqslant q}^{Q_3}(-1)^q\int_{2\pi q^2}^{2\pi(q+\alpha)^2}\psi(t)\cdot\left(\frac{2\pi}{t}\right)^{1/4}$$

$$\times\exp\Big[it\log\left(\frac{n}{y}\right)+\frac{it}{2}-2\pi in\alpha-\frac{i\pi\alpha^2}{2}+\frac{9}{8}\pi i$$

$$-i\pi(2y-2\alpha y-\alpha)\Big]dt,$$

$$B_4 = 2\,\mathrm{Re}\sum_{1\leqslant n}^{K}\frac{1}{n^{1/2}}\sum_{n\leqslant q}^{Q_4}(-1)^q\int_{2\pi(q+\alpha)^2}^{2\pi(q+1)^2}\psi(t)\cdot\left(\frac{2\pi}{t}\right)^{1/4}$$

$$\times\exp\Big[it\log\left(\frac{n}{y}\right)+\frac{it}{2}-2\pi in\alpha-\frac{i\pi\alpha^2}{2}+\frac{9}{8}\pi i+2\pi i\alpha y\Big]dt,$$

where $\psi(t)$ is as defined in Lemma P3.

$$C_1 = 2\,\mathrm{Re}\sum_{n=0}^{K_1}\frac{1}{(n+\alpha)^{1/2}}\sum_{n+n^{1/4}\leqslant q}^{Q_1}(-1)^{q-1}\int_{2\pi q^2}^{2\pi(q+\alpha)^2} i\psi(t)\cdot\left(\frac{2\pi}{t}\right)^{1/4}$$

$$\times\exp\Big[it\log\left(\frac{n+\alpha}{y}\right)+\frac{it}{2}+\frac{i\pi}{8}+\frac{i\pi\alpha^2}{2}+i\pi(2y-2\alpha y-\alpha)\Big]dt,$$

$$C_2 = 2\,\mathrm{Re}\sum_{n=0}^{K_1}\frac{1}{(n+\alpha)^{1/2}}\sum_{n+n^{1/4}\leqslant q}^{Q_2}(-1)^q\int_{2\pi(q+\alpha)^2}^{2\pi(q+1)^2}\psi(t)\left(\frac{2\pi}{t}\right)^{1/4}$$

$$\exp\Big[it\log\left(\frac{n+\alpha}{y}\right)+\frac{it}{2}+\frac{i\pi}{8}+\frac{i\pi\alpha^2}{2}-2\pi i\alpha y\Big]dt,$$

$$C_3 = 2\,\mathrm{Re}\sum_{n=1}^{K}\frac{1}{n^{1/2}}\sum_{n+n^{1/4}\leqslant q}^{Q_3}(-1)^q\int_{2\pi q^2}^{2\pi(q+\alpha)^2}\psi(t)\cdot\left(\frac{2\pi}{t}\right)^{1/4}$$

$$\times\exp\Big[it\log\left(\frac{n}{y}\right)+\frac{it}{2}-2\pi in\alpha-\frac{i\pi\alpha^2}{2}$$

$$+\frac{9}{8}\pi i-i\pi(2y-2\alpha y-\alpha)\Big]dt,$$

$$C_4 = 2\,\mathrm{Re}\sum_{n=1}^{K}\frac{1}{n^{1/2}}\sum_{n+n^{1/4}\leqslant q}^{Q_4}(-1)^q\int_{2\pi(q+\alpha)^2}^{2\pi(q+1)^2}\psi(t)\cdot\left(\frac{2\pi}{t}\right)^{1/4}$$

$$\times\exp\Big[it\log\left(\frac{n}{y}\right)+\frac{it}{2}-2\pi in\alpha-\frac{i\pi\alpha^2}{2}+\frac{9}{8}\pi i+2\pi i\alpha y\Big]dt,$$

$$B_5 = 2\,\mathrm{Re} \sum_{\substack{m \leqslant n \leqslant m + m^{1/4} \\ n \leqslant K_1}} \frac{\exp(2\pi i m \alpha)}{m^{1/2}(n+\alpha)^{1/2}}$$

$$\times \int_{2\pi(n+\alpha)^2}^{\infty} \exp[i\phi(t) - it \log\{m(n+\alpha)\}]\, dt$$

where $\phi(t) = t \log(t/2\pi) - t - \pi/4$.

$$B_6 = 2\,\mathrm{Re} \sum_{\substack{n < m \leqslant n + n^{1/4} \\ m \leqslant K}} \frac{\exp(2\pi i m \alpha)}{m^{1/2}(n+\alpha)^{1/2}}$$

$$\times \int_{2\pi m^2}^{\infty} \exp[i\phi(t) - it \log\{m(n+\alpha)\}]\, dt,$$

$$C_5 = 2\,\mathrm{Re} \sum_{m \leqslant n \leqslant K_1} \frac{\exp(2\pi i m \alpha)}{m^{1/2}(n+\alpha)^{1/2}}$$

$$\times \int_{T}^{\infty} \exp[i\phi(t) - it \log\{m(n+\alpha)\}]\, dt,$$

$$C_6 = 2\,\mathrm{Re} \sum_{n < m \leqslant K} \frac{\exp(2\pi i m \alpha)}{m^{1/2}(n+\alpha)^{1/2}}$$

$$\times \int_{T}^{\infty} \exp[i\phi(t) - it \log\{m(n+\alpha)\}]\, dt,$$

$$R(B_5) = 2\,\mathrm{Re} \sum_{m + m^{1/4} < n < K_1} \frac{\exp(2\pi i m \alpha)}{m^{1/2}(n+\alpha)^{1/2}}$$

$$\times \int_{2\pi(n+\alpha)^2}^{\infty} \exp[i\phi(t) - it \log\{m(n+\alpha)\}]\, dt,$$

$$R(B_6) = 2\,\mathrm{Re} \sum_{n + n^{1/4} < m \leqslant K} \frac{\exp(2\pi i m \alpha)}{m^{1/2}(n+\alpha)^{1/2}}$$

$$\times \int_{2\pi m^2}^{\infty} \exp[i\phi(t) - it \log\{m(n+\alpha)\}]\, dt.$$

From Lemma P3 and the remark after it, we get

*Lemma P6* :

$$\int_1^T |\zeta(\tfrac{1}{2} + it, \alpha)|^2\, dt = I_1 + I_2 + I_3 + I_4 + I_5 + O(T^{\epsilon})$$

where $g_i$ are defined as in the remark after Lemma P3 and

$$I_1 = \int_1^T |g_1(a,t) + g_2(a,t)|^2\, dt,$$

$$I_2 = \int_1^T |g_3(a,t)|^2\, dt,$$

$$I_3 = \int_1^T \{(g_1\bar{g}_3 + g_3\bar{g}_1)\}(a,t)\, dt,$$

$$I_4 = \int_1^T \{(g_2\bar{g} + g_3\bar{g}_2)\}(a,t)\, dt,$$

and $$I_5 = \int_1^T \{\bar{g}_1 g_4 + g_1\bar{g}_4 + \bar{g}_2 g_4 + g_2\bar{g}_4\}(a,t)\, dt.$$

## 3. Proof of theorem 3

We shall estimate $I_j$, $1 \leqslant j \leqslant 5$ one by one and first prove Theorem 3′.

*Theorem* 3′ : *With the notation of Theorem* 3 *if* $a \neq \frac{1}{2}$,

$$\int_1^T |\zeta(\tfrac{1}{2} + it, a)|^2\, dt = A(a,T) + A_1(a)\, T^{1/2} + 2V_3 + 2V_5 - 2V_{4,2} - 2V_{6,2}$$
$$- C_5 - C_6 + C_1 + C_2 + C_3 + C_4 + R(B_5) + R(B_6)$$
$$+ O(T^{3/8}) + O\left(\frac{\log T}{\| 2a \|}\right),$$

and for $a = \frac{1}{2}$

$$\int_1^T |\zeta(\tfrac{1}{2} + it, \tfrac{1}{2})|^2\, dt = A(\tfrac{1}{2}, t) + A_2\, T^{1/2} \log T + A_3\, T^{1/2} + 2V_3 + 2V_5$$
$$- 2V_{4,2} - 2V_{6,2} - C_5 - C_6 + R(B_5) + R(B_6)$$
$$+ C_1 + C_2 + C_3 + C_4 + O(T^{3/8}),$$

where

$$V_3 = \sum_{1 \leqslant m < n \leqslant K} \frac{\sin\left[T \log\left(\frac{n}{m}\right) + 2\pi(m-n)a\right]}{m^{1/2}\, n^{1/2} \log\left(\frac{n}{m}\right)},$$

$$V_5 = \sum_{0 \leqslant m < n \leqslant K_1} \frac{\sin\left[T \log\left(\frac{n+a}{m+a}\right)\right]}{(m+a)^{1/2}(n+a)^{1/2} \log\left(\frac{n+a}{m+a}\right)},$$

$$V_{4,2} = \sum_{m+m^{1/4} < n \leqslant K} \frac{\sin\left[2\pi n^2 \log\left(\frac{n}{m}\right) + 2\pi(m-n)a\right]}{m^{1/2}\, n^{1/2} \log\left(\frac{n}{m}\right)},$$

$$V_{6,2} = \sum_{m+m^{1/4}<n\leqslant K_1} \frac{\sin\left[2\pi(n+\alpha)^2\log\left(\frac{n+\alpha}{m+\alpha}\right)\right]}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}\log\left(\frac{n+\alpha}{m+\alpha}\right)}.$$

Then we shall deduce Theorem 3 from Theorem 3′.

*Lemma* 1 : $I_3 = B_1 + B_2 + O(1)$.

*Proof*: We divide the interval $[2\pi(n+\alpha)^2, T]$ into intervals of the type $[q, q+\alpha]$ and $[q+\alpha, q+1]$ where $q$ is a positive integer and then substitute the value of $\beta$ in each case.

*Lemma* 2 :

$$B_1 = -4\pi K \int_{(1-\alpha)/2}^{(1+\alpha)/2} G^2(u)\,du + C_1 + O(T^{3/8}).$$

*Proof*: Of the summation over $q$ in $B_1$, we make two parts as $n < q < n + n^{1/4}$ and $n + n^{1/4} \leqslant q \leqslant Q_1$. The second part immediately gives $C_1$. If the first part is defined to be $S(B_1)$,

$$S(B_1) = 2\,\mathrm{Re} \sum_{n\geqslant 0} \frac{1}{(n+\alpha)^{1/2}} \sum_{n<q<n+n^{1/4}} (-1)^{q-1} \int_{2\pi q^2}^{2\pi(q+\alpha)^2} i\psi(t)\left(\frac{2\pi}{t}\right)^{1/4}$$

$$\times \exp\left[it\log\left(\frac{n+\alpha}{y}\right) + \frac{it}{2} + \frac{i\pi}{8} + \frac{i\pi\alpha^2}{2}\right.$$

$$\left. + i\pi(2y - 2\alpha y - \alpha)\right] dt.$$

Let $q = n + r$, $1 \leqslant r < n^{1/4}$ and $(2\pi t)^{1/2} - (\alpha - 1)\pi = 2\pi(q+u)$, $r + u + (\alpha - 1)/2 = \omega$. Then clearly $\psi(t) = G(u)$ and

$$S(B_1) = 8\pi\,\mathrm{Re} \sum_{n\geqslant 0} \frac{1}{(n+\alpha)^{1/2}} \sum (-1)^{n+r-1} \int_{(1-\alpha)/2}^{(1+\alpha)/2} G(u)(n+\omega)^{1/2}$$

$$\times \exp\left[2\pi i(1-\alpha)(n+\omega) + \frac{5i\pi}{8} - i\pi\alpha + 2\pi i(n+\omega)^2\right.$$

$$\left.\times \log\left(\frac{n+\alpha}{n+\omega}\right) + i\pi(n+\omega)^2 + \frac{i\pi\alpha^2}{2}\right] du.$$

Let $$\cos\left[2\pi(n+\omega)(1-\alpha) + \frac{5\pi}{8} - \pi\alpha + 2\pi(n+\omega)^2\log\left(\frac{n+\alpha}{n+\omega}\right)\right.$$

$$\left. + \pi(n+\omega)^2 + \frac{\pi\alpha^2}{2}\right] = \cos X.$$

Then $$\cos X = \cos(-X) = \cos\left[-\pi n^2 + (2\omega^2 - 2\omega\alpha + \alpha^2 - 2\omega)\pi\right.$$

$$+ \pi a - \frac{\pi a^2}{2} - \frac{5\pi}{8} + \frac{2(\omega - a)^3}{3(n+a)} - \pi + O\left(\frac{r^4}{(n+a)^2}\right)\Bigg].$$

Now substituting the values of $\omega$ we get

$$\cos X = (-1)^n \cos \pi \left[ 2u^2 + 4u(r-1) + \frac{7}{8} + \frac{2}{3}\frac{\left(u + r - \frac{1+a}{2}\right)^3}{(n+a)} \right.$$

$$\left. + O\left(\frac{r^4}{n^2}\right)\right].$$

Let $\quad \pi\left[2u^2 + 4u(r-1) + \frac{7}{8}\right] = A, \quad \frac{2\pi}{3}\frac{\left(u + r - \frac{1+a}{2}\right)^3}{n+a} + O\left(\frac{r^4}{n^2}\right) = B,$

and use the formula $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and note that the substitution of 1 for $\cos B$ and

$\dfrac{2\pi\left(u + r - \frac{1+a}{2}\right)^3}{n+a}$ for $\sin B$ gives an error $O(T^{3/8})$.

Thus $\quad S(B_1) = 8\pi \sum_n \sum_{r < n^{1/4}} \frac{(-1)^{r-1}}{(n+a)^{1/2}} \int_{(1-a)/2}^{(1+a)/2} G(u)\left(n + r + u + \frac{a-1}{2}\right)^{1/2}$

$$\times \cos 2\pi\left[u^2 + 2u(r-1) + \frac{7}{16}\right] du - 8\pi \sum_n \sum_{r < n^{1/4}} \frac{(-1)^{r-1}}{(n+a)^{1/2}}$$

$$\times \int_{(1-a)/2}^{(1+a)/2} G(u)\left(n + r + u + \frac{a-1}{2}\right)^{1/2} \sin 2\pi\left[u^2 + 2u(r-1)\right.$$

$$\left. + \frac{7}{16}\right] \times \frac{2\pi}{3}\frac{\left(u + r - \frac{1+a}{2}\right)^3}{n+a} du + O(T^{3/8})$$

$$= S_1 + S_2 + O(T^{3/8}), \text{ say.}$$

Integrating by parts, $S_2$ is seen to be $O(T^{3/8})$. In $S_1$, we replace $(r + u - (1-a)/2)^{1/2}$ by $(n+a)^{1/2}$ and then change the variable $u$ to $1 - u$.
Then adding both expressions for $S_1$ and dividing by 2, we get

$$S(B_1) = 8\pi \sum_n \sum_r (-1)^{r-1} \int_{(1-a)/2}^{(1+a)/2} G(u) \cos 2\pi\left(u^2 - u + \frac{7}{16}\right)$$

$$\times \cos 2\pi(2ur - u)\, du + O(T^{3/8})$$

$$= 8\pi \sum_n \int_{(1-\alpha)/2}^{(1+\alpha)/2} G(u) \cos 2\pi\left(u^2 - u + \frac{7}{16}\right) \sum_{1 \leqslant r < n^{1/4}} (-1)^{r-1}$$

$$\times \cos 2\pi u\,(2\,(r-1)+1)\,du + O(T^{3/8})$$

$$= -4\pi \sum_n \int_{(1-\alpha)/2}^{(1+\alpha)/2} G(u) \cos 2\pi\left(u^2 - u - \frac{1}{16}\right)$$

$$\times \left[\frac{1-(-1)^R \cos 4\pi R u}{\cos 2\pi u}\right] du + O(T^{3/8})$$

where $R = [n^{1/4}]$.

Integration by parts of the second term inside the integral sign gives the proof of the Lemma.

*Lemma* 3 :

$$B_2 = -4\pi K \int_{\alpha/2}^{1-\alpha/2} G^2(u)\,du + C_2 + O(T^{3/8}).$$

*Proof*: Similar to that of Lemma 2.

*Remark*: It is interesting to compare the work of Balasubramanian [1] to see how, for the case $\alpha = 1$, we get finer error terms.

*Lemma* 4 :

$$I_4 = B_3 + B_4 + O(1).$$

*Proof*: Similar to that of Lemma 1.

*Lemma* 5 :

$$B_3 = -4\pi K \int_{(1-\alpha)/2}^{(1+\alpha)/2} G^2(u)\,du + C_3 + O(T^{3/8}).$$

*Proof*: Similar to that of Lemma 2.

*Lemma* 6 :

$$B_4 = -4\pi K \int_{\alpha/2}^{1-\alpha/2} G^2(u)\,du + C_4 + O(T^{3/8}).$$

*Proof*: Similar to that of Lemma 2.

*Lemma* 7 :

$$I_2 = 4\pi K\left[\int_{(1-\alpha)/2}^{(1+\alpha)/2} G^2(u)\,du + \int_{\alpha/2}^{1-\alpha/2} G^2(u)\,du\right] + O(T^{3/8}).$$

*Proof*: We note that

$$I_2 = \sum_{1 \leqslant q \leqslant Q} \left[ \int_{2\pi q^2}^{2\pi(q+a)^2} \psi^2(t) \left(\frac{2\pi}{t}\right)^{1/2} dt + \int_{2\pi(q+a)^2}^{2\pi(q+1)^2} \psi^2(t) \left(\frac{2\pi}{t}\right)^{1/2} dt \right] + O(1)$$

where $Q = K - 1$.

Hence $I_2 = \tilde{S}_1 + \tilde{S}_2 + O(1)$, where $\tilde{S}_1$ and $\tilde{S}_2$ are defined in an obvious way. For $\tilde{S}_1$ we use the substitution $\sqrt{2\pi t} - (a - 1)\pi = 2\pi(q + u)$ and proceed as in Lemma 2 to see that

$$\tilde{S}_1 = 4\pi Q \int_{(1-a)/2}^{(1+a)/2} G^2(u)\, du + O(T^{3/8}).$$

We let $\sqrt{2\pi t} - a\pi = 2\pi(q + u)$ in $\tilde{S}_2$ and get

$$\tilde{S}_2 = 4\pi Q \int_{a/2}^{1-a/2} G^2(u)\, du + O(T^{3/8}).$$

*Lemma* 8:

$$I_5 = O(T^{1/4} \log T).$$

*Proof*: We assume that

$$\int_0^T | g_1 + g_2 |^2 (a, t)\, dt = O(T \log T),$$

which we will prove later.

$$I_5 \leqslant 2 \left| \int_1^T \{(g_1 + g_2) \bar{g}_4\}(a, t)\, dt \right|,$$

and $$\int_{T/2}^T \{(g_1 + g_2)\bar{g}_4\}(a, t)\, dt \leqslant \{ \int_{T/2}^T | (g_1 + g_2)(a, t) |^2\, dt$$

$$\times \int_{T/2}^T O(t^{-3/2})\, dt\}^{1/2} = O(T^{1/4} \log T).$$

Replacing $T$ by $T/2$, $T/4, \cdots, T/2^n \cdots$ and adding, we get the Lemma.

*Lemma* 9 :

$$I_1 = V_1 + V_2 + 2V_3 - 2V_4 + 2V_5 - 2V_6 + V_7 + O(1)$$

where $$V_1 = \sum_{n=1}^{K_1} \frac{T - 2\pi(n + a)^2}{n + a} + \frac{T - 1}{a}, \quad V_2 = \sum_{m=1}^{K} \frac{T - 2\pi m^2}{m},$$

$$V_3 = \sum_{1 \leqslant m < n \leqslant K} \frac{\sin\left[T \log\left(\frac{n}{m}\right) + 2\pi(m - n)a\right]}{m^{1/2} n^{1/2} \log\left(\frac{n}{m}\right)},$$

$$V_4 = \sum_{1 \leqslant m < n \leqslant K} \frac{\sin\left[2\pi n^2 \log\left(\frac{n}{m}\right) + 2\pi (m-n) a\right]}{m^{1/2} n^{1/2} \log\left(\frac{n}{m}\right)},$$

$$V_5 = \sum_{0 \leqslant m < n \leqslant K_1} \frac{\sin\left[T \log\left(\frac{n+a}{m+a}\right)\right]}{(m+a)^{1/2}(n+a)^{1/2} \log\left(\frac{n+a}{m+a}\right)},$$

$$V_6 = \sum_{0 \leqslant m < n \leqslant K_1} \frac{\sin\left[2\pi (n+a)^2 \log\left(\frac{n+a}{m+a}\right)\right]}{(m+a)^{1/2}(n+a)^{1/2} \log\left(\frac{n+a}{m+a}\right)},$$

and $V_7 = V_{7,1} + V_{7,2}$;

where $$V_{7,1} = 2 \operatorname{Re} \sum_{1 \leqslant m < n \leqslant K_1} \frac{\exp(2\pi i m a)}{m^{1/2}(n+a)^{1/2}} \int_{2\pi(n+a)^2}^{T} \exp i [\phi(t) - t \log\{m(n+a)\}]\, dt,$$

and $$V_{7,2} = 2 \operatorname{Re} \sum_{0 \leqslant n < m \leqslant K} \frac{\exp(2\pi i m a)}{m^{1/2}(n+a)^{1/2}} \int_{2\pi m^2}^{T} \exp i [\phi(t) - t \log\{m(n+a)\}]\, dt.$$

*Proof*: Follows from the definitions of $g_1(a, t)$ and $g_2(a, t)$. We get

$$\int_1^T |g_2(a,t)|^2 dt = V_2 + 2V_3 - 2V_4 + O(1),$$

$$\int_1^T |g_1(a,t)|^2 dt = V_1 + 2V_5 - 2V_6 + O(1)$$

and $$\int_1^T (g_1 \bar{g}_2 + \bar{g}_1 g_2)(a,t)\, dt = V_7 + O(1).$$

*Lemma* 10 : $V_{7,1} = B_5 + R(B_5) - C_5$, where $B_5$, $R(B_5)$ and $C_5$ are as defined in § 2.

*Proof*: Easy. We only write

$$\int_{2\pi(n+a)^2}^{T},$$

as $$\left[\int_{2\pi(n+a)^2}^{\infty} - \int_T^{\infty}\right],$$

and then for the first part the summation

$$\sum_{0 < m \leqslant n \leqslant K_1}$$

is split into

$$\sum_{1 \leqslant m \leqslant n \leqslant m + m^{1/4}} + \sum_{m + m^{1/4} < n \leqslant K_1}$$

*Lemma* 11 :

$$B_5 = S(B_5) + O(T^{3/8}),$$

where $$S(B_5) = 4\pi \operatorname{Re} \sum_{m=1}^{K} \sum_{0 \leqslant r \leqslant m^{1/4}} \exp[2\pi i(2m\alpha + 2r\alpha + \alpha^2]$$

$$\times \int_{r+\alpha}^{\infty} \exp[i\pi(\theta^2 - \tfrac{1}{4})]\, d\theta.$$

*Proof* : We use Lemma P1.

*Lemma* 12 :

$$V_{7,2} = B_6 + R(B_6) - C_6.$$

*Proof* : Similar to that of Lemma 10.

*Lemma* 13 :

$$B_6 = S(B_6) + O(T^{3/8})$$

where $$S(B_6) = 4\pi \operatorname{Re} \sum_{n=0}^{K_1} \sum_{1 \leqslant r \leqslant n^{1/4}} \int_{r-\alpha}^{\infty} \exp[i\pi(\theta^2 - \tfrac{1}{4})]\, d\theta.$$

*Proof* : Similar to that of Lemma 10.

*Lemma* 14 : Let

$$S(r, \alpha) = \sum_{r^4 \leqslant m \leqslant K} \exp(4\pi i m\alpha)$$

then $$S(B_5) = \operatorname{Re} \sum_{0 \leqslant r \leqslant K^{1/4}} S(r, \alpha) \int_{r+\alpha}^{\infty} \exp[i\pi(\theta^2 - \tfrac{1}{4})] \cdot \exp[2\pi i(\alpha^2 + 2r\alpha)]\, d\theta$$

For $a \neq \frac{1}{2}$,

$$S(B_5) = O\left(\frac{\log T}{\| 2\alpha \|}\right).$$

*Proof* : The proof follows after changing the order of summation over $r$ and $m$ and noting that $S(r, \alpha)$ is a geometric series.

*Lemma* 15 :

$$S(B_6) = B_{6,1} + B_{6,2} + O(T^{3/8}),$$

where $$B_{6,1} = 4\pi(K_1 + 1) \operatorname{Re} \int_{1-\alpha}^{1} \frac{\exp[i\pi(u^2 - u - \frac{1}{4})]}{2\cos \pi u}\, du,$$

and $$B_{6,2} = 4\pi(K_1 + 1) \operatorname{Re} \sum_{r=1}^{\infty} \int_{r}^{\infty} \exp[i\pi(\theta^2 - \tfrac{1}{4})]\, d\theta.$$

*Proof* :

$$S(B_6) = 4\pi \operatorname{Re} \sum_{n=0}^{K_1} \sum_{1 \leqslant r \leqslant n^{1/4}} \int_{r-\alpha}^{r} \exp[i\pi(\theta^2 - \tfrac{1}{4})]\, d\theta$$

$$+ 4\pi \operatorname{Re} \sum_{n=0}^{K_1} \sum_{1 \leqslant r \leqslant n^{1/4}} \int_{r}^{\infty} \exp[i\pi(\theta^2 - \tfrac{1}{4})]\, d\theta.$$

The second term gives $B_{6,2}$ modulo an error $O(T^{3/8})$. The first term is

$$4\pi \operatorname{Re} \int_{1-a}^{1} \frac{\exp[i\pi(u^2 - u - \frac{1}{4})]}{2\cos \pi u} [1 + \exp[\pi i(2u+1) R] \, du, \quad R = [n^{1/4}].$$

This easily gives $B_{6,1}$.

*Lemma* 16 : For $\alpha = \frac{1}{2}$

$$S(B_5) + S(B_6)$$

$$= -\frac{1}{4\sqrt{2\pi}} T^{1/2} \log T + A_3 T^{1/2} + 2\sqrt{2\pi}\, H(\tfrac{1}{2})\, T^{1/2} + O(T^{3/8}).$$

*Proof* :

$$S(B_5) + S(B_6) = -4 \sum_{1 \leqslant m \leqslant K} \sum_{1 \leqslant r \leqslant m^{1/4}} \left\{\frac{1}{2r+1}\right\} + \frac{A_0}{\sqrt{2\pi}} T^{1/2} + O(T^{3/8}).$$

Now we use the well-known formula

$$\sum_{r=1}^{R} \frac{1}{r} = \log R + \gamma + O\left(\frac{1}{R}\right)$$

and get the result.

*Lemma* 17 :

$$V_4 = - \sum_{\substack{m=1 \\ 1 \leqslant r \leqslant m^{1/4}}}^{K} \frac{\sin \pi r(1+2\alpha)}{r} + V_{4,2} + O(T^{3/8})$$

and
$$V_6 = \sum_{\substack{m=1 \\ 1 \leqslant r \leqslant m^{1/4}}}^{K} \frac{\sin \pi r(1+2\alpha)}{r} + V_{6,2} + O(T^{3/8}).$$

*Proof*: We split the sum over $n$ in $V_4$ into

$$\sum_{m < n \leqslant m + m^{1/4}} + \sum_{m + m^{1/4} < n \leqslant K}.$$

In the first sum, $n = m + r$ gives

$$\sin\left[2\pi n^2 \log\left(\frac{n}{m}\right) + 2\pi(m-n)\,\alpha\right] = -\sin[\pi r(1+2\alpha)] + O\left(\frac{r^3}{m}\right),$$

and in the denominator

$$m^{1/2}(m+r)^{1/2} \log\left(1 + \frac{r}{m}\right)$$

is replaced by $r$. The second sum can be treated similarly. The above results give

*Lemma* 18 : For $\alpha = \frac{1}{2}$

$$I_1 = V_1 + V_2 + 2V_3 + 2V_5 - 2V_{4,2} - 2V_{6,2} - \frac{1}{4\sqrt{2\pi}} T^{1/2} \log T$$
$$+ R(B_5) + R(B_6) + \frac{T^{1/2}}{\sqrt{2\pi}} [\tfrac{1}{4} \log 2\pi - 4 \log 2 + \tfrac{1}{2} - 2\gamma]$$
$$- C_5 - C_6 + O(T^{3/8})$$

and for $\alpha \neq \frac{1}{2}$

$$I_1 = V_1 + V_2 + 2V_3 + 2V_5 - 2V_{4,2} - 2V_{6,2} + S(B_6) - C_5 - C_6$$
$$+ R(B_5) + R(B_6) + O(\log T/\|2\alpha\|) + O(T^{3/8}).$$

*Proof of Theorem* 3′ : It is easy to see that

$$V_1 + V_2 = 2T \log K + T[C(\alpha) + \gamma - 1] - \frac{1}{\alpha} + O(1).$$

Now we only have to sum up the results of Lemmas 1 through 18 and note that for $\alpha \neq \frac{1}{2}$

$$S(B_6) = 2\sqrt{2\pi}\, T^{1/2} \left[ \frac{1}{2} \int_{(1-\alpha)}^{1} \frac{\cos(u^2 - u - \frac{1}{4})\pi}{\cos \pi u} du + \sum_{r=1}^{\infty} \int_r^{\infty} \cos \pi \left( u^2 - \frac{1}{4} \right) du \right],$$

and Theorem 3′ is proved.

To deduce Theorem 3 from Theorem 3′, we need the following lemmas.

*Lemma* 19 :

$$V_3 = O(T^{5/12} \log^2 T), \quad V_5 = O(T^{5/12} \log^2 T).$$

*Proof* : Consider $V_3$ and split the sum over $n$ into

$$\sum_{m<n<2m} + \sum_{2m \leqslant n \leqslant K} .$$

For the second sum $\log(n/m) \gg \log 2$; $\log(n/m)$ is monotonic in $n$ for fixed $m$. Now consider $N < N' < 2N$, $2\pi f(x) = T \log x - 2\pi x \alpha$ and use Lemma P5. The first sum we write as

$$\sum_{K/2 < m \leqslant K} + \sum_{K/4 < m \leqslant K/2} + \cdots .$$

The summation is over $O(\log T)$ terms and each term is of the form

$$\sum_m \sum_{m<n<m+\lambda} + \sum_m \sum_{m+\lambda \leqslant n < 2m} = T_1 + T_2.$$

For $T_1$, let

$$f(x) = T \log \left( \frac{x+q}{x} \right) - 2\pi q \alpha$$

and use Lemma P4 to get

$$T_1 = O(M^{-1/2}T^{1/2}\lambda^{1/2}) + O(M^{3/2}T^{-1/2}).$$

Treating $M > T^{1/3}$ and $M \leqslant T^{1/3}$ separately, we get $T_1 = O(T^{5/12})$.

For $T_2$,

$$f(x) = \frac{T}{2\pi}\log x - x\alpha,$$

and use Lemma P5 to get

$$\sum_n \frac{n^{iT}\exp(-2\pi i n\alpha)}{n^{1/2}\log\left(\frac{n}{m}\right)} = \sum_{m+\lambda<n\leqslant m+2\lambda} + \sum_{m+2\lambda<n\leqslant m+3\lambda} + \dots = O(T^{1/6}\log T).$$

Hence $T_2 = O(T^{5/12}\log T)$.

*Lemma* 20 :

$$C_5 + C_6 = O(T^{5/12}\log^2 T) + O\left(\frac{\log T}{\alpha}\right).$$

*Proof*: Use integration by parts to get

$$C_5 + C_6 = 2\sum_{m,n} \frac{\sin\left[T\log\left(\frac{T}{2\pi mn'}\right) - T - \frac{\pi}{4} + 2\pi m\alpha\right]}{m^{1/2}n'^{1/2}\log\left(\frac{T}{2\pi mn'}\right)}$$

$$+ 2\sum_{m,n}\frac{1}{m^{1/2}n'^{1/2}}\int_T^\infty \frac{\sin[\phi(t) - t\log mn']\,\phi''(t)}{\left\{\log\left(\frac{t}{2\pi mn'}\right)\right\}^2}dt.$$

where $n' = n + \alpha$.

Note that $\phi''(t) = t^{-1}$ and the second term on the right side is $O(\log^2 T)$. For the first term we proceed as in Lemma 19 and note that the first term is

$$\sum_{m<n<2m} + \sum_{2m<n} + \sum_{n<m<2n} + \sum_{2n\leqslant m} + \sum_{n=m}.$$

Each of the first four summations, treated separately, give a term $O(T^{5/12}\log^2 T)$ and the last one is $O(\log T/\alpha)$.

*Remark* : It is not possible to apply the multiple average integral method of Balasubramanian [1] to $C_5 + C_6$ to reduce the error term to $O(T^{1/3})$, though this method applied to $V_3$ and $V_4$ does work and gives error terms $O(T^{1/3})$.

*Lemma* 21 :

$$R(B_5) + R(B_6) + C_1 + C_2 + C_3 + C_4 - 2V_{4,2} - 2V_{6,1_2} = O(T^{3/8}).$$

*Proof* : Using simple trigonometric formulae and using integration by parts, we get

$$C_1 = 2 \sum_{n+n^{1/4}<m} \frac{\cos \frac{\pi}{8}(4a^2 - 5)}{\cos \pi\alpha}$$

$$\times \frac{\sin\left[2\pi(m+\alpha)^2 \log\left(\frac{n+\alpha}{m+\alpha}\right) - \pi\frac{\alpha^2}{2} + \pi\alpha + \frac{5}{8}\pi\right]}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}\log\left(\frac{n+\alpha}{m+\alpha}\right)}$$

$$- 2 \sum_{n+n^{1/4}<m} \frac{\cos \frac{\pi}{8}(4a^2 - 5)}{\cos \pi\alpha}$$

$$\times \frac{\sin\left[2\pi m^2 \log\left(\frac{n+\alpha}{m}\right) + \pi\frac{\alpha^2}{2} - \pi\alpha + \frac{5}{8}\pi - 2\pi m\alpha\right]}{m^{1/2}(n+\alpha)^{1/2}\log\left(\frac{n+\alpha}{m+\alpha}\right)}$$

$$+ O(T^{3/8}) = C_{1,1} + C_{1,2} + O(T^{3/8}).$$

On the same lines we get similar expressions for $C_2$, $C_3$ and $C_4$. Then we can see that

$$C_{1,1} + C_{2,2} = 2V_{6,2} \qquad \text{and} \qquad C_{3,2} + C_{4,1} = 2V_{4,2}.$$

Also, integration by parts shows that

$$C_{1,2} + C_{2,1} + R(B_6) = O(\log^2 T),$$

and $$C_{3,1} + C_{4,2} + R(B_5) = O(\log^2 T).$$

*Proof of Theorem* 3 : In view of Lemma 21, it only remains to show that $A_1(\alpha) < -\sqrt{2\pi}\, H(\alpha)$.

$$A_1(\alpha) = -2\sqrt{2\pi}\, H(\alpha) + \sqrt{2\pi} \int_{(1-\alpha)}^{1} \frac{\cos(u^2 - u - \frac{1}{4})\pi}{\cos \pi u}\, du$$

$$+ 2\sqrt{2\pi} \sum_{r=1}^{\infty} \int_{r}^{\infty} \cos \pi\left(u^2 - \frac{1}{4}\right) du.$$

For $0 \leqslant a \leqslant \frac{1}{2}$,

$$\int_{(\frac{1}{2}-\alpha)}^{(\frac{1}{2}+\alpha)} \frac{\cos(u^2 - u - \frac{1}{4})\pi}{\cos \pi u}\, du = 0.$$

Also, it is not difficult to prove that for $\alpha \geqslant \frac{1}{2}$

$$\int_{(1-\alpha)}^{1} \frac{\cos\left(u^2 - u - \frac{1}{4}\right)\pi}{\cos \pi u}\, du < 0 \tag{1}$$

and for $0 < \alpha \leqslant 1$

$$- H(\alpha) + 2 \sum_{r=1}^{\infty} \int_{r}^{\infty} \cos \pi\left(u^2 - \frac{1}{4}\right) du < 0. \tag{2}$$

This completes the proof of Theorem 3.

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## Note added in proof

In January 1981, K Ramachandra and R Balasubramanian have considered these problems in a paper entitled 'A hybrid version of a theorem of Ingham'. Their approach is simpler but their results are weaker than those proved here.