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Stresses in spherical shells of heterogeneous dielectrics

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Abstract. Principles of elastostatics are followed here to investigate the displacement and stresses of a dielectric body between two concentric spherical surfaces which are at constant potentials. The dielectric is heterogeneous in character so that its specific inductive capacity varies as any function of the distance from the centre. The bounding surfaces are subjected to internal and external mechanical pressures of different magnitudes.

Keywords. Dielectrics; heterogeneity; spherical shell; condenser; stresses.

1. Introduction

It is well known that electrostrictive dielectrics, under the influence of appropriate electric fields exhibit elastic properties [1], [4], [5]. The electrostrictive effects are considered in this paper in a body of dielectric of non-uniform specific inductive capacity (SIC) vary as any function of space variables. The space occupied by the dielectric is between two concentric spherical surfaces always maintaining a constant electric voltage difference. Both the spherical surfaces are subjected to normal mechanical pressures. The present paper considers the displacement and stresses of the dielectric from the point of view of mechanics of continuous media with the help of equations of elasticity and electricity together with the equations of electromechanical interaction. Use of such dielectrics in a spherical condenser of high capacities is well recognised.

2. Fundamental equations

We consider the space between two concentric spherical surfaces of radii r_1 and r_2 ($r_2 > r_1$) and the surfaces are at constant potentials V_1 and V_2 respectively. The space is occupied with a heterogeneous dielectric whose SIC varies as any function of the distance from the centre of the sphere. In addition, the dielectric is held strained by internal and external mechanical pressures p_1 and p_2 applied normally on the bounding surfaces.

For the formulation of the problem the following equations of electricity and elasticity are taken into account.

The electric field is governed by the equations [3]

$$\text{rot } \mathbf{E} = 0, \tag{1a}$$

$$\text{div } \mathbf{D} = 0, \tag{1b}$$

$$\mathbf{D} = K\,\mathbf{E}, \tag{1c}$$

where **D**, **E**, K are the electric displacement vector, electric intensity vector and SIC respectively. If V be then electric potential then from (1a) the electric intensity vector may be written as

$$\mathbf{E} = -\text{ grad } V. \tag{2}$$

Using (1b) and (1c) one can have

$$\text{div}\,(K \text{ grad } V) = 0. \tag{3}$$

We take the spherical polar co-ordinates (r, θ, ϕ) as the co-ordinates of reference. Let the variable SIC K obey the following law

$$K = K_0/r^2 \psi\,(r), \tag{4}$$

where the integrability of the function $\psi\,(r)$ is assumed so that one can use

$$\psi\,(r) = \phi'\,(r). \tag{5}$$

Due to spherical symmetry (3) with the help of (4) and (5) becomes

$$\frac{d}{dr}\left[\frac{K_0}{r^2\phi'\,(r)} \cdot r^2 \frac{dv}{dr}\right] = 0.$$

It indicates clearly the invariability of the expression

$$\left[\frac{K_0}{\phi'\,(r)} \cdot \frac{dv}{dr}\right].$$

The electric potential at any point may then be generally given by

$$V = A_1 \phi\,(r) + A_2, \tag{6}$$

where A_1 and A_2 are the constants of integration which can be evaluated from the prescribed electrical boundary conditions

$$\left.\begin{array}{l} V = V_1 \text{ at the surface } r = r_1 \\ V = V_2 \text{ at the surface } r = r_2 \end{array}\right\}. \tag{7}$$

Following (6) and (7) one can get A_1 and A_2 as

$$A_1 = \frac{V_1 - V_2}{\phi\,(r_1) - \phi\,(r_2)},$$

$$A_2 = \frac{V_2 \phi\,(r_1) - V_1 \phi\,(r_2)}{\phi\,(r_1) - \phi\,(r_2)}. \tag{8}$$

The general form of the electric potential V under the prescribed boundary conditions may be found from (6) and (8)

$$V = \phi(r)\frac{V_1 - V_2}{\phi(r_1) - \phi(r_2)} + \frac{V_2\phi(r_1) - V_1\phi(r_2)}{\phi(r_1) - \phi(r_2)}. \tag{9}$$

The components of the electric intensity can be obtained from (2) and (9). We are only interested in the radial displacement u of the dielectric; the strain components are given by

$$S_{rr} = \frac{du}{dr}, \quad S_{\theta\theta} = S_{\phi\phi} = \frac{u}{r}, \quad S_{\theta\phi} = S_{r\phi} = S_{r\theta} = 0. \tag{10}$$

The constitutive relations of the electrostrictive material in terms of displacement and electric intensity are [2]

$$T_{rr} = (\lambda + 2G)\frac{du}{dr} + 2\lambda\frac{u}{r} + (a+b)E_r^2,$$

$$T_{\theta\theta} = T_{\phi\phi} = \lambda\frac{du}{dr} + 2(\lambda + G)\frac{u}{r} + aE_r^2,$$

$$T_{r\theta} = 0, \; T_{\theta\phi} = 0, \; T_{r\phi} = 0. \tag{11}$$

T's are the stress components, a and b are the electrostrictive scalar quantities, λ and G are the material constants. To obtain u we make use of the equations of equilibrium together with the mechanical boundary conditions, namely,

$$\begin{aligned} T_{rr} &= -p_1 \text{ on the surface } r = r_1, \\ &= -p_2 \text{ on the surface } r = r_2. \end{aligned} \tag{12}$$

The relevant stress equation of equilibrium is given by

$$\frac{\partial T_{rr}}{\partial r} + \frac{2}{r}[T_{rr} - T_{\theta\theta}] = 0. \tag{13}$$

On using (11), (13) becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) - 2u = -\frac{2r^2}{(\lambda + 2G)}\left[(a+b)E_r\frac{\partial E_r}{\partial r} + \frac{b}{r}E_r^2\right]. \tag{14}$$

3. Solution of the problem

To understand the radial displacement we transform the non-homogeneous differential equation (14) by setting

$$r = \exp\eta, \tag{15}$$

so that (15) is transformed to

$$\begin{aligned}(D^2 + D - 2)u &= -\frac{2}{(\lambda + 2G)}\exp\eta\left[(a+b)E_r\frac{\partial E_r}{\partial\eta} + bE_r^2\right], \\ &= F(\eta) \text{ (say)},\end{aligned} \tag{16}$$

where $D \equiv d/d\eta$.

The solution of (16) may be set as

$$u = A_3 (\exp \eta) + A_4 (\exp \eta)^{-2} + \frac{F(\eta)}{D^2 + D - 2}, \tag{17}$$

which is the most general form of the deformation. A_3 and A_4 are arbitrary constants.

The electric intensity in most cases varies exponentially or sinusoidally and we choose

$$F(\eta) = \exp(m\eta + \epsilon), \tag{18}$$

in which ϵ may be treated as the phase shift constant and m as the attenuation constant. They may be real or imaginary quantities.

$F(\eta) = \exp(m\eta + \epsilon)$ corresponds to

$$E_r^2 = A_1^2 [\phi'(r)]^2 = [A\, r^{m-1+\epsilon} + B\, r^{-2b/(a+b)}]. \tag{19}$$

From (18), (15) and (17), the radial displacement stands as

$$u = A_3 r + \frac{A_4}{r^2} + \frac{F_0 r^m}{m^2 + m - 1}, \tag{20}$$

in which F_0 is the known constant $F_0 = \exp \epsilon$. To evaluate the constants of integration A_3 and A_4 of (20) we use the mechanical boundary conditions stated in (12) along with (11) as

$$(\lambda + 2G)\frac{du}{dr} + 2\lambda \frac{u}{r} + (a+b) A_1^2 [\phi'(r)]^2 = -p_1 \quad \text{on} \quad r = r_1$$

$$(\lambda + 2G)\frac{du}{dr} + 2\lambda \frac{u}{r} + (a+b) A_1^2 [\phi'(r)]^2 = -p_2 \quad \text{on} \quad r = r_2.$$

to yield

$$A_3 = \frac{(p_1 r_1^3 - p_2 r_2^3) - m_1 F_0 (r_2^{m+2} - r_1^{m+2}) - (a+b) A_1^2 [\{\phi'(r_2)\}^2 r_2^3 - \{\phi'(r_1)\}^2 r_1^3]}{(3\lambda + 2G)(r_2^3 - r_1^3)},$$

and

$$A_4 = \frac{r_1^3 r_2^3 [(p_1 - p_2) - m (r_2^{m-1} - r_1^{m-1}) - (a+b) A_1^2 [\phi'(r_2)]^2 - [\phi'(r_1)]^2]}{4G (r_2^3 - r_1^3)}$$

where $$m_1 = \frac{\lambda(m+2) + 2G}{m^2 + m - 2}. \tag{21}$$

Now we are in a position to write the full expression of stresses in the following form

$$T_{rr} = \frac{(p_1 r_1^3 - p_2 r_2^3) - m_1 F_0 (r_2^{m+2} - r_1^{m+2}) - (a+b) A_1^2 \{\phi'(r_2)\}^2 r_2^3 - \{\phi'(r_1)\}^2 r_1^3]}{(r_2^3 - r_1^3)}$$

$$- \frac{r_1^3 r_2^3}{2r^3 (r_2^3 - r_1^3)} [(p_1 - p_2) - m_1 (r_2^{m-1} - r_1^{m-1}) - (a+b) A_1^2 [\phi'(r_2)]^2$$

$$- [\phi'(r_1)]^2] + \frac{F_0 r^{m-1}}{m^2 + m - 2} \{\lambda (m+2) + 2G\} + (a+b) E_r^2 ,$$

$$T_{\theta\theta} = \frac{(p_1 r_1^3 - p_2 r_2^3) - m_1 F_0 (r_2^{m+2} - r_1^{m+2}) - (a+b) A_1^2 [\{\phi'(r_2)\}^2 r_2^3 - \{\phi'(r_1)\}^2 r_1^3]}{(r_2^3 - r_1^3)}$$

$$+ \frac{r_1^3 r_2^3}{2r^3 (r_2^3 - r_1^3)} [(p_1 - p_2) - m_1 (r_2^{m-1} - r_1^{m-1}) - (a+b) A_1^2 [[\phi'(r_2)]^2$$

$$- [\phi'(r_1)]^2] + \frac{F_0 r^{m-1}}{m^2 + m - 2} \{\lambda (m+2) + 2G\} + a E_r^2 . \tag{22}$$

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