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Natural convection at a heated semiinfinite vertical plate with temperature dependent heat sources or sinks

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Abstract. Numerical results are presented for the transient and steady-state velocity field and the temperature field. These results were obtained by solving the partial differential equations describing the conservation of mass, momentum and energy by an explicit finite-difference method in time dependent form. It has been observed that the velocity components u, v (absolute) and the temperature θ have larger values in the presence of heat sources than in their absence. Opposite is the phenomenon in the presence of heat sinks. A worthwhile result obtained from the detailed examination of the problem is that the presence of heat sources delays the attainment of steady-state condition.

Keywords. Fluid dynamics; heat sources; heat transfer; laminar flow; natural convection.

1. Introduction

From technological point of view, unsteady flow is always important, for it has many practical applications. Pohlhausen [13] developed an analytical solution and Ostrach [11] a numerical solution for the isothermal vertical plate at steady-state. Their results agree well with the experimental values of Schmidt and Beckmann [13] for a uniform heat flux density at the plate. For very short times the fluid velocities are very small, and the analytical solution for conduction alone is presumed to be applicable. Siegel [14] studied the transient case by an integral method and obtained an estimate of the time required to attain steady-state. Gebhart [6] developed an approximate solution for the transient behaviour with a constant heat flux density at the plate. Recently Hellums and Churchill [7] gave a complete transient solution, in terms of composite variables, for free convection in an unconfined fluid initially at rest and at uniform temperature, adjacent to a semiinfinite vertical plate at a different uniform temperature. The fluid motion develops slowly following the development of nonuniformity in the temperature field. However, when the temperature differences are appreciably large, the volumetric heat generation or absorption term may exert strong influence on the heat transfer and as a consequence, on the fluid flow as well.

The analysis of temperature field as modified by the generation or absorption of heat in moving fluids is important in view of several physical problems such as (i) problems dealing with chemical reactions [18], (ii) problems concerned with dissociating fluids [9] and [8]. In fact, literature is replete with examples dealing with the heat transfer in laminar flow of viscous fluids. Heat generation has been assumed to be constant or a function of space variables by some authors. Some others have considered directly the frictional heating and the expansion effect. Sparrow and Cess [15] have obtained solutions of the steady flow and heat transfer of the stagnation point flow taking into account the temperature dependent heat generation (absorption). Topper [16] has analysed the piston flow in pipes with circular cross-section when the rate of heat generation depends linearly on the local temperature. Foraboschi and Federico [4] have assumed volumetric rate of heat generation as

$$\begin{aligned} Q &= Q_0 (T - T_0) \quad \text{when } T \geqslant T_0 \\ Q &= 0 \qquad\qquad\quad\;\; \text{when } T < T_0 \end{aligned} \tag{I}$$

in their study of the steady-state temperature profiles for linear parabolic- and piston-flow in circular tubes. Taking into account the heat generation of type (I), Foraboschi and Cocchi [3] have further investigated the transient state temperature profiles. The relation, (I) as explained by Foraboschi and Federico [4], is valid as an approximation of the state of some exothermic process and having T_0 as the onset temperature. When the inlet temperatures are not less than T_0 they used $Q = Q_0 (T - T_0)$ and studied its effect on the heat transfer in laminar flow of non-Newtonian heat generating fluids. Very recently Moalem [10] has studied the effect of temperature dependent heat sources ($Q' \propto 1/(a + bT)$) occurring in electrical heating, on the steady-state heat transfer within a porous medium.

To the best knowledge of the author very little attention has been paid to the study of the effect of heat sources or sinks on the unsteady free convection flow and heat transfer problems such as convection studies with internal heat sources proposed for earth's Mantle [5], [12], [17] and for the outer region of star interiors [1]. This analysis is therefore an humble attempt to throw adequate light on the effect of the heat sources or sinks on the free convection flow past a heated semiinfinite vertical plate. The volumetric rate of heat generation (absorption) is taken as $Q \propto (T - T_i)$, which is similar to one considered by Sparrow and Cess [15] and Foraboschi and Federico [4].

2. Formulation of the problem

Consider a semiinfinite heated vertical plate in an infinite fluid that is initially cold and at rest (see figure 1). The following assumptions were made in the analysis of the problem (i) all the fluid properties are constant except the density in the buoyancy force term, (ii) the flow is laminar, unsteady and two dimensional and (iii) the viscous dissipation and the work done by pressure are sufficiently small in comparison with the heat flow by conduction.

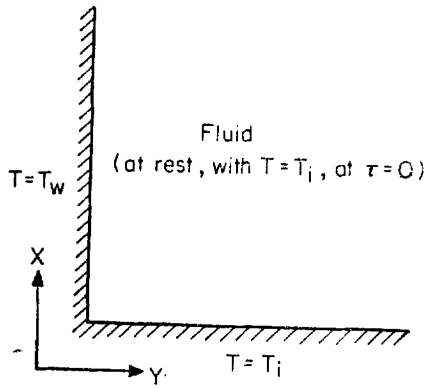

Figure 1. Flow configuration.

Under these assumptions the equations of momentum, mass and energy in the fluid at all times are described approximately by the following equations

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \beta g\,(T - T_i) + \nu\frac{\partial^2 U}{\partial Y^2}\,, \tag{1}$$

$$\partial U/\partial X + \partial V/\partial Y = 0, \tag{2}$$

$$\rho C_p\left(\frac{\partial T}{\partial \tau} + U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y}\right) = k\,\frac{\partial^2 T}{\partial Y^2} + Q^*\,(T - T_i), \tag{3}$$

where T_i is the initial temperature of the fluid and the other symbols have their usual meanings. The last term in the energy equation is the heat generation (absorption) term. The initial and boundary conditions relevant to the problem are

$$\begin{aligned} &U = V = 0,\ T = T_i \quad \text{at} \quad X = 0,\\ &U = V = 0,\ T = T_w \quad \text{at} \quad Y = 0,\\ &U = V = 0,\ T = T_i \quad \text{as} \quad Y \to \infty,\\ &U = V = 0,\ T = T_i \quad \text{at} \quad Z = 0. \end{aligned} \tag{4}$$

Defining non-dimensional variables

$$\begin{aligned} t &= Z\,(g\beta\triangle T)^{2/3}/\nu^{1/3},\\ x &= X\,(g\beta\triangle T/\nu^2)^{1/3},\\ y &= Y\,(g\beta\triangle T/\nu^2)^{1/3},\\ u &= U/(\nu g\beta\triangle T)^{1/3},\\ v &= V/(\nu g\beta\triangle T)^{1/3},\\ \theta &= (T - T_i)/\triangle T, \end{aligned}$$

eqs (1)–(3) and the conditions (4) can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \theta + \frac{\partial^2 u}{\partial y^2}, \tag{5}$$

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{6}$$

$$\Pr\left(\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y}\right) = \alpha\theta + \frac{\partial^2 \theta}{\partial y^2}, \tag{7}$$

$$\begin{aligned} &u = v = 0,\ \theta = 0 \quad \text{at} \quad x = 0,\\ &u = v = 0,\ \theta = 1 \quad \text{at} \quad y = 0,\\ &u = v = 0,\ \theta = 0 \quad \text{as} \quad y \to \infty,\\ &u = v = 0,\ \theta = 0 \quad \text{at} \quad t = 0, \end{aligned} \tag{8}$$

where $\alpha = Q^* \nu^{4/3}/k\,(g\beta\triangle T)^{2/3}$, the non-dimensional heat source or sink parameter,

$\Pr = \mu C_p/k$, the Prandtl number,

$\triangle T = T_w - T_i$, the temperature difference.

3. Method and solution

Now, the simultaneous nonlinear partial differential equations (5), (6) and (7), with conditions (8), are to be solved for the dependent variables u, v and θ as a function of x, y and t and in particular, to obtain the steady-state solution, if it exists. Although the primary goal is to obtain the steady-state solution, for which both $\partial u/\partial t$ and $\partial\theta/\partial t$ are zero, solution has been obtained considering the corresponding unsteady-state problem, already formulated in §2. Successive steps in time can then be regarded as successive approximations towards the steady-state solution. The solution was obtained by numerical integration and the integrations were carried out on the time dependent form of the equations by an explicit finite-difference method, using an EC 1030 computer, as explained in Carnahan *et al* [2]. The space under investigation has been restricted to finite dimensions. Here, a plate of height $x_{max} = 100$ and $y_{max} = 25$ as corresponding to $y = \infty$ has been considered. The velocities and temperature fields were calculated with $\Pr = 0{\cdot}733$ and $\alpha = 0{\cdot}1, 0$ and $-0{\cdot}1$ for a 100×100 grid at $t = 0{\cdot}5, 1{\cdot}0, 1{\cdot}5, \ldots, 80$ and are summarised in figures 2 to 4 for values of $t = 10, 20, 30, \ldots, 80$. An examination of the complete results so presented revealed little change in u, v and θ after $t = 50$ for all values of the heat source parameter α. Thus the results for $t = 80$ are essentially steady-state values.

4. Discussion of the results

The equations (5)–(7) subject to the conditions (8) were solved by the finite-difference method, mentioned in §3. The results thus obtained are presented in figures 2–4 for $Pr = 0{\cdot}733$ and $\alpha = 0{\cdot}1, 0, -0{\cdot}1$. It is worth mentioning here that the

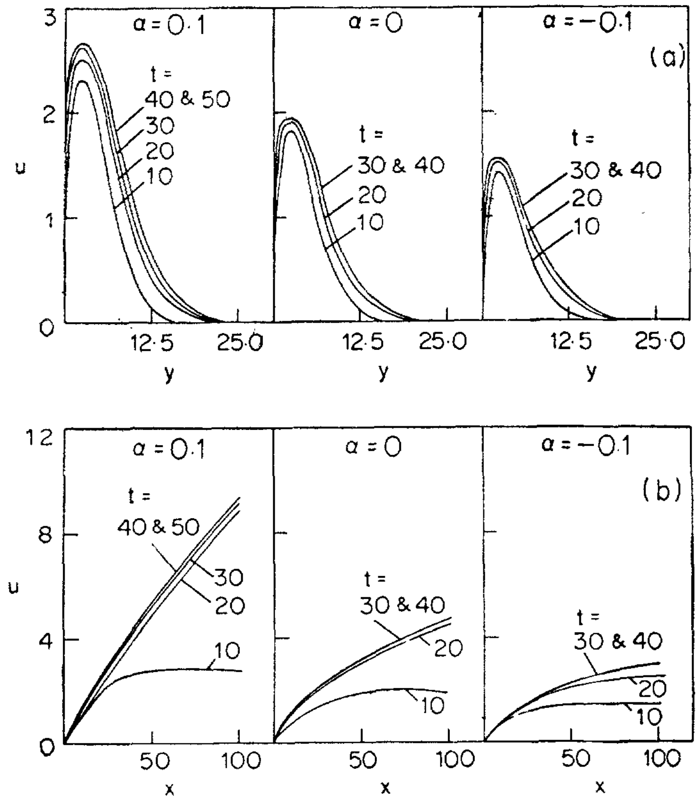

Figure 2. Transient and steady-state velocity profiles (*a*) at $x = 20$, (*b*) at $y = 5$.

results of Carnahan *et al* [2] for fluid velocity and fluid temperature are a special case of the solution presented in this paper for $\alpha = 0$.

In figure 2 the transient and the steady-state values for the velocity component u, for $\alpha = 0{\cdot}1$, 0, $-0{\cdot}1$, are shown for two cases (a) with fixed x and (b) fixed y. From figure 2, it can be observed that the values of velocity component u have increased considerably in the presence of heat sources. Opposite is the phenomenon in the presence of heat sinks. Observations from figure 2 yielded that the velocity component u increases monotonically, as the nondimensional time increases upto a certain stage and then it decreases slightly and thereafter remains unchanged. That is, in the presence of heat sources at $t = 50$, and at $t = 40$ in the absence of heat sources, the velocity u reaches the steady-state condition. These steady-state values for the velocity u reveal that the field of velocity component u starts increasing to a certain extent, near the plate, from its zero value and then it decreases steadily to the value zero in the y increasing direction. But u increases in the x increasing direction from its zero value.

Figure 3 describes the behaviour of the velocity component v, perpendicular to the plate. The negative values for v, at all points in the fluid, indicate that the velocity v is directed towards the plate. Here also, as in the case of u, the values of v (absolute) increase monotonically up to $t = 50$ in the presence of heat sources and up to $t = 40$ in their absence and thereafter remains unchanged. These steady-state results, for v, reveal that the velocity v (absolute) is enhanced in the presence

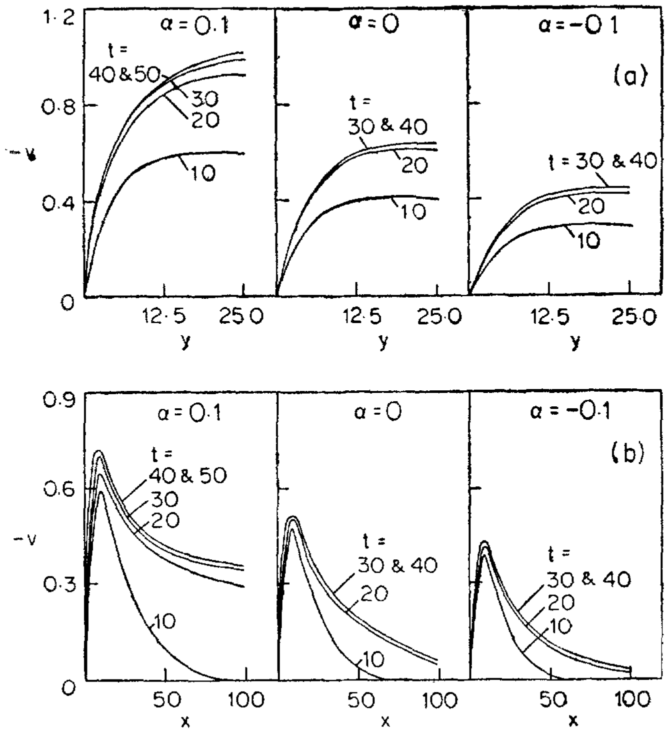

Figure 3. Transient and steady-state velocity profiles (a) at $x = 20$, (b) at $y = 5$.

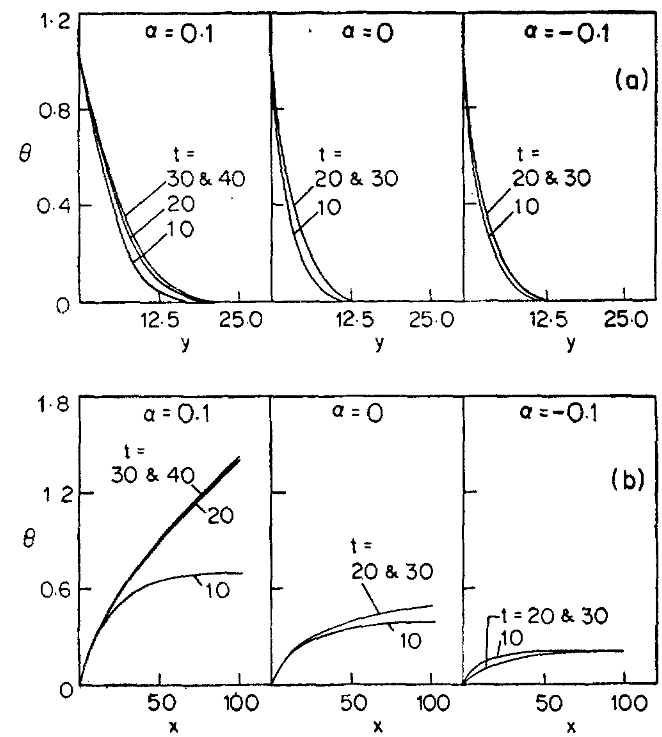

Figure 4. Transient and steady-state temperature profiles (a) at $x = 20$, (b) at $y = 5$.

of heat sources than in their absence. Also, v increases in the x increasing direction but it decreases in the y increasing direction.

The transient and the steady-state temperature profiles are shown in figure 4, for $\alpha = 0\cdot 1, 0, -0\cdot 1$, for two cases of (a) x fixed and (b) y fixed. It has been observed that the temperature increases monotonically and it reaches the steady-state condition at $t = 40$ in the presence of heat sources and at $t = 30$ in the presence of heat sinks. From the steady-state values of θ it can be seen that, as in the case of u, here also, the values of θ are considerably enhanced in the presence of heat sources than in their absence. Reverse is the phenomenon in the presence of heat sinks. It can also be observed that the maximum temperature occurs near the plate and this happens both in the presence of heat sources and in the presence of heat sinks. Finally it can be said that the heat sources or sinks have a strong influence on the heat transfer and as a consequence on the fluid flow as well.

5. Conclusions

(i) The fluid velocity components u, v (absolute) and the fluid temperature θ increase considerably in the presence of heat sources than in their absence, at all times and at all points in the fluid, and quite opposite is the phenomenon in the presence of heat sinks.

(ii) In the absence of heat sources the velocity component v, perpendicular to the plate, is always directed towards the plate and this is true even in the presence of heat sources or sinks.

(iii) The heat sources play an important role in delaying the velocity and the temperature to reach the steady-state condition.

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