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EOL and ETOL array languages

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Abstract. In this paper we study the families of ETOL and EOL array languages. Standard forms for ETOL and EOL array systems are defined and closure properties of the families are studied. Relations of these families with other developmental array languages and other array languages are studied.

Keywords. EOL array systems ; ETOL array systems ; normal form ; synchronized form ; closure properties ; hierarchy.

1. Introduction

In the theory of formal languages two generation mechanisms among others have been investigated in the past : Chomsky grammars and Lindenmayer-systems, particularly the latter field, has become a research area of enormous interest in the last few years [3, 5, 6].

Renewed interest has been evident in the last twelve years in extending the above two generation mechanisms to two dimensions [1, 2, 7 to 11]. However, all these constructs were such that the graphs or arrays were either allowed to grow only at the edges or surfaces or substitution for nodes were allowed only sequentially. But for biological reasons simultaneous rewriting is required and it should be possible to add new cells in the interior of the array.

In [1] cells divide simultaneously into at most two cells and new structures are defined by orienting the newly formed boundaries according to the configuration of all the neighbouring cells. In [2] there is an extension of parallel rewriting on string of symbols to graph generating systems.

In [4], we have defined a new array model in which parallel rewriting of every symbol in a rectangular array is considered where each symbol is replaced by an array of the same size and where we do not make any distinction between terminals and nonterminals. The language generated by it is the set of all arrays derived from the axiom, including the axiom, where the axiom is a rectangular array.

In this paper we generalize the notion of TOL array systems by adding an extra component called the *target* alphabet which is a subset of the total alphabet of a system and the language generated by it is defined as the set of all arrays over the target alphabet which can be derived from the axiom. It is interesting to note that unlike the string languages [3], the ETOL array languages are not closed under row catenation, row +, column catenation, column +, like the other array languages [9, 10, 11]. ETOL array languages and EOL array languages are closed under pictorial transformations such as reflection about the right most vertical, half turn, reflection about the base, transpose and conjugation.

The paper is divided into five sections. In the next section we define all necessary concepts and investigate some basic properties of ETOLAS and EOLAS. In § 3, we search for sub-classes of the class of ETOLAS which are powerful enough to generate the whole class of ETOLAL. In § 4, we consider the closure and non closure properties of the ETOLAL and EOLAL. It is interesting to observe that ETOLAL and EOLAL are closed only under union and array homomorphism whereas a subfamily DEOLAL is not closed even under these two operations. In § 5, we compare different subfamilies of ETOLAL and compare them with the matrix models and Kolam arrays.

2. Definitions and basic properties

In this section, we review some basic definitions needed for this paper. For the definitions of AFM and array languages the reader is referred to [9, 10].

Notation : Let I be an alphabet, a finite nonempty set of symbols. A matrix (or array) over I is an $m \times n$ rectangular array of symbols from $I\,(m, n \geqslant 1)$. The set of all matrices over I (including the empty matrix $\wedge$) is denoted by I^{**} and $I^{++} = I^{**} - \{\wedge\}$.

Definition 2.1

$$\text{If} \quad X = \begin{matrix} a_{11} \dots a_{1n} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ a_{m1} \dots a_{mn} \end{matrix}, \qquad Y = \begin{matrix} b_{11} \dots b_{1n'} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ b_{m'1} \dots b_{m'n'} \end{matrix}$$

The column catenation $X \oplus Y$ is defined only when $m = m'$ and is given by

$$\begin{matrix} a_{11} \dots a_{1n} b_{11} \dots b_{1n'} \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ a_{m1} \dots a_{mn} b_{m1} \dots b_{mn'} \end{matrix}$$

and the row catenation $X \ominus Y$ is defined only when $n = n'$ and is given by

$$\begin{matrix} a_{11} \dots a_{1n} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ a_{m1} \dots a_{mn} \end{matrix}$$

$b_{11} \ldots b_{1n}$

$\ldots\ldots\ldots$

$\ldots\ldots\ldots$

$b_{m'1} \ldots b_{m'n}$

When there is no ambiguity and the meaning is clear ① or ⊖ is left out.

Definition 2.2. If M and M' are two sets of matrices, the column product $M \mathbin{①} M' = \{X \mathbin{①} Y / X \text{ in } M, Y \text{ in } M'\}$ and the row product $M \ominus M' = \{X \ominus Y / X \text{ in } M, Y \text{ in } M'\}$.

Definition 2.3. An extended OL–array system (EOLAS) is a 4-tuple $G = (V, P, \omega, \Sigma)$ where :

1. V is a finite set (called the *alphabet* of G),
2. $\Sigma \subseteq V$ is a finite set called *terminal* or *target* alphabet.
3. $\omega \in V^{++}$ called *axiom* of G.
4. P is finite set of pairs (a, x) with a in V and x in V^{**} of dimensions $r \times s$ such that for each a in V at least one such pair is in P. The pairs (a, x) are called the rules or productions and are written as $a \underset{P}{\rightarrow} x$ or $a \rightarrow x$.

The elements of $V - \Sigma$ are called *auxiliary symbols* or *nonterminals*. EOLAS is called a *OLAS* if $V = \Sigma$.

The EOLAS is called deterministic if for every $a \in V$ there is exactly one α in V^{**} such that $(a, \alpha) \in P$ and the array system is denoted by DEOLAS. EOLAS is called propagating if there is no rule of the type $a \rightarrow \lambda$ in P.

Definition 2.4. An extended tabled OL-array system (ETOLAS) is a 4-tuple $G = (V, \mathscr{P}, \omega, \Sigma)$ such that V, Σ, ω are same as that in EOLAS. $\mathscr{P}$ consists of a finite set $\{P_1, \ldots, P_f\}$ for some $f \geqslant 1$ and each P_i is a finite subset of $V \times V^{**}$ with the following two conditions :

1. $(\forall_P)_{\mathscr{P}} (\forall_a)_V (\exists \alpha)_{V^{**}} ((a, \alpha) \in P)$,
2. $(\forall \langle a, \alpha \rangle)_P$, α's are of the same size.

ETOLAS is called deterministic if for each P_i and each α in V there is exactly one a in V^{++} such that $(a, \alpha) \in P_i$ and it is denoted by DETOLAS. ETOLAS is called propagating if there is no rule of the type $a \rightarrow \lambda$ in any P in $\mathscr{P}$ and it is denoted by PETOLAS. ETOLAS is called a TOLAS if $V = \Sigma$.

Definition 2.5. Let $G = (V, \mathscr{P}, \omega, \Sigma)$ be an ETOLAS (EOLAS).

$$\text{Let} \quad u = \begin{matrix} a_{11} \ldots a_{1n} \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ a_{m1} \ldots a_{mn} \end{matrix} \quad \text{and } v = \begin{matrix} M_{11} \ldots M_{1n} \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ M_{m1} \ldots M_{mn} \end{matrix}$$

$a_{ij} \in V$, $M_{ij} \in V^{**}$, and M_{ij}, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ are arrays of the same dimension. Then we say $u \Rightarrow v$ iff there exists a table P_i such that $a_{ij} \to M_{ij}$ are in P_i. $\underset{G}{\Rightarrow}{}^*$ denotes the reflexive transitive closure of $\underset{G}{\Rightarrow}$.

Definition 2.6. Let $G = (V, \mathscr{P}, \omega, \Sigma)$ be an ETOLAS (EOLAS). The language generated by G is defined as

$$L(G) = \{x \in \Sigma^{++} / \omega \underset{G}{\Rightarrow} {}^* x\}.$$

A language $L \subseteq \Sigma^{**}$ is called a ETOLAL (EOLAL) iff there exists a ETOLAS (EOLAS) G such that $L = L(G)$. The family of ETOLAL (EOLAL) is denoted by $\mathscr{F}$ ETOLAL ($\mathscr{F}$EOLAL).

Now let us study a few basic properties of ETOLAS (EOLAS) and ETOL array languages (EOL array languages). We just state the first three lemmas as the proofs are simple.

Lemma 2.1. If $G = (V, \mathscr{P}, \omega, \Sigma)$ is an ETOLAS, then $L(G) = L(H) \cap \Sigma^{**}$, where $H = (V, \mathscr{P}, \omega, V)$ is a TOLAS.

Lemma 2.2. If G is an ETOLAS (EOLAS), we can construct an equivalent ETOLAS (EOLAS) $H = (V, \mathscr{P}, \omega, \Sigma)$ such that $\omega \in V - \Sigma$. From now onwards we can take the axiom to be a single letter $S \in V - \Sigma$.

Lemma 2.3. There is an algorithm which for every ETOLAS (EOLAS) G constructs an ETOLAS (EOLAS) H such that $L(H) = L(G) \cup \{\lambda\}$.

Lemma 2.4. There exists an algorithm which for every ETOLAS (EOLAS) G and every coding h produces an ETOLAS (EOLAS) H such that $L(H) = h(L(G))$.

Proof. We shall consider ETOLAS and EOLAS separately.

(i) Let $G = (V, \mathscr{P}, S, \Sigma)$ be an ETOLAS and let h be a coding from Σ into Σ_1 where we can take $V \cap \Sigma_1 = \emptyset$.

Let $H = (V \cup \Sigma_1 \cup \{F\}, \bar{\mathscr{P}} \cup P_0, S, \Sigma_1)$ be an ETOLAS such that $F \notin V \cup \Sigma_1$ and

$$P_0 = \{a \to h(a) : a \in \Sigma \text{ and } h(a) \text{ is defined}\}$$

$$\{a \to F : a \in \Sigma \text{ and } h(a) \text{ is not defined or } a \notin \Sigma\}.$$

$\bar{P}_i = P_i \cup \{a \to M'_{r_i s_i} : a \notin V,\ M'_{r_i s_i}$ is an array of F's of dimension $r_i \times s_i\}$ and $\bar{\mathscr{P}} = \{\bar{P} / P \in \mathscr{P}\}$.

By the construction of H, it is clear that $L(H) = h(L(G))$.

(ii) Let $G = (V, P, S, \Sigma)$ be an EOLAS and let h be a coding from Σ into Σ_1, where $V \cap \Sigma_1 = \emptyset$.

Let $H = (V \cup \Sigma_1 \cup \{F\}, \bar{P}, S, \Sigma_1)$ be an EOLAS such that $F \notin V \cup \Sigma_1$.

Let $\bar{P} = P \cup \{a \to h(a) : a \in \Sigma \text{ and } h(a) \text{ is defined})\} \cup$

$\{a \to F : (a \in \Sigma \text{ and } h(a) \text{ is not defined}) \text{ or } a \notin \Sigma\}$

$\cup \{a \to M'_{r_i s_i}, F \to M'_{r_i s_i}, a \in \Sigma_1\}$,

where $M'_{r_i s_i}$ are arrays of F of dimension $r_i \times s_i$ and for every $a \in V$ we have a rule $\{a \to M_{r_i s_i}$ in $P\}$. Then clearly $L(H) = h(L(G))$.

Lemma 2.5. Every finite matrix language is an EOL array language.

Proof. Let $L = \{M_1, M_2, \ldots, M_k\}$ be a finite matrix language over the alphabet Σ. Let S and F be not in Σ. Now consider a EOLAS $G = (\Sigma \cup \{S, F\}, \text{P}, S, \Sigma)$, where $P = \{S \to M_i : l \leqslant i \leqslant k\} \cup \{a \to M_i' : l \leqslant i \leqslant k,\ a \in \Sigma \cup \{F\}\}$., where M_i' is an array of F of dimension equal to that of M_i. Then clearly $L = L(G)$.

Let us prove the membership problem of OLAS, EOLAS, TOLAS and ETOLAS.

Definition 2.7. For a OLAS $G = (\Sigma, P, \omega)$ and for any non-negative integer n. we define the finite matrix language $L_m(G)$ by induction on n. $L_0(G) = \{\omega\}$, $L_{n+1}(G) = \{y/(\exists z)(z \in L_n(G) \text{ and } z \Rightarrow y, y \in \Sigma^{++})\}$.

Definition 2.8. For any TOLAS $G = (\Sigma, \mathscr{P}, \omega)$, for any non-negative integer n, we define the finite language $L_n(G)$ by induction on n. $L_0(G) = \{\omega\}$, $L_{n+1}(G) = \{y/\text{there exists a } z \text{ in } L_n(G) \text{ and a } P \text{ in } \mathscr{P} \text{ such that } z \underset{P}{\Rightarrow} y, y \in \Sigma^{++}\}$.

Lemma 2.6. The membership problem for OLAS (TOLAS) is solvable.

Let $G = (\Sigma, P, \omega)$ be the given OLAS and let M be the given array. Define for all n, $K_n(G) = \{\alpha/\alpha \in L_m(G) \text{ for some } m \in \{0, 1, \ldots, n\} \text{ and } |\alpha| \leqslant |M|\}$, where $|\alpha|$ denotes the dimension of the array and it is denoted by (r, s). Then $K_{n+1}(G) = K_n(G) \cup \{\alpha/(\exists y)_{K_n(G)}(y \Rightarrow \alpha) \text{ and } |\alpha| \leqslant |M|\}$. Hence we get finite sets $K_0(G)$ $K_1(G), \ldots$. We have finitely many arrays α in Σ^{++} such that $|\alpha| \leqslant |M|$. So we find a r such that $K_m(G) = K_r(G)$ for all $m \geqslant r$. Therefore $M \in L(G)$ iff $M \in \overset{r}{\underset{i=1}{\cup}} K_i(G)$. Hence by the above procedure we decide in a finite number of steps whether or not $M \in L(G)$.

Theorem 2.1. There is an algorithm which, for any EOL (ETOL) array system G and any array M, decides whether or not M is in $L(G)$.

Proof. We give the proof for EOLAS, proof for ETOLAS is analogous.

Let $G = (V, P, S, \Sigma)$ be an EOLAS. Let $\bar{G} = (V, P, S)$ be a OLAS. Then M is in $L(G)$ iff M is in $L(\bar{G})$ and M is in Σ^{++}. But it is trivial to decide whether or not M is in Σ^{++}. Hence the theorem follows from Lemma 2.6.

3. Normal forms of ETOLAS and EOLAS

In this section we search for subclasses of the class of ETOLAS (EOLAS) which are powerful enough to generate the entire class of ETOLAL (EOLAL).

Definition 3.1. An ETOLAS $G = (V, \mathcal{P}, S, \Sigma)$ is said to be in normal form if

1. $S \in V - \Sigma$ and S will not appear on the right side of any production in any table.

2. There exists a unique table P_I (called the initial table) such that $S \underset{P_I}{\rightarrow} \alpha$ for some $\alpha \in (V - \Sigma)^{++}$, $\alpha \neq S$ and if $\alpha \neq S$, the only production for a in P_I is a M'_{r_s} where M'_{r_s} is an array of F of dimension equal to that of α. (F is a distinguished symbol of $V - \Sigma$ called the rejection symbol. An array of F's is called the rejection array).

3. There exists a unique table P_T (called the terminal table) different from P_I such that if $a \in V - \Sigma - \{S, F\}$ and $a \rightarrow \alpha$ is in P_T then $\alpha \in \Sigma \cup \{F\}$ and if $a \in \Sigma$ and $a \rightarrow \alpha$ is in P_T, then $\alpha = a$. If $a \in \{S, F\}$ then $a \rightarrow \alpha$ implies $\alpha = F$.

4. For all other tables $P_i \neq P_I, P_T$, $l \leqslant i \leqslant n$, $a \rightarrow \alpha$ is in P_i for all $a \in \Sigma$ then $\alpha = M'_{r_i s_i}$ where $M'_{r_i s_i}$ is the rejection array. If $a \rightarrow \alpha$ is in P_i and $a \in V - \Sigma - \{S, F\}$ then $\alpha \in (V - \Sigma)^{++}$. If $a \rightarrow \alpha$ is in P_i and $a \in \{S, F\}$ then $\alpha = M'_{r_i s_i}$. The only production with F on the left side in every table is of the type $F \rightarrow M'_{r_i s_i}$, where $M'_{r_i s_i}$ is the rejection array.

Theorem 3.1. There exists an algorithm which for every ETOLAS G produces an equivalent ETOLAS H which is in the normal form.

Proof. Let $G = (V, \mathcal{P}, \omega, \Sigma)$ be an ETOLAS. Let $G' = (V \cup \{S, F\}, \mathcal{P}', S, \Sigma)$ be an ETOLAS where $\{S, F\} \notin V$, and $\mathcal{P}' = P_0 \cup \{P'_i ; P_i \in \mathcal{P}\}$, where $\# \mathcal{P} = n$. Let $P_0 = \{S \rightarrow \omega, a \rightarrow M'_{r_0 s_0} : a \in V \cup \{F\}\}$, $P'_i = P_i \cup \{S \rightarrow M'_{r_i s_i}, F \rightarrow M'_{r_i s_i}\}$, $l \leqslant i \leqslant n$, where $M'_{r_i s_i}$ are rejection arrays. It is clear that $L(G) = L(G')$. Let $V_1 = \{(\bar{a} / a \in V \cup \{S\}) \cup \{F\}\}$, $P_1 = \{\bar{S} \rightarrow \bar{\omega} : S \rightarrow \omega \in P_0\} \cup \{a \rightarrow M'_{r_0 s_0} : a \in (V_1 - \{\bar{S}\}) \cup \Sigma\}$. $\bar{P}'_i = \{\bar{a} \rightarrow \bar{\alpha} : a \rightarrow \alpha \in P_i\} \cup \{\bar{S} \rightarrow M'_{r_i s_i}, F \rightarrow M'_{r_i s_i}\} \cup \{a \rightarrow M'_{r_i s_i} : a \in \Sigma\}$, $l \leqslant i \leqslant n$, and $P_T = \{\bar{a} \rightarrow a : \bar{a} \in \Sigma\} \cup \{a \rightarrow a : a \in \Sigma\} \cup \{a \rightarrow F : a \notin \bar{\Sigma} \cup \Sigma\}$. Then $H = (V_1 \cup \Sigma, \bar{\mathcal{P}}, \bar{S}, \Sigma)$ is an ETOLAS which satisfies all the requirements for H to be normal where $\bar{\mathcal{P}} = \{P_I, P_T\} \cup \{\bar{P}'_i : P_i \in \mathcal{P}\}$. It follows from construction that $L(G) = L(H)$.

After constructing the normal form of an ETOLAS, we now construct a synchronized ETOLAS and synchronized EOLAS.

Definition 3.2. An ETOLAS (EOLAS) $G = (V, \mathcal{P}, S, \Sigma)$ ($G = (V, P, S, \Sigma)$) is said to be synchronized iff it has the following property : For any $M_1, M_2 \in V^{++}$ such that $M_1 \overset{*}{\Rightarrow} M_2$ and $M_1 \neq M_2$, if $M_1 \notin (V - \Sigma)^{++}$ then $M_2 \notin \Sigma^{++}$ that is, if an array contains an element of Σ, then all other arrays derivable from it contain some element not in Σ.

Theorem 3.2. For every ETOLAS there exists an equivalent synchronized ETOLAS.

Proof. Let $G = (V, \mathcal{P}, S, \Sigma)$ be any ETOLAS. By theorem 3.1 we may assume that G is in the normal form. In the terminal table P_T of G replace each production of the type $a \to a$ where $a \in \Sigma$, by the production $a \to F$, where F is the rejection symbol. Then the system so obtained is the synchronized ETOL array system. Hence the theorem.

Theorem 3.3. There is an algorithm which for any EOLAS G produces a synchronized EOLAS $\bar{G}$ such that $L(G) = L(\bar{G})$.

Proof. Let $G = (V, P, \omega, \Sigma)$ be an EOLAS. Let $\bar{\Sigma} = \{\bar{a}/a \in \Sigma\}$, $(V - \Sigma) \cap \bar{\Sigma} = \emptyset$. If $a \neq b$ then $\bar{a} \neq \bar{b}$ in $\bar{\Sigma}$. Let $\bar{V} = V \cup \bar{\Sigma} \cup \{F, S\}$ where $S, F \notin V - \bar{\Sigma}$. If α is in V^{++}, we associate the array $\bar{\alpha}$ in $(\bar{V})^{++}$ as follows :

$$\text{If} \quad \alpha = \begin{matrix} \alpha_{11} \cdots \alpha_{1n} \\ \cdots\cdots\cdots \\ \alpha_{m1} \cdots \alpha_{mn} \end{matrix} \quad \text{then} \quad \bar{\alpha} = \begin{matrix} \beta_{11} \cdots \beta_{1n} \\ \cdots\cdots\cdots \\ \beta_{m1} \cdots \beta_{mn} \end{matrix} \quad \text{where}$$

$\beta_{ij} = \bar{a}_{ij}$ if a_{ij} is in Σ and $\beta_{ij} = a_{ij}$ if a_{ij} is not in Σ. Let $\bar{G} = (\bar{V}, \bar{P}, S, \Sigma)$ be an EOLAS where $P = \{a \to \bar{\alpha}/a \in V - \Sigma,\ a \to \alpha \text{ in } P\} \cup \{a \to \alpha/a \in V - \Sigma,\ a \to \alpha \text{ in } P\} \cup \{\bar{a} \to \bar{\alpha}/a \in \Sigma,\ a \to \alpha \text{ in } P\} \cup \{\bar{a} \to \alpha/a \in \Sigma,\ a \to \alpha \text{ in } P\} \cup \{a \to \alpha'/a \in \Sigma \cup \{F\},\ a \to \alpha$ in P where α' is the rejection array where dimension is equal to that of $\alpha\} \cup \{S \to \omega,\ S \to \bar{\omega}\} \cup \{a \to \alpha'/a \in \bar{V} - \{S\},\ \alpha'$ is the rejection array where $|\alpha| = |\omega|\}$. Then clearly $\bar{G}$ is synchronized EOLAS and $L(G) = L(\bar{G})$.

4. Closure properties of the class of ETOLAL and EOLAL

In this section we consider a number of operations like union, row catenation, column catenation, row +, column +, array homomorphism, complementation, on the class of ETOLAL and EOLAL. In string languages the family of EOL languages and the family of ETOL languages form an abstract family of languages [3]. But in array languages, the class of ETOLAL and the class of EOLAL are not closed under row catenation, row + , column catenation, column +.

Like the other known family of array languages [4, 9, 10], the family of ETOLAL and family of EOLAL are closed under quarter-turn, transpose, half turn, reflection about the rightmost vertical, reflection about the base and conjugation. For basic definitions refer [10].

Theorem 4.1. The family of ETOLAL (EOLAL) is closed under union and array homomorphism.

Proof. (i) *Union.* Let $G_1 = (V_1, \mathscr{P}_1, S_1, \Sigma_1)$ and $G_2 = (V_2, \mathscr{P}'_1, S_2, \Sigma_2)$ be any two ETOL array systems. Without loss of generality let us assume G_1 and G_2 to be in normal form and that $(V_1 - (\Sigma_1 \cap \Sigma_2) - F) \cap (V_2 - (\Sigma_1 \cap \Sigma_2) - F) = \emptyset$. Let S be a new symbol such that $S \notin V_1 \cup V_2$. Let $P_0 = \{S \to S_1, S \to S_2\} \cup \{a \to F/a \in V_1 \cup V_2\}$. If $P_j \in \mathscr{P}_1$, then $\bar{P}_j = P_j \cup \{a \to M'/a \in V_2 \cup \{S\}\}$. If $P'_j \in \mathscr{P}'_1$, then $\bar{P}'_j = \{a \to M''/a \in V_1 \cup \{S\}\} \cup P'_j$, where $M'(M'')$ are rejection arrays whose dimensions equal the dimension of the right side of P_j (P'_j).

Let $G_3 = (V_1 \cup V_2 \cup \{S\}, \mathscr{P}_2, S, \Sigma_1 \cup \Sigma_2)$ be an ETOLAS where $\mathscr{P}_2 = P_0 \cup \{\bar{P}_j / P_j \in \mathscr{P}_1\} \cup \{\bar{P}'_j / P'_j \in \mathscr{P}'_1\}$. Then clearly $L(G_3) = L(G_1) \cup L(G_2)$;

(ii) *Array homomorphism*. Let $G = (V, \mathscr{P}, S, \Sigma)$ be an ETOLAS. Let $\Sigma = \{a_1, a_2, \ldots, a_n\}$. Define an array homomorphism H from Σ into $\triangle$ as :

$$H(a) = \begin{matrix} b_{11} \ldots b_{1n} \\ \ldots \\ b_{m1} \ldots b_{mn} \end{matrix} \in \triangle^{++} \text{ and } a \in \Sigma. \text{ Let } P_0 = \{a \to H(a) / a \in \Sigma\},$$

$h(a) \in \triangle^{++}\} \cup \{b \to M'_{mn} / b \in \triangle \cup \{F\} \cup (V - \Sigma),\ |M'_{mn}| = |H(a)|$ and M'_{mn} is the rejection array$\}$. Let $\bar{P}_{r_j} = P_{r_j} \cup \{b \to M' / b \in \triangle \cup \{F\},\ M' \in \{F\}^{++},\ |M'|$ is equal to the dimension of the right side of the rules of $P_{r_j}\}$. Let $G_1 = (V \cup \triangle \cup \{F\}, \bar{\mathscr{P}}_r, S, \triangle)$ be an ETOLAS where $\bar{\mathscr{P}}_r = P_0 \cup \{\bar{P}_{r_j} / P_{r_j} \in \mathscr{P}_r\}$. Then clearly $L(G_1) = H(L(G))$.

Proof for showing that the $\mathscr{F}$EOLAL is closed under union and homomorphism is analogous.

Theorem 4.2. The family of ETOLAL (EOLAL) is not closed under row catenation, column catenation, row +, column +, complementation.

Proof. (i) *Row Catenation* : Let $G_1 = (V_1 P_1, S_1, \Sigma_1)$ and $G_2 = (V_2, P_2, S_2, \Sigma_2)$ be two ETOLAS, where

$$P_1 = \{S_1 \to S_1 S_1,\ S_1 \to aa,\ a \to aa\} \text{ and}$$

$$P_2 = \left\{S_2 \to \begin{matrix} S_2 S_2 \\ S_2 S_2 \end{matrix},\ S_2 \to \begin{matrix} bb \\ bb \end{matrix},\ b \to \begin{matrix} bb \\ bb \end{matrix}\right\}$$

Then it is clear that

$$\begin{matrix} L(G_1) \\ L(G_2) \end{matrix} = \left\{ \begin{matrix} aa \\ bb \\ bb \end{matrix},\ \begin{matrix} aaaa \\ bbbb \\ bbbb \\ bbbb \end{matrix},\ \ldots \right\}$$

cannot be generated by any ETOLAS.

(ii) *Column catenations* : Let $G_3 = (V_3, P_3, S_3, \Sigma_3)$ be an ETOLAS, where

$$P_3 = \left\{S_3 \to \begin{matrix} S_3 \\ S_3 \end{matrix},\ S_3 \to \begin{matrix} a \\ a \end{matrix},\ a \to \begin{matrix} a \\ a \end{matrix}\right\}.$$

Then it is clear that

$$L(G_3) \cdot L(G_2) = \left\{ \begin{matrix} abb \\ abb \end{matrix},\ \begin{matrix} abbbb \\ abbbb \\ abbbb \\ abbbb \end{matrix},\ \ldots \right\}$$

cannot be generated by any ETOLAS.

(iii) *Row +* : We can show that $(L(G_3))_+$ cannot be generated by any ETOLAS.

(iv) *Column* + : We can show that $(L(G_1))^+$ cannot be generated by any ETOLAS.

(v) *Complementation* : It is clear that the complement of $L(G_1)$ cannot be generated by any ETOLAS. Hence the theorem.

Theorem 4.3. The family of ETOLAL (EOLAL) is closed under quarter-turn, transpose, half-turn, reflection about the rightmost vertical, reflection about the base and conjugation.

Proof. Let $G = (V, \mathcal{P}, \omega, \Sigma)$ be a ETOLAS. Consider a ETOLAS $G_1 = (V, \mathcal{P}_1, \omega_1, \Sigma)$ where $\omega_1 = T(\omega)$, $T(A)$ denotes transpose of A. $\mathcal{P}_1 = \{P_1/P \in \mathcal{P}\}$ and $P_1 = \{a \to T(\alpha)/a \to \alpha \text{ in } P\}$. Then clearly $L(G_1) = T(L(G))$.

The proof for the other operations is analogous.

We observe that the family of EOLAL is closed under union and homomorphism, but it is of interest to note that a subfamily DEOLAL is not closed under these two operations. Before proving the nonclosure of $\mathcal{F}$ DEOLAL under union and array homomorphism, let us prove the following two lemmas.

Lemma 4.1. Let $a, b \in \Sigma$ and $a \neq b$. (i) If L is a DEOLAL then $\left\{\begin{matrix} aa \\ aa \end{matrix}, \begin{matrix} ab \\ ab \end{matrix}, \begin{matrix} ba \\ ba \end{matrix}\right\}$ is not a subset of L; (ii) $\left\{a, \begin{matrix} aa \\ aa \end{matrix}\right\}$ is not a DEOLAL.

Proof. Let $G = (V, P, S, \Sigma)$ be a DEOLAS. Define a mapping μ from V^{++} into V^{++} as follows : $\mu^\circ(a) = a$, $\mu^1(a) = x$ if $a \underset{G}{\Rightarrow} x$. For $n \geq 1$, $\mu^{n+1}(\alpha) = \mu^1(\mu^n(\alpha))$. We also define for $n \geq 0$, $m \geq 0$, $\mu^{n+m}(\alpha) = \mu^n(\alpha).\ \mu^m(\alpha)$, $\mu^n(\alpha\beta) = \mu^n(\alpha) \cdot \mu^n(\beta)$,

$$\mu^n\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\mu^n(\alpha)}{\mu^n(\beta)} = \mu^n(\alpha) \ominus \mu^n(\beta)$$

where $\alpha, \beta \in V^{++}$. Let us assume that

$$\left\{\begin{matrix} aa \\ aa \end{matrix}, \begin{matrix} ab \\ ab \end{matrix}, \begin{matrix} ba \\ ba \end{matrix}\right\} \subseteq L(G),$$

i.e. we find i, j and k such that

$$S \overset{i}{\Rightarrow} \begin{matrix} aa \\ aa \end{matrix} \text{ i.e., } \mu^i(S) = \begin{matrix} aa \\ aa \end{matrix},\ S \overset{j}{\Rightarrow} \begin{matrix} ab \\ ab \end{matrix} \text{ i.e., } \mu^j(S) = \begin{matrix} ab \\ ab \end{matrix},$$

$$S \overset{k}{\Rightarrow} \begin{matrix} ba \\ ba \end{matrix} \text{ i.e. } \mu^k(S) = \begin{matrix} ba \\ ba \end{matrix}.$$

Let us show that $i > j$. On the contrary let us take $i < j$ then

$$\begin{matrix} ab \\ ab \end{matrix} = \mu^j(S) = \mu^{j-i}(\mu^i(S)) = \frac{\mu^{j-i}(a) \cdot \mu^{j-i}(a)}{\mu^{j-i}(a) \cdot \mu^{j-i}(a)}.$$

Hence

$$a = \mu^{j-i}(a) \text{ and } b = \mu^{j-i}(a)$$

that is $a \overset{j-i}{\Longrightarrow} a$ and $a \overset{j-i}{\Longrightarrow} b$. This is not possible as the system is deterministic. Hence $i > j$. Similarly we show $i > k$. Next let us assume that $j > k$.

$$\begin{matrix} ab \\ ab \end{matrix} = \mu^j(S) = \mu^{j-k}(\mu^k(S)) = \mu^{j-k}\begin{pmatrix} ba \\ ba \end{pmatrix} = \begin{matrix} \mu^{j-k}(b) \cdot \mu^{j-k}(a) \\ \mu^{j-k}(b) \cdot \mu^{j-k}(a) \end{matrix}.$$

Hence $a = \mu^{j-k}(b)$ and $b = \mu^{j-k}(a)$ that is $b \overset{j-k}{\Longrightarrow} a$ and $a \overset{j-k}{\Longrightarrow} b$. Consider

$$\begin{matrix} aa \\ aa \end{matrix} = \mu^i(S) = \mu^{i-k}(\mu^k(S)) = \mu^{i-k}\begin{pmatrix} ba \\ ba \end{pmatrix} = \begin{matrix} \mu^{i-k}(b) \cdot \mu^{i-k}(a) \\ \mu^{i-k}(b) \cdot \mu^{i-k}(a) \end{matrix}.$$

Therefore $a = \mu^{i-k}(b)$ and $a = \mu^{i-k}(a)$. Hence $b \overset{i-k}{\Longrightarrow} a$ and $a \overset{i-k}{\Longrightarrow} a$.

Consider

$$\begin{matrix} aa \\ aa \end{matrix} = \begin{matrix} \mu^{i-k}(a) \cdot \mu^{i-k}(b) \\ \mu^{i-k}(a) \cdot \mu^{i-k}(b) \end{matrix} = \begin{matrix} \mu^{i-k}(ab) \\ \mu^{i-k}(ab) \end{matrix} = \mu^{i-k}\begin{pmatrix} ab \\ ab \end{pmatrix} = \mu^{i-k}(\mu^j(S))$$

$$= \mu^{i-k+j}(S) = \mu^{j-k}(\mu^i(S)) = \mu^{j-k}\begin{pmatrix} aa \\ aa \end{pmatrix} = \begin{matrix} \mu^{j-k}(a) \cdot \mu^{j-k}(a) \\ \mu^{j-k}(a) \cdot \mu^{j-k}(a) \end{matrix}.$$

Therefore $a = \mu^{j-k}(a)$ or $a \overset{j-k}{\Longrightarrow} a$. But we have already proved that $a \overset{j-k}{\Longrightarrow} b$. Hence a contradiction. Even if we take $j < k$, by a similar argument we arrive at a contradiction. Hence our assumption that $\left\{\begin{matrix} aa \\ aa \end{matrix}, \begin{matrix} ab \\ ab \end{matrix}, \begin{matrix} ba \\ ba \end{matrix}\right\} \subseteq L(G)$ is wrong. So, $\left\{\begin{matrix} aa \\ aa \end{matrix}, \begin{matrix} ab \\ ab \end{matrix}, \begin{matrix} ba \\ ba \end{matrix}\right\}$ is not a subset of DEOLAL;

(ii) Let $G = (V, P, S, \Sigma)$ be a DEOLAS, so that $\left\{a, \begin{matrix} aa \\ aa \end{matrix}\right\} \subseteq L(G)$. Then $S \overset{i}{\Rightarrow} a$ and $S \overset{j}{\Rightarrow} \begin{matrix} aa \\ aa \end{matrix}$, i.e., $\mu^i(S) = a$ and $\mu^j(S) = \begin{matrix} aa \\ aa \end{matrix}$. If $j < i$ then $a = \mu^i(S)$ $= \mu^{i-j}(\mu^j(S)) = \mu^{i-j}\begin{pmatrix} aa \\ aa \end{pmatrix}$ which is not possible. Hence $j > i$. Now $\begin{matrix} aa \\ aa \end{matrix} = \mu^j(S)$ $= \mu^{j-i}(\mu^i(S)) = \mu^{j-i}(a)$. Hence $a \overset{j-i}{\Longrightarrow} \begin{matrix} aa \\ aa \end{matrix}$.

$$\text{Hence} \quad \begin{matrix} aaaa \\ aaaa \\ aaaa \\ aaaa \end{matrix} = \begin{matrix} \mu^{j-i}(a) \cdot \mu^{j-i}(a) \\ \mu^{j-i}(a) \cdot \mu^{j-i}(a) \end{matrix} = \begin{matrix} \mu^{j-i}(aa) \\ \mu^{j-i}(aa) \end{matrix} = \mu^{j-i}\begin{pmatrix} aa \\ aa \end{pmatrix} = \mu^{j-i}(\mu^j(S)) = \mu^{2j-i}(S).$$

Therefore $S \overset{2j-i}{\Longrightarrow} (aaaa)_4$. Hence $(aaaa)_4 \notin L(G)$. So $\left\{a, \begin{matrix} aa \\ aa \end{matrix}\right\}$ is not a DEOLAL.

Lemma 4.2. Let $a, b \in \Sigma$ and $a \neq b$. If L is a DEOLAL then $\left\{\begin{matrix} aa \\ aa \end{matrix}, \begin{matrix} aa \\ bb \end{matrix}, \begin{matrix} bb \\ aa \end{matrix}\right\}$ is not a subset of L.

Proof. Proof is similar to lemma 4.1.

Theorem 4.4. DEOLAL is not closed under union and array homomorphism.

Proof. (i) *Union*: Let $G_1 = \left(V, P_1, \begin{smallmatrix} ab \\ ab \end{smallmatrix}, V\right)$ and $G_2 = \left(\{a\}, P_2, \begin{smallmatrix} aa \\ aa \end{smallmatrix}, \{a\}\right)$ be two DEOLASs where $P_1 = \{a \to b, b \to a\}$ and $P_2 = \{a \to a\}$. Then $L(G_1) = \left\{\begin{smallmatrix} ab \\ ab \end{smallmatrix}, \begin{smallmatrix} ba \\ ba \end{smallmatrix}\right\}$ and $L(G_2) = \left\{\begin{smallmatrix} aa \\ aa \end{smallmatrix}\right\}$. But $L(G_1) \cup L(G_2) = \left\{\begin{smallmatrix} ab \\ ab \end{smallmatrix}, \begin{smallmatrix} ba \\ ba \end{smallmatrix}, \begin{smallmatrix} aa \\ aa \end{smallmatrix}\right\}$ is not a DEOLAL follows from lemma 4.1.

(ii) *Array homomorphism*: Let $G_3 = (V, P_3, cd, V)$ be a DEOLAS where $P_3 = \{c \to a, \ d \to b, \ a \to b, \ b \to a\}$. Then $L(G_3) = \{cd, ab, ba\}$. Define a homomorphism h as $h(a) = h(c) = h(d) = \begin{smallmatrix} a \\ a \end{smallmatrix}$, $h(b) = \begin{smallmatrix} b \\ b \end{smallmatrix}$. Then

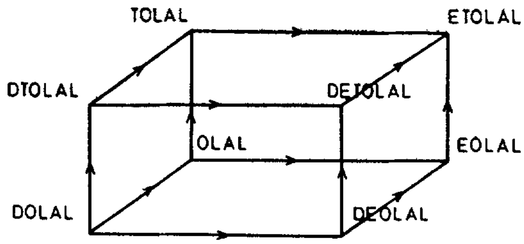
$$h(L(G_3)) = \left\{\begin{matrix} aa \\ aa \end{matrix}, \begin{matrix} ab \\ ab \end{matrix}, \begin{matrix} ba \\ ba \end{matrix}\right\}$$

which is not a DEOLAL follows from lemma 4.2.

5. Inter-relations among some subfamilies of ETOLAL

In this section, we investigate the hierarchy between the subfamilies of ETOLAL.

Theorem 5.1. The following diagram holds.



A solid line denotes strict inclusion in the direction indicated. If two families $\mathscr{F}_1$ and $\mathscr{F}_2$ are not connected by a path following the arrows in the diagram, then the families are incomparable but not disjoint.

Proof. Inclusions follow from definitions. For proper inclusions, refer to the examples in table 1. As the proof techniques are similar, we just prove that Example 3 in the table is an ETOLAL which cannot be generated by any DETOLAS.

Let $G = \left(\{a, b, c, d\}, \{P_1, P_2\}, \begin{smallmatrix} aa \\ bb \end{smallmatrix}, \{a, b, c\}\right)$ be an ETOLAS, where $P_1 = \{a \to aa, b \to cc, c \to dd, b \to bb, d \to cc\}$, $P_2 = \{a \to ab, b \to ab, c \to dd, d \to cc\}$. Then

$$L(G) = \left\{\begin{matrix} aa \\ bb \end{matrix}, \begin{matrix} aaaa \\ bbbb \end{matrix}, \begin{matrix} aaaa \\ bbcc \end{matrix}, \begin{matrix} aaaa \\ ccbb \end{matrix}, \begin{matrix} aaaa \\ cccc \end{matrix}, \begin{matrix} aaaaaaaa \\ ccbbccbb \end{matrix}, \ldots\right\} = L.$$

Table 1.

Sl. No.	V	$\mathscr{P}$	Σ	ω	L in the family $\mathscr{F}_{a}$	L not in the family $\mathscr{F}_{a}$
1.	$\{a, b, c, d, e\}$	$\left\{a \to \substack{a\\a},\ b \to \substack{b\\b},\ c \to \substack{d\\d},\ d \to \substack{e\\e},\ e \to \substack{e\\e}\right\}$	$\{a, b, c, d\}$	abc	DEOLAL	DOLAL, OLAL, DTOLAL, TOLAL
2.	$\{S, F, a\}$	$\left\{S \to a,\ a \to F,\ F \to F,\ S \to \substack{aa\\aa},\ a \to \substack{FF\\FF},\ F \to \substack{FF\\FF}\right\}$	$\{a\}$	S	EOLAL	DEOLAL, OLAL, TOLAL
3.	$\{a, b, c, d\}$	$P_1 = (a \to aa, b \to cc,$ $c \to dd,\ b \to bb,\ d \to ee)$ $P_2 = (a \to ab,\ b \to ab,$ $c \to dd,\ d \to cc)$	$\{a, b, c\}$	aa bb	ETOLAL	EOLAL, DETOLAL, TOLAL
4.	$\{a, b, c, d\}$	$P_1 = (a \to aa,\ b \to cc,\ c \to dd,\ d \to dd)$ $P_2 = (a \to ab,\ b \to bc,\ c \to dd,\ d \to dd)$	$\{a, b, c\}$	abb	DETOLAL	EOLAL, TOLAL, DTOLAL, DEOLAL

If possible let there be a DETOLAS $G' = (V, \mathscr{P}, S, \{a, b, c\})$ so that $L(G') = L$. Without loss of generality, let us assume G' to be in the normal form. As $\begin{matrix} aa \\ bb \end{matrix}$ is the smallest array of L, we have $S \overset{*}{\Rightarrow} \begin{matrix} \bar{a}\,\bar{a} \\ \bar{b}\,\bar{b} \end{matrix} \underset{P_T}{\Rightarrow} \begin{matrix} aa \\ bb \end{matrix}$. Now $\begin{matrix} \bar{a}\,\bar{a}\,\bar{a}\,\bar{a} \\ \bar{c}\,\bar{c}\,\bar{b}\,\bar{b} \end{matrix}$ and $\begin{matrix} \bar{a}\,\bar{a}\,\bar{a}\,\bar{a} \\ \bar{b}\,\bar{b}\,\bar{c}\,\bar{c} \end{matrix}$ are not derivable from $\begin{matrix} \bar{a}\,\bar{a} \\ \bar{b}\,\bar{b} \end{matrix}$. Hence we have derivations

$$S \overset{*}{\Rightarrow} \begin{matrix} \bar{a}\,\bar{a}\,\bar{a}\,\bar{a} \\ \bar{c}\,\bar{c}\,\bar{b}\,\bar{b} \end{matrix} \underset{P_T}{\Rightarrow} \begin{matrix} aaaa \\ ccbb \end{matrix} \quad \text{and} \quad S \overset{*}{\Rightarrow} \begin{matrix} \bar{a}\,\bar{a}\,\bar{a}\,\bar{a} \\ \bar{b}\,\bar{b}\,\bar{c}\,\bar{c} \end{matrix} \underset{P_T}{\Rightarrow} \begin{matrix} aaaa \\ bbcc \end{matrix}.$$

where $V \supset \{\bar{a}, \bar{b}, \bar{c}\}$. If $S \underset{G'}{\overset{*}{\Rightarrow}} M \underset{G'}{\overset{*}{\Rightarrow}} \begin{matrix} aaaaaaaa \\ ccbbccbb \end{matrix}$, then

$$M \notin \left\{ \begin{matrix} \bar{a}\,\bar{a}\,\bar{a}\,\bar{a} \\ \bar{b}\,\bar{b}\,\bar{b}\,\bar{b} \end{matrix},\ \begin{matrix} \bar{a}\,\bar{a}\,\bar{a}\,\bar{a} \\ \bar{b}\,\bar{b}\,\bar{c}\,\bar{c} \end{matrix},\ \begin{matrix} \bar{a}\,\bar{a}\,\bar{a}\,\bar{a} \\ \bar{c}\,\bar{c}\,\bar{b}\,\bar{b} \end{matrix},\ \begin{matrix} \bar{a}\,\bar{a}\,\bar{a}\,\bar{a} \\ \bar{c}\,\bar{c}\,\bar{c}\,\bar{c} \end{matrix} \right\}.$$

So we have to introduce one more table in $\mathscr{P}$. Proceeding in this way to generate the words of L, we require an infinite number of tables, a contradiction. Hence L is not a DETOLAL. L is also not an EOLAL or a TOLAL.

Theorem 5.2. $\mathscr{F}_{\text{FML}} \subsetneq \mathscr{F}_{\text{EOLAL}}$.

Proof. Inclusion follows from lemma 2.5. For proper inclusion consider an EOL array system.

$$G = \left(\{S, A, a\},\ \left\{ S \to \begin{matrix} AA \\ AA \end{matrix},\ A \to \begin{matrix} aa \\ aa \end{matrix},\ a \to \begin{matrix} aa \\ aa \end{matrix} \right\},\ S, \{a\} \right).$$

Then $L(G)$ is not a FML.

In any infinite ETOLAL (EOLAL) the length and breadth of the array increase exponentially and not linearly. Whereas in the case of RML, $(R : X)\, A\, L\, (X = R, CF, CS)$ of [9, 10] the length or the breadth of the array or both increase linearly. Hence we have the following theorem.

Theorem 5.3. ETOLAL (EOLAL) is incomparable but not disjoint with RML and $(R : X)\, A\, L$.

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