On the relation of generalized Valiron summability to Cesàro summability

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Abstract. A family \((V_\rho^2)\) of summability methods, called generalized Valiron summability, is defined. The well-known summability methods \((B_\alpha, \gamma)\), \((E_\rho)\), \((T_\alpha)\), \((S_\rho)\) and \((V_\rho)\) are members of this family. In §3 some properties of the \((B_\alpha, \gamma)\) and \((V_\rho)\) transforms are established. Following Satz II of Faulhaber (1956) it is proved that the members of the \((V_\rho^2)\) family are all equivalent for sequences of finite order. This paper is a good illustration of the use of generalized Boral summability. The following theorem is established:

Theorem. If \(s_n (n \geq 0)\) is a real sequence satisfying

\[
\lim_{m \to \infty} \lim_{e \to 0^+} \min_{m \leq n \leq m + \epsilon} \left( \frac{s_n - s_m}{m^\rho} \right) > 0 (\rho > 0),
\]

and if \(s_n \to s (V_\rho)\) then \(s_n \to s (C, 2\rho)\).

Keywords. Generalized Valiron summability; Boral summability; Rajagopal’s theorem.

1. Introduction

Rajagopal ([4], Theorem 2) proved the following theorem connecting Borel and Cesàro summabilities; and, after him, Sitaraman ([5], Theorem II) proved the theorem with Borel summability replaced by summability \((S_\rho)\) defined as usual in §5:

Theorem A. If \(s_n (n \geq 0)\) is a real sequence satisfying

\[
\lim_{m \to \infty} \lim_{e \to 0^+} \min_{m \leq n \leq m + \epsilon} \left( \frac{s_n - s_m}{m^\rho} \right) > 0 (\rho > 0),
\]

and if \(s_n \to s (B)\) then \(s_n \to s (C, 2\rho)\).

In this paper we prove (Theorem 4) that Theorem A is extensible to a family \((V_\rho^2)\) of summability methods which include as special cases generalized Borel summability \((B_\alpha, \gamma)\) defined in §2 and the well-known summabilities \((E_\rho)\), \((T_\alpha)\), \((S_\rho)\) defined in the usual notation in §5. Of course Theorem A itself obviously

1 In homage to the late Professor C T Rajagopal.
includes the similar theorem for summability \((E_p)\) instead of summability \((B)\), since \((E_p) \subset (B)\). Valiron summability \((V_\nu)\) is also a special case of summability \((V_\nu^2)\), as pointed out in \(\S 5\), and the latter is the generalized Valiron summability of the title.

The Tauberian condition (1) reduces to a classic special case when \(\rho = 0\). A different special case of (1) and its further specialization are respectively

\[
\begin{align*}
  s_n - s_{n-1} &= O_k(n^{\rho-1/2}), \\
  s_n - s_{n-1} &= o(n^{\rho-1/2}).
\end{align*}
\]

(1 a)

(1 b)

Hardy and Littlewood originally proved the special case of Theorem A with (1 b) instead of (1), as stated by Hardy ([3], note on §§ 9·6-7). Their result was extended by Borwein [1] to generalized Borel summability, and an idea of his (Lemma 7) is used in the sequel.

2. Definitions

The \(V_\nu^k\) transform of a (generally complex) sequence \(s_n(n \geq 0)\) is the function defined by

\[
V_\nu^k(x) = \sum_{n=0}^{\infty} c_n(x) s_n, \quad x > 0,
\]

where \(c_n(x)\) satisfies the following three conditions:

(i) \(c_n(x) \geq 0\) for \(n = 0, 1, 2, \ldots, x > 0\);

(ii) there exist \(a > 0\) and \(\delta\) with \(\frac{1}{2} < \delta < \frac{3}{2}\) such that, for every positive integer \(k\), \(c_n(x)\) can be expressed as

\[
c_n(x) = \left(\frac{a}{\pi x}\right)^{1/2} \exp \left\{ -\frac{a}{x^2} (n - x)^2 + g_k + R_k \right\}
\]

whenever \(x\) is sufficiently large and \(|n - x| \leq x^3\), and where

\[g_k = \sum_{i=0}^{2k+1} \sum_{j=0}^{i+1} l_{ij} \frac{(n-x)^j}{x^i}, \quad l_{12} = 0,
\]

\(l_{ij}\) being independent of \(n\) and bounded as \(x \to \infty\),

\[R_k = O\left(\frac{|n-x|^{2k+1} + 1}{x^{2k}}\right)\quad \text{as} \quad x \to \infty,
\]

uniformly in \(n\) for \(|n - x| \leq x^3\);

(iii) for every \(\sigma > 0\)

\[\sum_{|n-x| > x^\delta} (n+1)^\sigma c_n(x) = o(1) \quad \text{as} \quad x \to \infty.
\]

We say that \(s_n\) is summable \((V_\nu^k)\) to \(s\) (finite), and write \(s_n \to s(V_\nu^k)\) if \(V_\nu^k(x) \to s\) as \(x \to \infty\).
The \((B_{a,\gamma})\) transform \((a > 0, \gamma \text{ real})\) of \(s_n\) is the function defined by
\[
B(x) = a \exp(-ax) \sum_{n=N}^{\infty} \frac{(ax)^{n-\gamma}}{\Gamma(na + \gamma)} s_n, \ x > 0,
\]
where \(N\) is the least positive integer such that \(Na + \gamma \geq 1\). We say that \(s_n\) is summable \((B_{a,\gamma})\) to \(s\) (finite), and write \(s_n \rightarrow s(B_{a,\gamma})\) if \(B(x) \rightarrow s\) as \(x \rightarrow \infty\).

The \(n\)th Cesàro sum and the \(n\)th Cesàro mean of \(s_n\), each of order \(r > -1\), are denoted by \(S^r_n\) and \(S^r_n\) respectively. Thus
\[
S^r_n = s^0_n = s_n; \quad S^r_n = \sum_{\ell=0}^{n} \left( \begin{array}{c} n - r - 1 \end{array} \right) s^r_{n-r}. \quad (r = n-n).
\]
We say that \(s_n\) is summable \((C, r)\) to \(s\) (finite), and write \(s_n \rightarrow s(C, r)\) if \(s^r_n \rightarrow s\) as \(n \rightarrow \infty\).

3. Preliminary results

In this section we study some properties of the \((B_{a,\gamma})\) and \(V^\gamma\) transforms.

Lemma 1. The \((B_{a,\gamma})\) transform is a \(V^\gamma\) transform with \(a = a/2\).

Proof. Borwein ([1], Lemma 2 (d)) has proved that \(c_n(x)\) defined as below satisfies condition (iii):
\[
c_n(x) = a \exp(-ax) \frac{(ax)^{n-\gamma}}{\Gamma(na + \gamma)} \text{ for } n \geq N \text{ and } c_n(x) = 0 \text{ for } n < N.
\]
To verify condition (ii), let \(\frac{1}{2} < \delta < \frac{3}{2}\), \(x\) be large, \(|n - x| \leq x^\delta\), and \(k\) be any positive integer. Writing \(h = n - x + (\gamma - 1)/a\) and using the formula
\[
\log \Gamma(y + 1) = \frac{1}{2} \log(2\pi) + \left( y + \frac{1}{2} \right) \log y - y + \sum_{r=1}^{\infty} \frac{(-1)^{r-1} B_r}{2r} y^{2r+1}
\]

\[+ O(y^{2\delta+1}) \text{ as } y \rightarrow \infty,
\]
with \(y = ax + ah\) we see that
\[
\log \left( a \left( \frac{2\pi x}{a} \right)^{1/2} \right) = \frac{1}{2} \log(2\pi) - ax + (ax + ah + \frac{1}{2}) \log(ax)
\]
\[- \log \Gamma(ax + ah + 1) = A_1 + A_2 + A_3
\]
where
\[
A_1 = ah + (ax + ah + \frac{1}{2}) \log(ax) - (ax + ah + \frac{1}{2}) \log(ax + ah), \quad (2)
\]
\[
A_2 = - \sum_{r=1}^{\infty} \frac{(-1)^{r-1} B_r}{2r} (ax + ah)^{-2r+1}, \quad (3)
\]
\[|A_3| \leq M(ax + ah)^{-2\delta-1}\]
for some constant $M$. By Taylor's theorem,

$$\log(1+y) = \sum_{r=1}^{2k} \frac{(-1)^{r-1}}{r} y^r + \frac{y^{2k+1}}{2k+1} (1 + \theta y)^{-2k-1},$$

where $0 \leq \theta = \theta(k, y) \leq 1$. Therefore, from (2),

$$A_1 = ah - \left( ax + ah + \frac{1}{2} \right) \log\left(1 + \frac{h}{x}\right)$$

$$= ah - ax \left( \frac{h}{x} - \frac{1}{2} \left( \frac{h}{x} \right)^2 + \sum_{r=0}^{2k} \frac{(-1)^{r-1}}{r} \left( \frac{h}{x} \right)^r \right)$$

$$- ah \left( \frac{h}{x} + \sum_{r=2}^{2k-1} \frac{(-1)^{r-1}}{r} \left( \frac{h}{x} \right)^r \right) - \frac{1}{2} \sum_{r=1}^{2k-1} \frac{(-1)^{r-1}}{r} \left( \frac{h}{x} \right)^r + A_4$$

$$= - a \frac{h^2}{2x} + \sum_{r=2}^{2k-1} u_r h^{r+1} x^{-r} + \sum_{r=1}^{2k-1} v_r h^r x^{-r} + A_4$$

where $u_r, v_r$ are independent of $h, x$ and

$$A_4 = \frac{a}{2k} \frac{h^{2r+1}}{x^{2k}} + \frac{1}{2k} \frac{h^{2k}}{x^{2k}} - \frac{a x + ah + \frac{1}{2}}{2k+1} \left( \frac{h}{x} \right)^{2k+1} (1 + \theta h/x)^{-2k-1}. \quad (5)$$

Again, Taylor's theorem gives

$$(1+y)^{-\mu} = \sum_{r=0}^{m} (-1)^r \binom{\mu + v - 1}{v} y^r + (-1)^{m+1} y^{m+1} \binom{\mu + m}{m+1}$$

$$\times (1 + \theta y)^{-\mu-m-1},$$

where $0 \leq \theta = \theta(\mu, m, y) \leq 1$. Using this with $y = h/x$, $\mu = 2r - 1$, $m = 2k - 2r$, $r = 1, \ldots, k$, we get from (3),

$$A_3 = \sum_{r=1}^{k} \frac{(-1)^r B_r}{(2r-1) 2r} \alpha^{2r+1} x^{-2r+1} \left\{ \sum_{p=0}^{2k-2r} \frac{(-1)^p}{p} \left( \frac{2r-1 + v - 1}{v} \right) \left( \frac{h}{x} \right)^p \right\}$$

$$+ \frac{(-1)^{2k-2r+1}}{2k-2r+1} \left( \frac{h}{x} \right)^{2k-2r+1} \left( \frac{2k-1}{2k-2r+1} \right) (1 + \theta h/x)^{-2k}$$

$$= \sum_{r=1}^{k} \sum_{p=0}^{2k-2r} w_{r, p} h^p x^{-2r+p-1} + A_5,$$

where $w_{r, p}$ are independent of $h, x$ and

$$A_5 = \sum_{r=1}^{k} \frac{(-1)^{r+1} B_r}{(2r-1) 2r} \alpha^{2r+1} \frac{h^{2r-2r+1}}{x^{2k}} \left( \frac{2k-1}{2k-2r+1} \right) (1 + \theta h/x)^{-2k}. \quad (6)$$
Thus we have proved that
\[
\log \left\{ c_\alpha (x) \left( \frac{2\pi x}{\alpha} \right)^{1/2} \right\} = -\frac{a}{2} h^2 + \sum_{i=1}^{2k+1} \sum_{j=0}^{i-1} l_{ij} h^j x^i + A_3 + A_4 + A_5,
\]
where \( l_{ij} \) are independent of \( h, x \) and \( l_{i2} = 0 \). Noting that \( h = n - x + (\gamma - 1)/\alpha \) and writing \( \alpha = 2\alpha \) we get
\[
\log c_\alpha (x) = \log \left( \frac{a}{\pi x} \right)^{1/2} - \frac{a}{x} \left\{ (n - x)^2 + 2(n - x) \left( \frac{\gamma - 1}{\alpha} \right) + \left( \frac{\gamma - 1}{\alpha} \right)^2 \right\}
+ \sum_{i=1}^{2k+1} \sum_{j=0}^{i-1} l_{ij} \frac{1}{x^{i+j}} \sum_{j=0}^{i} \left( \frac{j}{\alpha} \right) (n - x)^j \left( \frac{\gamma - 1}{\alpha} \right)^{i-j} + A_3 + A_4 + A_5
= \log \left( \frac{a}{\pi x} \right)^{1/2} - ax^{-1} (n - x)^2 + \sum_{i=1}^{2k+1} \sum_{j=0}^{i-1} l_{ij} \frac{(n - x)^j}{x^i} + A_3 + A_4 + A_5,
\]
(7)
where \( l_{ij} \) are independent of \( n, x \) and \( l_{i2} = l_{i2} = 0 \).

Since \( h = n - x + (\gamma - 1)/\alpha \), we have \(| h |/x < \frac{1}{2}\) and \( 1 + \theta h/x > \frac{1}{2}\) whenever \( 0 \leq \theta \leq 1 \), \( x \) is large and \( | n - x | \leq x^\delta \). Moreover,
\[
|h|^\nu \leq \left\{ | n - x |^{2k+1} + 1 \right\} \left\{ 1 + \frac{|\gamma - 1|}{\alpha} \right\}^{2k+1}, \; \nu = 0, 1, \cdots, 2k + 1.
\]
Supposing these estimates in (4), (5) and (6), we find that
\[
x^{2k} \left[ A_3 \right] + \left| A_4 \right| + \left| A_5 \right| \leq M \left[ | n - x |^{2k+1} + 1 \right]
\]
if \( | n - x | \leq x^\delta \) and \( x \) is large, \( M \) being a constant. This, in view of (7), completes the proof of the lemma.

Lemma 2. If \( c_\alpha (x) \) satisfies the conditions of a \( V^k_\alpha \) transform, then for \( \sigma > 0 \),
(a) \( c_\alpha (x) = \left\{ 1 + \epsilon_1 (n, x) \right\} \left( \frac{a}{\pi x} \right)^{1/2} \exp \left[ -ax^{-1} (n - x)^2 \right], \)
(b) \( \left( \frac{n}{x} \right)^\sigma c_\alpha (x) = \left\{ 1 + \epsilon_2 (n, x) \right\} \left( \frac{a}{\pi x} \right)^{1/2} \exp \left[ -ax^{-1} (t - x)^2 \right] dt, \)
where \( \epsilon_1 (n, x), \epsilon_2 (n, x) \to 0 \) as \( x \to \infty \) uniformly in \( n \) for \( | n - x | \leq x^\delta \);
(c) \( \sum_{n=1}^\infty \left( \frac{n}{x} \right)^\sigma c_\alpha (x) \to 1 \) as \( x \to \infty \);
(d) \( \theta_1 (N, x) = \sum_{n=N}^\infty \left( \frac{n}{x} \right)^\sigma c_\alpha (x) (\sqrt{n - \sqrt{N}}) \to 0, \)
(c) \( \theta_2 (N, x) = \sum_{n=N}^\infty \left( \frac{n}{x} \right)^\sigma c_\alpha (x) \to 0, \)
\((f)\) \(\theta_3(M, x) \equiv \sum_{n=1}^{M} \left(\frac{n}{x}\right)^{\sigma} c_n(x) \to 0,\)

as \(x, N, \sqrt{N} - \sqrt{x}, M, \sqrt{x} - \sqrt{M} \to \infty.\) (c) for \(\sigma = 0\) shows that \(V_{n}^\sigma\) is a positive regular transform of \(s_n.\)

Proof. (a) Taking \(k = 1\) in condition (ii) on \(c_n(x),\) we see that

\[
\exp(g_1 + R_1) = \exp \left\{ \frac{L_0}{x} + \frac{L_{11}}{x} \frac{(n - x)}{x} + O \left( \frac{n - x}{x^2} \right)^3 \right\} + O \left( \frac{n - x}{x^2} \right) ^{3}
\]

which proves (a), since \(\delta < 2/3.\)

(b) Noting that \((n/x)^\sigma = 1 + \epsilon'(n, x)\) and that

\[
\sum_{n=x}^{n+1} \exp \left\{ - ax^{-1} (t - x)^2 \right\} dt = \exp \left\{ - ax^{-1} (n - x)^2 \right\} \{1 + \epsilon'(n, x)\}
\]

we deduce (b) from (a).

(c) Write the sum in (c) as

\[
\left( \sum_{|n - x| \leq \delta \epsilon} + \sum_{|n - x| > \delta \epsilon} \right) \left(\frac{n}{x}\right)^{\sigma} c_n(x) = S_1 + S_2.
\]

From (b) it follows that

\[
S_1 = \int_{a-\delta}^{a+\delta} \left(\frac{a}{\pi x}\right)^{1/2} \exp \left\{ - ax^{-1} (t - x)^2 \right\} dt + o(1)
\]

\[
= \int_{-\delta}^{\delta} \left(\frac{a}{\pi}\right)^{1/2} \exp \left\{ - au^2 \right\} du + o(1),
\]

which tends to 1 as \(x \to \infty\) since \(\delta > \frac{1}{2}.\) On the other hand, \(S_2 \to 0\) as \(x \to \infty\) by our condition (iii) on \(c_n(x).\)

(d) Write \(\sqrt{N} - \sqrt{x} = u\) so that \(N - x = u (\sqrt{N} + \sqrt{x}) > u \sqrt{x}.\)

In view of condition (iii) on \(c_n(x)\) it suffices to prove that

\[
S = \sum_{N \leq n \leq x + \delta} \left(\frac{n}{x}\right)^{\sigma} c_n(x) (\sqrt{n} - \sqrt{N}) \to 0,
\]

as \(x, u \to \infty.\) Since \(\sqrt{n} - \sqrt{N} < (n - x) / \sqrt{x}\) for \(n \geq N > x,\) it follows from (b) that, for all large \(x,\)

\[
S \leq \sum_{N \leq n \leq x + \delta} \frac{n - x}{\sqrt{x}} 2 \int_{a}^{n+1} \left(\frac{a}{\pi x}\right)^{1/2} \exp \left\{ - ax^{-1} (t - x)^2 \right\} dt
\]
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\[ \leq 2 \left( \frac{a}{\pi x} \right)^{1/2} \sum_{n \leq x} \int_{n \leq t \leq n + 1} \frac{t - x}{\sqrt{x}} \exp \left[ -ax^{-1} (t - x)^2 \right] \, dt \]

\[ \leq 2 \left( \frac{a}{\pi x} \right)^{1/2} \int_{N}^{\infty} \frac{t - x}{\sqrt{x}} \exp \left[ -ax^{-1} (t - x)^2 \right] \, dt \]

\[ = 2 \left( \frac{a}{\pi} \right)^{1/2} \int_{N/\sqrt{x}}^{\infty} v \exp \left( -av^2 \right) \, dv \]

which tends to 0 as \( u \to \infty \).

Proofs of (e) and (f) are similar to that of (d).

Lemma 3. If \( s_n \) is a real sequence satisfying (1) then there exist positive constants \( K_1, K_2 \) such that, for \( n \geq m \geq 1 \),

\[ s_n - s_m \geq -K_1 \alpha (n - m) - K_2 \beta m. \]

This is proved exactly like Theorem 239 of Hardy [3] (see Rajagopal [4], Lemma 1).

Lemma 4. If \( s_n \) is a real sequence satisfying (1), and if

\[ V(s)(x) = O(x^\alpha)(x \to \infty), \]

then \( s_n = O(n^\alpha)(n \to \infty) \).

Proof. It may be remarked that the proof is applicable to any positive regular transform, in place of \( V(s)(x) \), for which the \( c_n(x) \) satisfy (c)–(f) of Lemma 2. We proceed on the lines of the proofs of Theorem 1 of Rajagopal [4] and Theorem 238 of Hardy [3]. Write, for \( n \geq 1 \)

\[ \sigma_n = \frac{s_n}{n^\beta}, \quad \sigma_1(n) = \max_{1 \leq r \leq n} \sigma_r, \text{ and } \sigma_2(n) = \max_{1 \leq r \leq n} (-\sigma_r), \]

and assume, for convenience, that \( s_0 = 0 \). Since

\[ x^\alpha V(s)(x) = \sum_{n=1}^{\infty} (n/x)^\beta c_n(x) \sigma_n, \]

it follows from (8) and Lemma 2 (c) that neither \( \sigma_n \to \infty \) nor \( \sigma_n \to -\infty \) is possible. The lemma is proved by showing that each of the following two cases contradicts (8):

(I) \( \sigma_1(n) \geq \sigma_2(n) \) for infinitely many \( n \) and \( \sigma_1(n) \to \infty \);

(II) \( \sigma_1(n) < \sigma_2(n) \) for all but a finite number of values of \( n \) and \( \sigma_2(n) \to \infty \).

Case I. Corresponding to a large positive number \( H \), choose the least \( M = M(H) \) for which \( \sigma_M = \sigma_1(M) > 2H \) and \( \sigma_1(M) \geq \sigma_2(M) \), and then the
least $N = N(H) > M$ for which $\sigma_N \leq \frac{1}{2} \sigma_M$. Define $x = x(H)$ by $2\sqrt{x} = \sqrt{M} + \sqrt{N}$ and write

$$x^{-p} V^k_a(x) = \left( \sum_{n=1}^{N-1} + \sum_{n=M}^{N-1} + \sum_{n=N}^{\infty} \right) (n/x)^p c_n(x) \sigma_n = S_1 + S_2 + S_3. \quad (9)$$

If $(M/N)^p \leq 2/3$ then

$$\sqrt{N} - \sqrt{M} \geq \left\{ 1 - (2/3)^{1/2p} \right\} \sqrt{N}. \quad (10)$$

If $(M/N)^p > 2/3$ it follows from Lemma 3 that

$$\begin{align*}
\sigma_N - \sigma_M (M/N)^p &> - K_1 (\sqrt{N} - \sqrt{M}) - K_2, \\
K_1 (\sqrt{N} - \sqrt{M}) &> - \sigma_N + \sigma_M (M/N)^p - K_2, \\
&> - \frac{1}{2} \sigma_M + \frac{3}{2} \sigma_M - K_2, \\
&> \frac{3}{2} H - K_2,
\end{align*}$$

by the choice of $M, N$. This, together with (10), shows that $\sqrt{N} - \sqrt{M} \to \infty$ as $H \to \infty$. Hence $\sqrt{N} - \sqrt{x}, \sqrt{x} - \sqrt{M} \to \infty$ as $H \to \infty$.

The estimates which follow are when $H \to \infty$ and so $x, M, N, \sqrt{x} - \sqrt{M}, \sqrt{N} - \sqrt{x} \to \infty$ as in Lemma 2. By the choice of $N, M,$

$$S_{n-1} = (N - 1)^p \sigma_{n-1} > (N - 1)^p \frac{1}{2} \sigma_M > (N - 1)^p H.$$

Hence, by Lemma 3, we have for $n \geq N$,

$$\begin{align*}
s_n &> - K_1 n^p \{ \sqrt{n} - \sqrt{(N - 1)} \} - K_2 (N - 1)^p + s_{n-1} \\
&> - K_1 n^p \{ \sqrt{n} - \sqrt{(N - 1)} \},
\end{align*}$$

and, therefore, by Lemma 2 (d), (e)

$$S_3 > - K_1 \sum_{n=N}^{\infty} (n/x)^p c_n(x) \{ \sqrt{n} - \sqrt{(N - 1)} \} = - K_1 \theta_1 (N - 1, x).$$

On the other hand, by Lemma 2 (c), (e), (f) and the choice of $M, N$,

$$S_2 \geq \frac{1}{2} \sigma_M \sum_{n=M}^{N-1} (n/x)^p c_n(x) = \frac{1}{2} \sigma_M \{ 1 + o (1) - \theta_1 (N, x) - \theta_3 (M - 1, x) \},$$

$$S_1 \geq - \sigma_2 (M) \sum_{n=1}^{N-1} (n/x)^p c_n(x) \geq - \sigma_2 (M) \theta_3 (M - 1, x).$$

Combining these estimates for $S_1, S_2, S_3$ we find from (9) that $x^{-p} V^k_a(x) \to \infty$ as $H \to \infty$ contradicting (8).

Case II. Corresponding to a large positive number $H$ choose the least $N = N(H)$ such that $\sigma_2 (n) > \sigma_1 (n)$ for $n \geq N$ and $\sigma_N = - \sigma_2 (N) < - 2H$; and then the last $M = M(H) < N$ for which $\sigma_M \geq \frac{1}{2} \sigma_N = - \frac{1}{2} \sigma_2 (N)$. Define $x$ as in case I, and write

$$x^{-p} V^k_a(x) = \left( \sum_{n=1}^{M} + \sum_{n=M+1}^{N} + \sum_{n=N+1}^{\infty} \right) (n/x)^p c_n(x) \sigma_n = S_1 + S_2 + S_3. \quad (11)$$
Using Lemma 3 we find, as in case I, that
\[
K_1 (\sqrt{N} - \sqrt{M}) > - \sigma_N + \sigma_M (M/N)^p - K_2 (M/N)^p \\
\geq - \sigma_N \left(1 - \frac{1}{2} (M/N)^p\right) - K_2 \\
> H - K_2,
\]
and that \(\sqrt{N} - \sqrt{x}, \sqrt{x} - \sqrt{M} \to \infty\) as \(H \to \infty\).

From Lemma 3, it follows that, for \(n \geq N\),
\[
\sigma_n > - \{ - \sigma_N + K_3 \} \frac{(N/n)^p}{(\sqrt{n} - \sqrt{N})} \\
\geq - \{ \sigma_3 (N) + K_3 \} - K_1 (\sqrt{n} - \sqrt{N}) \\
= - t_n,
\]
say. Thus \(\sigma_n < t_n\) for \(n \geq N\); while \(- \sigma_n \leq - \sigma_N < t_N\) for \(n < N\) by our choice of \(N\) and definition of \(\sigma_3 (n)\). Since \(t_n\) is an increasing function of \(n\), we thus have \(- \sigma_m \leq t_m \leq t_n\) for \(n \geq m \geq N\), and \(- \sigma_m < t_n \leq t_n\) for \(n \geq N > m\).
This implies that \(t_n \geq \sigma_3 (n)\) for \(n \geq N\) by the definition of \(\sigma_3 (n)\). Hence, by the choice of \(N\) and Lemma 2 (d), (e)
\[
\sigma_n \leq \sigma_3 (n) < \sigma_3 (N) \leq t_n = \sigma_3 (N) + K_2 + K_1 (\sqrt{n} - \sqrt{N}) \quad (n \geq N),
\]
\[
S_3 \leq \sum_{n=N+1}^{\infty} \left(\frac{n}{x}\right)^p c_n (x) \{ \sigma_3 (N) + K_2 + K_1 (\sqrt{n} - \sqrt{N}) \}
\leq \{ \sigma_3 (N) + K_3 \} \theta_3 (N, x) + K \theta_1 (N, x).
\]

On the other hand, by Lemma 2 (c), (e), (f) and by the choice of \(M, N\),
\[
S_2 \leq - \frac{1}{2} \sigma_2 (N) \sum_{n=M+1}^{N} \left(\frac{n}{x}\right)^p c_n (x)
\leq - \frac{1}{2} \sigma_2 (N) \{1 + o (1) - \theta_2 (N + 1, x) - \theta_2 (M, x)\},
\]
\[
S_1 \leq \sigma_1 (M) \sum_{n=1}^{M} \left(\frac{n}{x}\right)^p c_n (x) \leq \sigma_3 (N) \theta_3 (M, x),
\]
since \(\sigma_1 (M) \leq \sigma_1 (N) < \sigma_3 (N)\). Combining these estimates for \(S_1, S_2, S_3\) we find from (11) that \(x^p V^r_0 (x) \to - \infty\) as \(H \to \infty\) contradicting (8). This completes the proof.

Lemma 5. If \(s(t) = s_n\) for \(n \leq t < n + 1\) and \(s_n = O (1)\) \((n \to \infty)\), then
\[
V^r_0 (x) - \int_{0}^{\infty} (a/\pi x)^{1/2} \exp \{- ax^{-1} (t - x)^2\} s(t) dt \to 0 \quad (x \to \infty).
\]

Proof. It suffices to prove that
\[
\sum_{n=0}^{\infty} \left| c_n (x) - \int_{n}^{n+1} (a/\pi x)^{1/2} \exp \{- ax^{-1} (t - x)^2\} dt \right| \to 0 \quad (x \to \infty),
\]
Denoting the summand in (12) by $d_n(x)$, we find from the condition (iii) on $c_n(x)$ with $\sigma = 0$ that, as $x \to \infty$

$$
\sum_{|x-n| > \delta} d_n(x) \leq o(1) + \left( \int_{-\delta}^{0} + \int_{x+\delta}^{\infty} \right) (a/\pi x)^{1/2} \exp \left\{ -ax^{-1}(t-x)^2 \right\} dt
$$

$$
= o(1) + \left( \int_{-\delta}^{-\delta-1} + \int_{x+\delta-1}^{\infty} \right) (a/\pi)^{1/2} \exp (-au^2) du = o(1);
$$
on the other hand, using Lemma 2(b) with $\sigma = 0$,

$$
\sum_{|x-n| \leq \delta} d_n(x) \leq \sum_{|x-n| \leq \delta} \epsilon \int_{n}^{n+1} (a/\pi x)^{1/2} \exp \left\{ -ax^{-1}(t-x)^2 \right\} dt < \epsilon,
$$

for $x > x_0(\epsilon)$. This proves (12).

**Lemma 6.** If $s_n$ is a real sequence satisfying

$$
\lim_{m \to \infty} \lim_{e \to 0^+} \min_{m \leq n \leq m+e} (s_n - s_m) \geq 0,
$$

\begin{equation}
(a/\pi x)^{1/2} \int_{0}^{\infty} \exp \left\{ -ax^{-1}(t-x)^2 \right\} s(t) dt \to s(x \to \infty),
\end{equation}

where $s(t) = s_n$ for $n \leq t < n + 1$ and if $s_n = O(1) (n \to \infty)$, then $s_n \to s (n \to \infty)$.

Taking $s = 0$ this lemma is proved exactly like its case a $= \frac{1}{2}$ ([3], pp. 313–314).

**Lemma 7.** If $s_n \to s (B_{a,\gamma})$ then, for $r \geq 0$,

\begin{equation}
\alpha^{r+1} \Gamma(r+1) \exp \left\{ -ax \right\} \sum_{n=N}^{\infty} \frac{(ax)^{na+\gamma-1}}{\Gamma(na+\gamma+r)} S_n \to s (x \to \infty),
\end{equation}

\begin{equation}
t_r(n) \equiv \alpha^r \frac{\Gamma(r+1) \Gamma(na+\gamma)}{\Gamma(na+\gamma+r)} S_n \to s (B_{a,\gamma}).
\end{equation}

(14) is known ([1], Lemma 4) and (15) follows readily from (14).

**Lemma 8.** If $s_n \to s (B_{a,\gamma})$ and $s'_r = O(n^\sigma)$ for some $r \geq 0$, $\sigma \geq \frac{1}{2}$, then $s_{r+1} = O(n^{\sigma+1/2})$, and, more generally, for an integer $q$ such that $1 \leq q \leq 2\sigma$, $s_{r+q} = O(n^{\sigma-q/2})$.

**Proof.** We have identically, for $m < n$, 

$$
s_{r+1}^n - s_{r+1}^m = \sum_{\nu = m+1}^{n} (s_{\nu+1}^n - s_{\nu+1}^\nu) = \sum_{\nu = m+1}^{n} \frac{r + 1}{\nu} (s_{\nu}^\nu - s_{\nu}^{\nu+1}),
$$

(see [3], p. 122). By hypothesis, $s_r^\nu = O(\nu^{\sigma})$ as $\nu \to \infty$ and so $s_{r+1}^n = O(\nu^{\sigma+1})$. Hence as $m \to \infty$, we have uniformly for $m \leq n \leq m + \epsilon \sqrt{m}, 0 < \epsilon < 1$,

$$
s_{r+1}^n - s_{r+1}^m = \sum_{\nu = m+1}^{n} O(\nu^{\sigma-1}) = \epsilon O(m^{\sigma-1/2}).
$$

By the definition of $t_r(n)$ in (15) and Stirling's approximation,

$$
t_r(n) = s_r^\nu \left\{ 1 + \frac{1}{n} w_r(n) \right\}, w_r(n) = O(1),
$$

(17)
so that we can write

\[ t_{r+1}(n) - t_{r+1}(m) = s_n^{\sigma+1} - s_m^{\sigma+1} + \frac{1}{n} s_n^{\sigma+1} w_{r+1}(n) - \frac{1}{m} s_m^{\sigma+1} w_{r+1}(m). \]  

(18)

Using (16) in (18), we get uniformly for \( m \leq n \leq m + \epsilon \sqrt{m} \), \( 0 < \epsilon < 1 \), as \( m \to \infty \)

\[ t_{r+1}(n) - t_{r+1}(m) = \epsilon O(m^{\sigma-1/2}) + O(m^{\sigma-1}); \]

\[ \lim_{\epsilon \to 0} \lim_{m \to \infty} \max_{m \leq n \leq m + \epsilon \sqrt{m}} \left| t_{r+1}(n) - t_{r+1}(m) \right| m^{-\sigma+1/2} = 0. \]

This and \( t_r(n) \to s(B_{\sigma,\gamma}) \) which is a consequence of the hypothesis \( s_n \to s(B_{\sigma,\gamma}) \) by Lemma 7, together imply \( t_r(n) = O(n^{\sigma-1/2}) \) on appeal to Lemma 4 with \( t_r(n) \), \( \sigma - \frac{1}{2} \) instead of \( s_n \), \( \rho \) respectively and with the \( V_2^0 \) transform specialized to the \( (B_{\sigma,\gamma}) \) transform. Finally, we have also \( s_n^{\sigma+1} = O(n^{\sigma-1/2}) \) because of (17).

The more general conclusion of the lemma follows from successive repetitions \( q \) times of the above argument.

4. Theorems

Theorem 1. Suppose that \( s_n = O(n^\sigma) (n \to \infty) \) for some \( \sigma \geq 0 \). Then \( s_n \to s(V_2^0) \) if and only if

\[ \left( \frac{\sigma}{\pi x} \right)^{1/2} \sum_{n=0}^{\infty} \exp \left[ - ax^{-1} (n - x)^2 \right] s_n \to s(x \to \infty). \]

The proof is exactly like that of Satz II of Faulhaber [2]. Combining Theorem 1 with Lemma 1 we have

Theorem 2. Suppose that \( s_n = O(n^\sigma) (n \to \infty) \) for some \( \sigma \geq 0 \). Then \( s_n \to s(V_2^0) \) if and only if \( s_n \to s(B_{2\sigma,\gamma}) \).

Theorem 3. If \( s_n \) is a real sequence satisfying (13) and \( s_n \to s(V_2^0) \) then \( s_n \to s(n \to \infty) \).

Proof. Since (13) is the special case \( \rho = 0 \) of (1), it follows from Lemma 4 that \( s_n = O(1) \) and thereafter the desired conclusion is obvious from Lemmas 5 and 6.

Theorem 4. If \( s_n \) is a real sequence satisfying (1) and \( s_n \to s(V_2^0) \), then \( s_n \to s(C, 2\rho) \).

Proof. After Theorem 3, we need prove only the case \( \rho > 0 \). Also, \( s_n = O(n^{\rho}) \) by lemma 4, so that we may appeal to Theorem 2 and replace hypothesis \( s_n \to s(V_2^0) \) by \( s_n \to s(B_{2\sigma,\gamma}) \).

If \( \rho \) is the greatest integer less than \( 2\rho + 1 \) then

\[ 0 < \mu = 2\rho - (\rho - 1) \leq 1 \]

and it follows from the more general conclusion of Lemma 8 with \( r = 0 \) that

\[ s_n^{\mu+1} = O(n^\rho - (\rho - 1)/2) = O(n^{\mu/2}). \]
Hence, exactly as elsewhere ([4], proof of Theorem 2, 439-40) we obtain, for 
\( m \leq n \leq m + \varepsilon \sqrt{m}, 0 < \varepsilon < 1, \) uniformly as \( m \to \infty \)
\[
s_{n}^{2p} - s_{m}^{2p} = \varepsilon^{\mu} O(1) + \varepsilon O(m^{(\mu-1)/2}), \quad s_{m}^{2p} = O(m^{\mu/2}). \tag{19}
\]
Further, the definition of \( t_{r}(n) \) in (15) gives, by (17),
\[
t_{2p}(n) - t_{2p}(m) = s_{n}^{2p} - s_{m}^{2p} + \frac{1}{n} s_{n}^{2p} w_{2p}(n) - \frac{1}{m} s_{m}^{2p} w_{2p}(m)
\begin{align*}
&= \varepsilon^{\mu} O(1) + \varepsilon O(m^{(\mu-1)/2}) + O(m^{\mu/2-1})
\end{align*}
on account of (19). Thus
\[
\lim_{\varepsilon \to 0^+} \lim_{m \to \infty} \sup_{m \leq n \leq m + \varepsilon \sqrt{m}} \max_{r \geq s \geq t} |t_{2p}(n) - t_{2p}(m)| = 0,
\]
while our assumption \( s_{n} \to s(B_{2a}, \gamma) \) implies \( t_{2p}(n) \to s(B_{2a}, \gamma) \) by Lemma 7. Now appealing to Theorem 3, with \( s_{n} \) replaced by \( t_{2p}(n) \) and summability \( (V_{b}^{*}) \) by
\[
\text{summability } (B_{2a}, \gamma), \text{ we see that}
\begin{align*}
t_{2p}(n) \to s \quad \text{whence } \quad s_{n}^{2p} \to s
\end{align*}
as required, by (17).

5. Some standard cases of summability method \( (V_{b}^{*}) \)

In addition to the general Borel method \( (B_{a}, \gamma) \) and Borel method which is \( (B_{1}, 1) \),
the following are special cases of the method \( (V_{b}^{*}) \).

I. The Valiron method \( (V_{a}) \) which corresponds to the transforms of \( s_{n} \) given by
\[
V_{a}(x) = \left( \frac{a}{\pi x} \right)^{1/2} \sum_{n=0}^{\infty} \exp \left[ -ax^{-1}(n-x)^{2} \right] s_{n}, \quad x > 0,
\]
is a special case. For, \( V_{a}(x) \) is \( V_{b}^{*}(x) \) of \( \S \ 2 \), with \( g_{b} + R_{b} = 0 \) in condition
(ii) imposed on \( c_{a}(x) \), and with condition (iii) on \( c_{a}(x) \) satisfied in consequence
of a result stated by Faulhaber (Hilfssatz 1 with \( p = 0 \)).

II. The Euler method \( (E_{a}) \), \( p > 0 \), the Hardy-Littlewood method \( (T_{a}) \), \( 0 < a < 1 \),
and the method \( (S_{p}) \), \( 0 < \beta < 1 \), due to Meyer-Konig and Vermes, correspond
to transforms of \( s_{n} \) which may be written
\[
\sum_{n=0}^{\infty} c_{n, m} s_{n}, \quad m = 1, 2, 3, \ldots,
\]
where 
for \( (E_{a}) \) : \( c_{n, m} = 2^{-m} \binom{m}{n} (2^{p} - 1)^{m-n}, \)
for \( (T_{a}) \) : \( c_{n, m} = (1 - a)^{m+1} \binom{n}{m} a^{n-m}, \)
for \( (S_{p}) \) : \( c_{n, m} = (1 - \beta)^{m+1} \binom{m+n}{n} \beta^{n}, \)
with the convention that \( \binom{v}{r} = 0 \) for \( v < r \). In all these cases we may define

\[
c_n(x) = \begin{cases} 
c_{m,n} & \text{for } x_m \leq x < x_{m+1} \\
0 & \text{for } 0 < x < x_1, \end{cases}
\]  

(20)

where \( x_m = \lambda m \) with a suitable constant \( \lambda \) in each case. We can verify, using arguments of Faulhaber (proof of Hilfssatz 5) with trivial modifications, that the \( c_n(x) \) of (20) satisfy for \( x = x_m \) the conditions required of the \( c_n(x) \) in the definition of \( V^\lambda \) in §2, provided we choose

for (\( E_\lambda \)) : \( a = \frac{2^n}{(2^{n+1} - 2)}, \quad \lambda = 2^{-n}, \)

for (\( T_\alpha \)) : \( a = \frac{(1 - \alpha)}{2\alpha}, \quad \lambda = (1 - \alpha)^{-1}, \)

for (\( S_\beta \)) : \( a = \frac{(1 - \beta)}{2}, \quad \lambda = \beta (1 - \beta)^{-1}. \)

Finally we can verify that, if the \( c_n(x) \) of (20) satisfy the required conditions for \( x = x_m \), where \( x_m \) is any increasing unbounded sequence with \( x_{m+1} - x_m = O(1) \) then they satisfy the conditions for all \( x > 0 \). In particular, taking \( x_m = \lambda m \) we see that the transforms \( E_\lambda, T_\alpha, S_\beta \) are all special cases of the \( V^\lambda \) transform.

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**References**