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Finsler spaces with recurrent pseudocurvature tensor

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Abstract. Sinha [2] has defined pseudocurvature tensor field in an n-dimensional Finsler space F_n. In the present paper we have studied some recurrent properties of pseudocurvature tensor.

Keywords. Finsler space; pseudocurvature tensor; pseudorecurrent space.

1. Introduction

Consider an n-dimensional Finsler space F_n. In such a space Sinha [5] defined pseudo deviation tensor $T^i_j(x, \dot{x})$

$$T^i_j = -[H\delta^i_j + (1/n+1)(\dot{\partial}_k H^k_j - \dot{\partial}_j H)\dot{x}^i], \tag{1}$$

where H^i_j are positively homogeneous of degree one in $\dot{x}^i$ and H is scalar function defined by $H(x, \dot{x}) = 1/(n-1) H^i_i$. The pseudo curvature tensors T^i_{jk} and T^i_{jhk} are defined by [5]

$$T^i_{jk} = 1|3\,(\dot{\partial}_j T^i_k - \dot{\partial}_k T^i_j), \tag{2}$$

$$T^i_{jhk} = \dot{\partial}_j T^i_{hk}. \tag{3}$$

The pseudocurvature tensor satisfies the following identities [5]

$$T^i_{jhk} + T^i_{hkj} + T^i_{kjh} = 0, \tag{4a}$$

$$T^i_{jk} = -T^i_{kj}, \tag{4b}$$

$$T^i_{hk}\dot{x}^h = T^i_k, \quad T^i_{jhk}\dot{x}^j = T^i_{hk}, \tag{5a}$$

$$T^i_j \dot{x}^j = 0. \tag{5b}$$

Also we have a scalar function $T(x, \dot{x})$ defined by

$$T = 1/(n-1)\, T^i_i. \tag{6}$$

All indices take values from 1 to n. Usual summation convention holds for repeated indices. Here $\partial_j \equiv \partial/\partial x^j$ and $\dot{\partial}_j = \partial/\partial \dot{x}^j$. Also for any tensor S, $S_{(1)}$ is covariant derivative of the tensor S with respect to Berwald's connections. We shall consider covariant derivative only in this sense.

Further it has been shown [5] that

$$W^i_j = H^i_j + T^i_j, \tag{7a}$$

$$W^i_{jk} = H^i_{jk} + T^i_{jk}, \tag{7b}$$

$$W^i_{jhk} = H^i_{jhk} + T^i_{jhk}. \tag{7c}$$

Here W^i_j is projective deviation tensor [4] and W^i_{jh} and W^i_{jhk} are projective curvature tensors. H^i_j has been defined in [4], p. 125.

2. Finsler spaces with recurrent pseudocurvature tensor

Definition. (2.1) An F_n whose psuedocurvature tensor T^i_{jhk} satisfies

$$T^i_{jhk(l)} = K_l T^i_{jhk} \tag{8}$$

will be called pseudorecurrent space of first order or simply T-recurrent F_n and K_l is the associated recurrence vector.

Further if $T^i_{jhk(l)} = 0$ the space will be called T-symmetric and it will be called T-flat if $T^i_{jhk} = 0$.

Transvecting (8) first by $\dot{x}^j$ and then by $\dot{x}^h$ we get in view of (5)

$$T^i_{hk(l)} = K_l T^i_{hk}, \tag{9}$$

$$T^i_{k(l)} = K_l T^i_k. \tag{10}$$

Hence the tensors T^i_{hk} and T^i_k are also recurrent in a T-recurrent space.

Theorem (2.1). The recurrence vector K_l satisfies the following relation

$$K_{l(m)} - K_{m(l)} = H^i_{ml} \dot{\partial}_i (\log T). \tag{11}$$

Proof. Since (8) implies (10) we start with (10). Differentiating (10) covariantly we get

$$T^i_{k(l)(m)} = (K_{l(m)} + K_l K_m) T^i_k. \tag{12}$$

Interchanging l and m in (12) and subtracting the new equation from (12) we get

$$T^i_{k(l)(m)} - T^i_{k(m)(l)} = (K_{l(m)} - K_{m(l)}) T^i_k.$$

Using commutation formula (equation (6·10) on p. 126 of [4]) we get

$$(K_{l(m)} - K_{m(l)}) T^i_k = - (\dot{\partial}_r T^i_k) H^r_{lm} + T^r_k H^i_{rlm} - T^i_r H^r_{klm}. \tag{13}$$

Contracting i and k we get (11) in view of (6).

From (7c) we easily deduce;

Theorem (2.2). If an F_n satisfies any two of the following it satisfies the third also (i) The space is projectively flat; (ii) The space is H-flat (i.e., $H^i_{jhk} = 0$); (iii) The space is T-flat.

Corollary (2.1). In a projectively flat F_n T-recurrence implies H-recurrence and conversely.
Proof is easy.

Differentiating (7c) covariantly we have

$$W^i_{jhk(l)} = H^i_{jhk(l)} + T^i_{jhk(l)}. \tag{14}$$

From (14) and (8) we have

Theorem (2.3). If two of the following hold then the third also holds (i) the space is projectively recurrent; (ii) The space is H-recurrent (i.e., $H^i_{jhk(l)} = N_l H^i_{jhk}$); (iii) The space is T-recurrent.

We further see that

Theorem (2.4). If two of the following hold the third also holds: (i) The space is projectively symmetric; (ii) The space is H-symmetric (i.e., $H^i_{jhk(l)} = 0$): (iii) The space is T-symmetric.

We have already seen that in a T-recurrent F_n the tensors T^i_j and T^i_{jk} are also recurrent. For the converse part we have:

Theorem (2.5). Let T^j_{hk} be recurrent in an F_n.

$$T^j_{hk(l)} = K_l T^j_{hk}. \tag{15}$$

Then F_n will be T-recurrent with vector of recurrence K_l if

$$(\dot\partial_i K_l)\, T^j_{hk} = T^m_{hk} G^j_{mil} + T^j_{mh} G^m_{kil} - T^j_{mk} G^m_{hil}. \tag{16}$$

Proof. Differentiating (15) partially with respect to $\dot x^i$ and using commutation formula {(6·11b) of [4] (p. 127)} we immediately have the result in view of (8).

Theorem (2.6). The recurrence vector K_l in (15) is homogeneous of degree zero in $\dot x$.

Proof. Transvecting (16) with $\dot x^i$ and $\dot x^h$ we get

$$(\dot x^i \dot\partial_i K_l)\, T^j_k = 0 \tag{17}$$

since $G^j_{mil}\dot x^m = 0$ and G^j_{mil} is completely symmetric in m, i, l. Contracting j and k in (17) and noting $T \neq 0$ we get the result.

Theorem (2.7). Let

$$T^j_{k(l)} = K_l T^j_k. \tag{18}$$

Then T^j_{hk} is recurrent if

$$T^j_k(\dot\partial_h K_l) - T^j_h \dot\partial_k K_l + T^r_h G^j_{krl} - T^r_k G^j_{hrl} = 0. \tag{19}$$

Proof. It follows directly on the pattern of (15) in view of (2).

Transvecting (19) with $\dot x^h$ and noting (5b) we get $T^j_k \dot x^h \dot\partial_h K_l = 0$. Contracting j and k we get $\dot x^h \dot\partial_h K_l = 0$ which was proved earlier in theorem (13).

In case F_n is projectively flat we get a simple relation.

Theorem (2.8). In a projectively flat T-recurrent space we have

$$K_l T^i_{hk} + K_h T^i_{kl} + K_k T^i_{lh} = 0. \tag{20}$$

Proof. From (7*c*) we have $H^i_{jhk} + T^i_{jhk} = 0$. Differentiating covariantly with respect to x^l we get $H^i_{jhk(l)} + T^i_{jhk(l)} = 0$. Taking skew symmetric part in h, k, l and then transvecting the result with $\dot{x}^j$ we have

$$(H^i_{jhk(l)} + H^i_{jkl(h)} + H^i_{jlh(k)})\,\dot{x}^j + (T^i_{jhk(l)} + T^i_{jkl(h)} + T^i_{jlh(k)})\,\dot{x}^j = 0. \tag{21}$$

The first set vanishes due to (eq. (6.13) p. 128 [4]). Hence we get

$$T^i_{hk(l)} + T^i_{kl(h)} + T^i_{lh(k)} = 0.$$

(20) follows immediately from it and from (9).

3. Pseudorecurrent spaces of the second order

Riemannian recurrent spaces of second order have been considered by several authors [1], [2], [3]. On the same pattern of definition we have
Definition (3.1). An F_n whose pseudocurvature tensor T^i_{jhk} satisfies

$$T^i_{jhk(l)(m)} = a_{lm} T^i_{jhk} \tag{22}$$

will be called pseudorecurrent spaces of second order or simply T-birecurrent space with associated tensor of recurrence a_{lm}.

Further if $a_{lm} = 0$ the space will be called T-bisymmetric. Contracting i and k in (22) we get

$$T_{jh(l)(m)} = a_{lm} T_{jh}. \tag{23}$$

Theorem (3.1). The recurrence tensor a_{lm} is non-symmetric.
Proof. From (23) we immediately have

$$T_{jh(l)(m)} - T_{jh(m)(l)} = a^*_{lm} T_{jh}, \tag{24}$$

where $a^*_{lm} = a_{lm} - a_{ml}$.

Transvecting (24) with $\dot{x}^j \dot{x}^h$ we get

$$T_{(l)(m)} - T_{(m)(l)} = a^*_{lm} T.$$

Using commutation formula (6.10) in [4], p. 126, we get

$$T a^*_{lm} = -(\dot{\partial}_r T) H^r_{lm} = (\dot{\partial}_r T) H^r_{ml}.$$

Thus $a^*_{lm} \neq 0$ which proves the theorem. Further

$$a^*_{lm} = \dot{\partial}_r (\log T) H^r_{ml}.$$

Theorem (3.2). Every T-recurrent space is T-birecurrent.

Proof. Differentiating (8) covariantly with respect to x^m we get (22) with

$$a_{lm} = K_{l(m)} + K_l K_m.$$

Theorem (3.3). If a T-recurrent space is T-bisymmetric then the recurrence vector is itself recurrent.

Proof. If a T-recurrent space is T-bisymmetric, then

$$K_{l(m)} + K_l K_m = 0.$$

The statement follows from the above.

Theorem (3.4). In a projectively flat T-birecurrent space we have

$$a_{lm} T^i_{hk} + a_{hm} T^i_{kl} + a_{km} T^i_{lh} = 0. \tag{25}$$

Proof. It follows exactly on the pattern of the proof of (20).

Theorem (3.5). If a projectively flat T-birecurrent space is also T-recurrent then

$$a^*_{lm\,(n)} = K_n a^*_{lm}. \tag{26}$$

Proof. Transvecting (22) with $\dot{x}^j$ we immediately get

$$T^i_{hk(l)(m)} = a_{lm} T^i_{hk}. \tag{27}$$

From this we immediately get

$$T^i_{hk(l)(m)} - T^i_{hk(m)(l)} = a^*_{lm} T^i_{hk}.$$

or using commutation formula we have

$$(-\dot{\partial}_r T^i_{hk})\, H^r_{lm} + T^r_{hk} H^i_{rlm} - T^i_{rk} H^r_{hlm} - T^i_{hr} H^r_{klm} = a^*_{lm} T^i_{hk}.$$

Taking covariant derivative of this and using corollary (8) we get on simplification

$$a^*_{lm\,(n)}\, T^i_{hk} = a^*_{lm} K_n T^i_{hk}.$$

From this (26) follows.

In conclusion we would like to mention that theorems analogous to theorems (2.3) and (2.4) and corollary (2.1) can be given for T-birecurrent spaces also. We omit the statement to avoid repetition.

References

[1] Chakki M S and Roychaudhary A N 1967 *Indian J. Math.* **9** 279
[2] Lichnerowicz A 1952 *Proc. Int. Congr. Math.* **2** 216
[3] Roter W 1964 *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* **12** 207
[4] Rund H 1959 *Differential geometry of Finsler spaces* (Berlin: Springer–Verlag)
[5] Sinha B B 1971 *Prog. Math.* **5** 88