On propagation and attenuation of Love waves

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Abstract. The period equation for Love waves is derived for a layered medium, which is composed of a compressible, viscous liquid layer sandwiched between homogeneous, isotropic, elastic solid layer and homogeneous, isotropic half space. In general, the period equation will admit complex roots and hence Love waves will be dispersive and attenuated for this type of model. The period equation is discussed in the limiting case when thickness $H_L$ and coefficient of viscosity, $\eta_L$, of the liquid layer tend to zero so as to maintain the ratio $P = H_L/\eta_L$ a constant. Numerical values for phase velocity, group velocity, quality factor ($Q$) and displacement in the elastic layer and half space have been computed as a function of the frequency for first and second modes for various values of the parameter $P$. It is shown that Love waves are not attenuated when $P = 0$ and $\infty$. The computed values of $Q$ for first and second modes indicate that when $P \neq 0$ or $\infty$ the value of $Q$ attains minimum value as a function of dimensionless angular frequency.

Keywords. Love waves; period equation; viscous compressible liquid; wave equation.

1. Introduction

Viscous liquids like elastic solids can transmit [5] both compressional and shear waves. Most liquids are viscous, even though the viscosity may be small. For example, water has coefficient of viscosity 0.0178 in CGS units. Whenever a wave propagates in a viscous fluid, the amplitude of the wave attenuates with increasing distance from a reference point. Hence by introducing a viscous liquid layer of finite thickness in between homogeneous, isotropic elastic solid and homogeneous elastic half space, it is possible to study the dispersion and attenuation of surface waves. Banghar et al [3] have studied the propagation of Rayleigh waves in a system composed of viscous compressible liquid layer sandwiched, between two homogeneous isotropic solid half spaces. The object of this paper is to study the propagation of Love waves in a system composed of a compressible viscous, liquid layer sandwiched between homogeneous, elastic Layer and homogeneous isotropic elastic, half space. The period equation for Love waves for the above mentioned model is derived and discussed for a particular limiting case.
2. Theory

The waves studied propagate in a stratified medium composed of a viscous compressible liquid layer of finite thickness \( H_2 \) sandwiched between homogeneous isotropic solid layer and elastic half space. Figure 1 shows the geometry of the system under consideration. Let \((x, y, z)\) be a Cartesian co-ordinate system such that \(x\) and \(y\) are horizontal while \(z\) is taken positive downwards. We shall assume the wave propagation to be two-dimensional (i.e., wave propagating only in \(x-z\) plane). It is also assumed that all variables of state are independent of \(y\).

3. Basic equations

Isotropic homogeneous elastic solid. The equations of motion (in the absence of body forces) can be written as

\[
\rho \left( \frac{\partial^2 \mathbf{S}}{\partial t^2} \right) = (\lambda + \mu) \nabla \cdot \mathbf{S} + \mu \nabla^2 \mathbf{S},
\]

where \(\mathbf{S} = (u, v, w)\) is a displacement vector,

\[
\nabla \cdot \mathbf{S},
\]

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2},
\]

\(\rho\) is mass density and \(\lambda\) and \(\mu\) are Lamé constants. For Love waves

\[
\mathbf{S} = (0, v, 0),
\]

\(\nabla \cdot \mathbf{S} = 0\).

Hence, the equation of motion for Love waves reduces to

\[
\rho \frac{\partial^2 v}{\partial t^2} = \mu \nabla^2 v.
\]

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Figure 1. Diagram showing the geometry of the problem under investigation.
Propagation and attenuation of Love waves

Equation (6) can be written as
\[ \frac{\partial^2 \mathbf{v}}{\partial t^2} = \beta^2 \nabla^2 \mathbf{v}, \]  
where \( \beta = (\mu/p)^{1/3} \) is the velocity of shear waves in an elastic solid. Let \( \mu_1, \rho_1, \beta_1 \) represent rigidity, density and shear wave velocity in a solid layer, while \( \mu_3, \rho_3, \beta_3 \) represent corresponding quantities in elastic half space such that
\[ \rho_1 \beta_1^2 = \mu_1, \]
\[ \rho_3 \beta_3^2 = \mu_3. \]

Displacements \( \mathbf{v}_1 \) and \( \mathbf{v}_3 \) in solid layer and half space will satisfy
\[ \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \beta_1^2 \nabla^2 \mathbf{v}_1, \]
and
\[ \frac{\partial^2 \mathbf{v}_3}{\partial t^2} = \beta_3^2 \nabla^2 \mathbf{v}_3. \]

4. Viscous, compressible liquid

The equations of motion (in the absence of body forces) for viscous compressible liquid [5] can be written as
\[ \rho_2 \left( \frac{\partial \mathbf{V}}{\partial t} \right) = -\nabla p + \frac{\eta_2}{3} \nabla (\nabla \cdot \mathbf{V}) + \eta_2 \nabla^2 \mathbf{V}, \]
where \( \rho_2 \) is density of the liquid, \( \eta_2 \) the coefficient of viscosity and \( p \) is the overpressure, and \( \mathbf{V} = (U_2, V_2, W_2) \) is the velocity vector. For Love waves,
\[ \mathbf{V} = (0, V_2, 0), \]
and
\[ \nabla \cdot \mathbf{V} = 0. \]

Therefore (11) reduces to a single equation, which can be written as
\[ \rho_2 \left( \frac{\partial V_2}{\partial t} \right) = \eta_2 \nabla^2 V_2. \]

It may be mentioned here that the constancy of overpressure follows from (11). This constant overpressure is taken as zero.
\[ \therefore \quad p = 0. \]

Hence, compressibility of the liquid does not affect the propagation of Love waves. Equation (14) can also be represented as
\[ \frac{\partial V_2}{\partial t} = \nu \nabla^2 V_2, \]
where \( \nu = \eta_2/\rho_2 \) is the coefficient of kinematic viscosity.
5. Solutions of wave equations

We assume that time dependence is given by a factor \( \exp (iwt) \), hence (9), (10) and (16) can be represented by

\[
(\nabla^2 + k_{\beta_1}^2) v_1 = 0, \quad (17)
\]

\[
(\nabla^2 + k_{\beta_2}^2) v_2 = 0, \quad (18)
\]

and

\[
(\nabla^2 + k_{\beta_3}^2) V_2 = 0, \quad (19)
\]

where

\[
k_{\beta_1}^2 = \frac{\omega^2}{\beta_1^2}, \quad (20)
\]

\[
k_{\beta_2}^2 = \frac{\omega^2}{\beta_2^2}, \quad (21)
\]

\[
k_{\beta_3}^2 = -i\omega/v, \quad (22)
\]

and \( \omega = 2\pi f \) is the angular frequency.

Solutions of (17), (19) and (18), for the geometry under consideration, are written as

\[
v_1 = [A \cos v_3 z + B \sin v_3 z] \exp [i (\omega t - kx)], \quad (23)
\]

\[
V_2 = i\omega \ [C_1 \cos v_3 z + D \sin v_3 z] \exp [i(\omega t - kx)], \quad (24)
\]

and

\[
v_3 = E \exp (-V_3 x) \exp [i (\omega t - kx)] \quad (25)
\]

where

\[
v_3^2 = k_{\beta_3}^2 - k_3^2,
\]

\[
v_3^2 = k_{\beta_3}^2 - k_3^2 = -\left[ k^2 + \frac{i\omega}{v} \right],
\]

\[
v_3^2 = k^2 - k_{\beta_3}^2,
\]

Real \( v_3 > 0 \),

and \( C = \omega/k \), \quad (26)

and \( k \) is a horizontal wave number, \( C \) is horizontal phase velocity and \( A, B, C_1, D \) and \( E \) are constants to be determined.

It may be pointed out that while writing the solution of wave equation in the elastic half space, we have taken into consideration Sommerfeld [7] condition. Further we remark, that the form of \( V_2 \) given by (24) was chosen so that constants \( A, B, C_1, D \) and \( E \) have the same dimensions.

6. Boundary conditions and period equation

The boundary conditions [4] are the vanishing of normal and tangential stresses at the free surface \( z = 0 \), and continuity [2] of velocities and normal and tangential
stresses at the interface \( z = H_1 \) and \( z = H_1 + H_2 \). Following the notation used by Ewing et al. [4] for normal and tangential stresses, the boundary conditions are

\[ P_{xx} = 0 \text{ at } z = 0, \]

\[
egin{bmatrix}
[ P_{xx} ]_1 = [ P_{xx} ]_2 \\
\text{at } z = H_1,
\end{bmatrix}
\]

and

\[
egin{bmatrix}
[ P_{xx} ]_3 \equiv [ P_{xx} ]_3 \\
V_2 = i\omega v_2
\end{bmatrix}
\text{at } z = H_1 + H_2
\]

In the present case we have

\[ P_{xx} \text{ and } P_{xy} \text{ are identically zero,} \]

and \( P_{xy} = \mu (\partial \sigma / \partial z) \) for elastic solids

while \( P_{xx} = \eta (\partial V_2 / \partial z) \) for viscous fluid.

Using (27), (28), (29) and (30), we obtain

\[ \mu_1 v_1 B = 0, \]

\[ - \mu_1 v_1 A \sin v_1 H_1 + \mu_1 v_1 B \cos v_1 H_1 = i\omega \eta_2 \left[ - c_1 v_2 \sin v_2 H_1 + D v_2 \cos v_2 H_1 \right], \]

\[ A \cos v_1 H_1 + B \sin v_1 H_1 = C_1 \cos v_2 H_1 + D \sin v_2 H_1, \]

\[ i\omega \eta_2 \left[ - c_1 v_2 \sin v_2 (H_1 + H_2) + D v_2 \cos v_2 (H_1 + H_2) \right] = \mu_3 v_3 E \]

\[ \times \exp \left[ - \nu/3 (H_1 + H_2) \right], \]

\[ C_1 \cos v_2 (H_1 + H_2) + D \sin v_2 (H_1 + H_2) = E \exp \left[ - \nu_3 (H_1 + H_2) \right]. \]

Equation (31) are five homogeneous equations. In order to have values of \( A, B, C_1, D \) and \( E \) different from zero, we obtain

\[
\begin{vmatrix}
\mu_1 v_1 & 0 & 0 & 0 & 0 \\
0 & - \mu_1 v_1 & \text{sin } v_1 H_1 & \text{sin } v_2 H_1 & \cos v_2 H_1 \\
\text{cos } v_1 H_1 & \text{cos } v_2 H_1 & - \text{cos } v_2 H_1 & - \text{sin } v_2 H_1 & 0 \\
0 & 0 & - c_1 v_2 \sin v_2 (H_1 + H_2) & c_1 v_2 \cos v_2 (H_1 + H_2) & 0 \\
0 & 0 & \text{cos } v_2 (H_1 + H_2) & \text{sin } v_2 (H_1 + H_2) \cos v_2 (H_1 + H_2) & -1
\end{vmatrix}
\]

Expanding the determinant represented by (32), we obtain

\[ i\omega \eta_2 \left[ \mu_1 v_1 \tan v_1 H_1 - \mu_3 v_3 \right] \]

\[ + \left[ \mu_1 v_1 \mu_3 v_3 \tan v_1 H_1 - \omega^2 \eta_2^2 v_2^2 \right] \tan v_2 H_2 = 0. \]
Equation (33) represents the period equation for Love waves in a layered medium composed of a compressible, viscous, liquid layer, sandwiched between homogeneous, isotropic solid layer and homogeneous, isotropic, solid half space. In general (33) will admit complex roots.

Using (23), (24), (25) and (31), it can be shown that

\[ \nu_1 = A \cos \nu x \exp \left[ i(\omega t - kx) \right], \]

and

\[ \nu_3 = A \exp \left[ \nu_3 (H_1 + H_2 - z) + i(\omega t - kx) \cos \nu_1 H_1 \cos \nu_2 H_2 \right] \]

\[ + \left[ 1 + \frac{i\mu_1 \nu_1}{\omega \eta_2 \nu_2} \tan \nu_1 H_1 \tan \nu_2 H_2 \right]. \tag{34} \]

7. Discussion

We shall discuss the period (33) in the following limiting cases.

Thickness \(H_2\), and coefficient of viscosity \(\eta_2\) of the compressible liquid layer tend to zero separately in such a way such that the ratio of thickness to coefficient of viscosity is finite, i.e., \(H_2 \to 0, \eta_2 \to 0\), such that

\[ P = H_2/\eta_2 = \text{a constant.} \]

With these approximations, (33) reduces to

\[ [1 - (iP \nu_3)/\omega] \tan \nu_1 H_1 = \left( \eta_2 \nu_3/\nu_1 \right). \tag{35} \]

With the same approximation the displacements in the elastic layer and elastic half space, given by (34), reduce to

\[ \nu_1 = A \cos \nu x \exp \left[ i(\omega t - kx) \right], \]

and

\[ \nu_3 = A \exp \left[ -\nu_3 (z - H_1) + i(\omega t - kx) \right] \left[ 1 + \frac{i\mu_1 \nu_1 P}{\omega} \tan \nu_1 H_1 \right] \cos \nu_1 H_1. \tag{36} \]

It may be pointed out that for \(P = 0\) (thickness of liquid layer becomes zero for a finite coefficient of viscosity) equation (35) gives

\[ \tan \nu_1 H_1 = \left( \eta_2 \nu_3/\nu_1 \right). \tag{37} \]

which is the period equation [4] for Love waves for a case when a homogeneous isotropic solid layer is in welded contact with homogeneous, isotropic half space. Using (35) numerical values of phase and group velocity for first and second modes of Love waves as a function of \(\gamma = 2fH_1/\beta_1\) for the case \(\rho_0/\rho_1 = 1.0\) and \(\mu_0/\mu_1 = 1.25\) were computed and are shown in figure 2. When \(P\) tends to infinity (i.e., coefficient of viscosity tends to zero for a finite thickness of the liquid layer), equation (35) reduces to

\[ \tan \nu_1 H_1 = 0. \tag{38} \]
Equation (38) is the condition of existence of Love waves [6] in a homogeneous, isotropic solid layer overlying an inviscid fluid layer [4]. It may be pointed out that no Love wave energy is coupled to the lower half space, because a perfect fluid does not transmit Love waves. Using (38) numerical values of phase and group velocity for first and second modes of Love waves as a function of frequency have been calculated and are shown in figure 3. It may be noted, that in this case Love modes show inverse dispersion and the group velocity varies from 0 at long periods to \( \beta_1 \) at short periods.

Displacement \( v_1 \) and \( v_2 \) given by (36) were calculated for first and second modes for some values of \( P \) as a function of depth and are shown in figures 7 and 8. It is interesting to note that for \( P \) about 20, elastic solid layer decouples from the rest of the system under consideration. Figure 8 also shows that the node in displacement for second mode is dependant on the value of \( P \).

For all values of \( P \neq 0 \) and \( P \neq \infty \) (35) will have complex roots. We shall choose the complex values of \( k \) and \( C \) in such a way such that \( \omega \) is real. We introduce here quality factor \( Q \), which is defined as \( 1/Q = 2Uk^*/k,C \) [1] where \( k_r \) and \( k^* \) are real and imaginary parts of a complex wave number \( k \) while \( U \) and \( C \) are respectively the real parts of the complex group velocity and phase velocity. Equation (34) was solved numerically for phase velocity, group velocity and \( Q \) for
various values of $P$, as a function of the angular frequency for first and second modes of Love waves, assuming $\rho_0/\rho_1 = 1.0$ and $\mu_0/\mu_1 = 1.25$.

Figures 4, 5 and 6 show the variation of $Q$ for first and second modes of love waves as a function of non-dimensional angular frequency for various values of $P$. It may be pointed out that the value of $Q$ for a given $P$ first decreases, attains a minimum value and then increases as a function of the angular frequency. Table 1 shows the values of the real part of the phase and group velocity for first mode as a function of angular frequency for various values of $P$. It is noted from this table, that the value of group velocity is more than that of phase velocity for certain frequencies and for some $P$'s.
Figure 4. Variation of $Q$ as a function of non-dimensional frequency for first mode of Love waves for some values of the parameter $P$.

Figure 5. Variation of $Q$ as a function of non-dimensional frequency for second mode of Love waves for some values of the parameter $P$. 

\[ \gamma_1 = \frac{\omega H_1}{\beta_1} \]
Figure 6. Comparison of $Q$ for first and second mode of Love waves for some values of the parameter $P$. 
Figure 7. Variation of displacement for first mode of Love waves as a function of depth for some values of the parameter $P$ and $\omega H_1/\beta_1 = 6.1$. 

$\frac{z}{H_1}$

$\omega H_1/\beta_1 = 6.1$
Figure 8. Variation of displacement for second mode of Love waves as a function of depth for some values of the parameter $P$ and $\omega H_1/\beta_1 = 12.02$.

8. Conclusions

The period equation for Love waves is derived for a layered system composed of a compressible liquid layer sandwiched between homogeneous, isotropic, elastic solid layer and elastic half space. The period equation is solved for the case when thickness and coefficient of viscosity of the liquid layer tend to zero such that their ratio is held constant ($P = H_2/\eta_2 = 0$). It is shown that the case $P = 0$ corresponds to the case of solid elastic layer in welded contact with half space. When $P$ tends to infinity, it corresponds to the case in which the overlying elastic solid layer is not coupled to half space. It is inferred that for any layered model composed of a compressible, viscous liquid layer overlain by homogeneous isotropic elastic layers and underlain by homogeneous isotropic half space, Love waves will not only be dispersive but also are subject to attenuation as they propagate. This problem is of great relevance to compute the response of structures due to Love waves generated by crustal earthquakes.
Table 1. Variation of real part of phase and group velocity as a function of non-dimensional frequency and parameter \( P \) for first mode of Love waves for a model for which \( \rho_0/\rho_1 = 1.0, \mu_0/\mu_1 = 1.25. \)

<table>
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<th>( \omega H_k/\beta_1 )</th>
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<th>( P = 1.0 )</th>
<th>( P = 2.0 )</th>
<th>( P = 5.0 )</th>
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<td>( C/\beta_1 )</td>
<td>( U/\beta_1 )</td>
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Propagation and attenuation of Love waves.
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