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Polarisations on an abelian variety

By

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1. Statement of the theorem and preliminaries

We shall prove the following.

Theorem 1.1. Let X be an abelian variety over an algebraically closed field k and let Aut (X) denote the automorphism group of X. If, for any integer $d \neq 0$, P_d denotes the subset of the Neron-Severi group of X consisting of elements of degree d, then there are only finitely many orbits under the natural action of Aut (X) on P_d.

For the definition of degree, see § 1.6.

In particular, if P denotes the set of principal polarisations on X, there are only finitely many orbits under the natural action of Aut (X) on P. This result was conjectured in the case $k = \mathbf{C}$, by Martens ([3], p. 47).

On applying Torelli's theorem we have

Corollary 1.2. There are, upto isomorphism, only finitely many smooth irreducible curves over k, having a given abelian variety as the Jacobian.

In the proof of theorem 1.1, we will use

Theorem 1.3. (Borel and Harish-Chandra) Let G be a reductive algebraic group defined over $\mathbf{Q}$ and let Γ be an arithmetic subgroup of G [2]. Let $\rho : G \to GL(V)$ be a rational representation defined over $\mathbf{Q}$ and L a lattice of V invariant under Γ. If O is a closed orbit of G, then $O \cap V$ consists of a finite number of orbits of Γ.

For a proof see ([1], theorem in § 9.11) or ([2], theorem 6.9).

1.4. *Aut (X) as an arithmetic group.* Let B denote the ring of endomorphisms of X and let $B_K = B \otimes K$ for any field K containing $\mathbf{Q}$. Then $B_{\mathbf{Q}}$ is a finite dimensional semi-simple algebra which contains B as a lattice ([4], theorem 3 on p. 176; also p. 178). Moreover there is a homogeneous polynomial function, with rational coefficients, ϕ defined over B such that $\phi(ab) = \phi(a)\phi(b)$ for $a, b \in B$ and $\phi(a) = \deg a$, for $a \in B$, where deg a denotes the degree of the endomorphism a ([4], theorem 2, p. 174; also p. 63).

Lemma 1.5. Let $\alpha \in B_K$, with $\phi(\alpha) \neq 0$. Then α is a unit of B_K.

Proof. For $\alpha \in B$, $\phi(\alpha) = \deg \alpha = n \neq 0$, there is an element β of B such that $\alpha\beta = \beta\alpha = n_X$ ([4], p. 169) proving that α is a unit of $B_{\mathbf{Q}}$.

Because ϕ is homogeneous, this proves the lemma for $K = Q$. The general case follows from this. In fact, let $B = \Pi S_i$ where S_i are simple algebras over

Q. It then follows that $\phi = \Pi N_i^{k_i}$ with $k_i > 0$ where N_i is the reduced norm in S_i. Since the determinant of the regular representation of $S_i \otimes_{\mathbf{Q}} K$ is a positive power of $N_{i,K}$ (the extension of N_i to $S_i \otimes_{\mathbf{Q}} K$), we see that an element a of $S_i \otimes K$ is a unit if and only if $N_{i,k}^{k_i}(a) \neq 0$. Thus a is a unit if and only if $\phi(a) \neq 0$.

Putting

$$G_K = \{a \in B_K \mid \phi(a) = 1\}$$

for all fields K containing **Q**, one gets a reductive algebraic group G defined over **Q**, as $B_{\mathbf{Q}}$ is a semi-simple algebra. Moreover G contains

$$\text{Aut }(X) = \{a \in B \mid a \text{ is unit}\} = \{a \in B \mid \phi(a) = \deg a = 1\}$$

as an arithmetic group.

1.6. *Rosati involution and the action of Aut (X) on the Neron-Severi group*

In this section, we construct a representation of G defined over **Q**, which when restricted to the arithmetic subgroup Aut (X) becomes the natural representation of Aut (X) on the Neron-Severi group.

Let $\quad NS(X) = \text{Pic } X/\text{Pic}^\circ X$

denote the Neron-Severi group of X. Let $n \in NS(X)$ and L a line bundle representing n. The homomorphism

$$\phi_L : X \to \text{Pic}^\circ X = \hat{X} \text{ (the dual abelian variety)}$$

defined by $\phi_L(x) = T_x^*(L) \otimes L^{-1}$ depends only on n and will be denoted by ϕ_n. We define

$$\deg n = \begin{cases} \deg \phi_n & \text{if } \phi_n \text{ is an isogeny} \\ 0 & \text{otherwise.} \end{cases}$$

Fix, once for all, an element L_0 in $NS(X)$ corresponding to an ample line bundle. Let $d_0 = \deg L_0$. For any endomorphism f of X, put

$$f' = \phi_{L_0}^{-1} \circ \hat{f} \circ \phi_{L_0} \in B_{\mathbf{Q}},$$

where $\hat{f} : \hat{X} \to \hat{X}$ in the transpose of f. The map $f \mapsto f'$ extends to an involution $\theta_{\mathbf{Q}}$ of $B_{\mathbf{Q}}$, i.e., $\theta_{\mathbf{Q}}$ is a linear isomorphism of $B_{\mathbf{Q}}, \theta_{\mathbf{Q}}^2 = \text{identity}$, $\theta_{\mathbf{Q}}(ab) = \theta_{\mathbf{Q}}(b)\,\theta_{\mathbf{Q}}(a)$, and $\theta_{\mathbf{Q}}(a) = a'$ for $a \in B$ ($\theta_{\mathbf{Q}}$ is called the Rosati involution associated to L_0). Let $\theta_K = \theta_{\mathbf{Q}} \otimes Id_K$; then θ_K is an involution of B_K and the $\theta_K : G_K \to G_K, \mathbf{Q} \subset K$, define a morphism $\theta : G \to G$.

If $\quad n \in NS(X)$, put $\rho(n) = \phi_{L_0}^{-1} \circ \phi_n$.

Then ρ gives an injection of $NS(X)$ into B and $\rho(NS(X)) = F$ is a lattice in

$$\mathbf{Q} \cdot F = S_{\mathbf{Q}} = \{a \in B_{\mathbf{Q}} \mid \theta_{\mathbf{Q}}(a) = a\} \text{ ([4], § 20, p. 190).}$$

Let $\quad S_K = \{b \in B_K \mid \theta_K(b) = b\}.$

Consider the following representation π of G on S defined over $\mathbf{Q}$: for $g \in G_K$ and $s \in S_K$ put $\pi(g^{-1})s = \theta(g)sg$. For $n \in NS(X)$ and $g \in \mathrm{Aut}(X)$, let $g^*(n)$ denote the pull-back of n by g. We then have

$$\theta(g)\rho(n)g = \phi_{L_0}^{-1}\hat{g}\phi_{L_0} \circ \phi_{L_0}^{-1}\phi_n \circ g = \phi_{L_0}^{-1}\hat{g}\phi_n g = \phi_{L_0}^{-1}\phi_{g^*n} = \rho(g^*n).$$

Thus $\pi(\mathrm{Aut}\, X)$ leaves the lattice $\rho(NS(X))$ in $B_{\mathbf{Q}}$ invariant and the action π of $\mathrm{Aut}(X)$ on $\rho(NS(X))$ coincides with the natural action of $\mathrm{Aut}\, X$ on $NS(X)$.

2. Algebras with involution

In this section, we study algebras with involution over an algebraically closed field K of characteristic zero. All the involutions encountered will be denoted by θ.

Let V, W be vector spaces of dimension n over K and $(,) : V \times W \to k$ be a non-degenerate bilinear pairing. If $A : V \to V$ is an endomorphism, denote by A' the endomorphism of W defined by $(AX, Y) = (X, A'Y)$ for $X \in V$, $Y \in W$. Then $A \mapsto A'$ gives an isomorphism of End V onto End W. Put $C_n = \mathrm{End}\, V \times \mathrm{End}\, W$ and

$$\theta(A, B) = (B', A').$$

Now if $V = W$ and $A'' = A$ for all A in End V, then the bilinear form $(,)$ is either symmetric or skew-symmetric. The algebra End V with the involution $\theta(A) = A'$ will be denoted by A_n in the first case and by B_n in the second case.

Lemma 2.1. Let $a \in A_n$, B_n or C_n be a unit such that $\theta(a) = a$; then we may write $a = \beta\theta(j)j$ where β is in the centre and j satisfies:

(i) if $a \in A_n$ or B_n, $\det(j) = 1$,

(ii) if $a \in C_n$, $j = (j_1, j_2)$, then $\det j_1 = \det j_2 = 1$.

Proof: If $a \in C_n$, a is of the form (A, A'); put $A = \beta B$ with $\beta \in K^*$, $\det B = 1$, and let $j = (B, 1)$. Then $a = \beta^2\theta(j)j$.

If $a \in A_n$ (resp. B_n), the form (aX, Y) is a non-degenerate bilinear symmetric (resp. antisymmetric) form on V, and so is (X, Y).

Hence there exists $g \in \mathrm{Aut}\, V$ such that $(aX, Y) = (gX, gY)$ for all $X, Y \in V$. Again, putting $g = \beta j$ with $\beta \in K^*$ and $\det j = 1$, we have $a = \beta^2\theta(j)j$.

Lemma 2.2. Let B be a semi-simple algebra with involution θ over K. Let a be a unit of B with $\theta(a) = a$. Then we may write $a = \beta\theta(j)j$, where β is in the centre and $N(j) = 1$ for any homogeneous polynomial function N on B satisfying $N(ab) = N(a)N(b)$ for all $a, b \in B$ and $N(1) = 1$.

Proof: It is well known that any such algebra with involution is a product of algebras with involution of the type A_n, B_n or C_n ([5], p. 595]. Write B as such a product. Then N is a product of powers of the determinant functions in the various factors. The lemma now follows from lemma 2.1.

3. Proof of theorem 1.1

In view of theorem 1.3 and the observations in § 1.6, the theorem will be a consequence of

Lemma 3.1. For any algebraically closed field K of characteristic zero, there are only finitely many G_K-orbits in the closed set

$$S_d = \{\alpha \in S_K \mid \phi(\alpha) = d/d_0\},$$

where $S_K = S_Q \otimes_Q K$

and $S_Q = (\alpha \in B_Q / \theta_Q(\alpha) = \alpha)$.

Moreover, each of these orbits is closed.

Proof: Let Z' be the centre of B_K and $Z = Z' \cap S_K$, $R = G_K \cap Z$. If $\alpha \in S_K$ with $\phi(\alpha) = d/d_0$, $(d \neq 0)$, α is a unit by lemma 1.5. Using lemma 2.2, we may write $\alpha = \theta(j)\beta j$ with $\beta \in Z$ and $\phi(j) = 1$.

So, in order to show that there are finitely many G_K-orbits in S_d, it suffices to show that there exist only finitely many R-orbits in $S_d \cap Z$. If $\alpha \in S_d \cap Z$, α is a unit by lemma 1.5 and so $S_d \cap Z = \alpha R$, from which it follows that the R-orbits in $S_d \cap Z$ can be put in bijective correspondence with R/R^2. Now R is the kernel of $\phi \mid Z^*$ so that R is the product of a torus and a finite abelian group, proving the first part of the lemma.

To prove that the orbits are closed it is enough to show that they are equidimensional or, what is the same, that the isotropy groups in question have the same dimension. For this, it is enough, by the preceding considerations, to look at the isotropy groups of points in $S_d \cap Z$. But here any isotropy group clearly consists of elements $g \in G_K$ such that $\theta(g)g =$ identity.

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