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Temperature distribution in a laminar circular jet

S S TAK and J L BANSAL
Department of Mathematics, University of Jodhpur, Jodhpur 342 001

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Abstract. The effect of frictional heat on the temperature distribution in a laminar circular jet has been studied. It is found from the analysis and the graphs that as the Prandtl number decreases from unity the overall temperature difference near the axis of the jet increases but as we move away from the axis it goes on decreasing. The reverse phenomenon happens in the case of increasing Prandtl number.

Keywords. Jet flow; laminar boundary layer; forced convection.

1. Introduction

The velocity distribution in a laminar incompressible circular jet was obtained by Schlichting [5] and Bickley [1] and the corresponding temperature distribution, neglecting the heat due to dissipation, has been obtained by Yih [6]. The details of this may also be found in Loitsianski [2].

In the present paper we have extended the solution of Yih [6] to include the heat generated due to friction. A similar solution of the energy equation is being tried. The resulting nonhomogeneous linear differential equation is reduced to Gauss' hypergeometric equation by a proper transformation of the similarity variable, and the solution, for arbitrary values of the Prandtl number, is obtained.

2. Governing equations

Let an incompressible fluid be discharged through a circular orifice and then mixed with the same surrounding fluid which is at rest and has a temperature T_∞. Taking the origin in the orifice and the axes of coordinates are selected in such a way that the axis of x is the axis of the jet, and the radial distance is denoted by r. The axial and radial velocity components are denoted by u and v respectively. Owing to the assumption of a constant pressure, the boundary layer equations are (Loitsianski [2]).

$$\frac{\partial}{\partial x}(ur) + \frac{\partial}{\partial r}(vr) = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{2}$$

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial r}=\frac{\nu}{\sigma}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{\nu}{c_pT_\infty}\left(\frac{\partial u}{\partial r}\right)^2, \tag{3}$$

where

$$\theta=[(T-T_\infty)/T_\infty],\ \sigma=(\mu c_p/k),\ \text{(Prandtl number)}, \tag{4}$$

and the other symbols have their usual meanings. The boundary conditions are

$$r=0:\ (\partial u/\partial r)=0,\ v=0,\ (\partial\theta/\partial r)=0, \tag{5}$$

$$r=\infty:\quad u=0;\quad \theta=0\,.$$

Besides these boundary conditions, the following integral conditions should also be satisfied:

$$2\pi\rho\int_0^\infty ru^2dr=\text{const.}=J_0\ \text{(say)}, \tag{6}$$

and

$$\frac{d}{dx}\int_0^\infty ru\theta dr=\frac{\nu}{c_pT_\infty}\int_0^\infty r\left(\frac{\partial u}{\partial r}\right)^2dr. \tag{7}$$

The condition (6) is the known momentum conservation law and may also be deduced from (2) by integrating it with respect to r between the limits 0 to ∞, taking into account the continuity eq. (1) and the boundary condition (5) respectively. In a similar manner the condition (7) is obtained from the energy equation (3).

3. Analysis

An exact solution of the velocity distribution, which we shall require, may be found in Schlichting [5] and the results are as follows:

$$u=\frac{\nu a^2}{x}\frac{f'(\xi)}{\xi};\ v=\frac{\nu a}{x}\left[f'(\xi)-\frac{f(\xi)}{\xi}\right];\ \xi=ar/x,$$

where

$$f(\xi)=\frac{\xi^2}{1+(\xi^2/4)},\ a=\left[\frac{3J_0}{16\pi\rho\nu^2}\right]^{1/2} \tag{8}$$

and a prime denotes differentiation with respect to ξ.

It will be convenient to represent the general solution of the energy equation (3) by the superposition of two solutions of the form:

$$\theta=\theta_1+\theta_2, \tag{9}$$

where θ_1 is the general solution in the absence of frictional heat and θ_2, a particular solution, when the frictional heat is taken into consideration.

Therefore, θ_1 satisfies the equation

$$u\frac{\partial\theta_1}{\partial x}+v\frac{\partial\theta_1}{\partial r}=\frac{\nu}{\sigma}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta_1}{\partial r}\right), \tag{10}$$

with the boundary conditions:

$$r=0:\partial\theta_1/\partial r=0,\ \ r=\infty:\ \theta_1=0, \tag{11}$$

and the integral condition (7) becomes:

$$\frac{d}{dx}\int_0^\infty ru\theta_1 dr=0,$$

or

$$2\pi\int_0^\infty ru\theta_1 dr=\text{const.}=H_0/T_\infty\ \text{(say)}. \tag{12}$$

The solution of the above equation has already been obtained by Yih [6], the details of which may be found in Loitsianski [2], and the result is as follows:

$$\theta_1=\frac{C_0H_0}{\nu T_\infty x}\left(1+\frac{\xi^2}{4}\right)^{-2\sigma}, \tag{13}$$

where

$$C_0=(1+2\sigma)/8\pi. \tag{14}$$

Thus we have to determine the solution of θ_2, which satisfies the differential equation:

$$u\frac{\partial\theta_2}{\partial x}+v\frac{\partial\theta_2}{\partial r}=\frac{\nu}{\sigma}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta_2}{\partial r}\right)+\frac{\nu}{c_pT_\infty}\left(\frac{\partial u}{\partial r}\right)^2, \tag{15}$$

with the boundary conditions:

$$r=0:\frac{\partial\theta_2}{\partial r}=0,\quad r=\infty:\theta_2=0. \tag{16}$$

The integral condition (7), with $\theta=\theta_2$, will be identically satisfied.

Before we investigate the solution of (15) for arbitrary values of the Prandtl number σ, it may be noted that when $\sigma=1$,

$$\theta_2=-u^2/(2c_pT_\infty), \tag{17}$$

which is known as Crocco' integral. Now, for arbitrary values of Prandtl number; guided by the Crocco' integral; we postulate the following form for θ_2:

$$\theta_2=-\frac{16\nu^2\alpha^4}{c_pT_\infty}x^{-2}h(\xi). \tag{18}$$

Substituting (8), (18) and its derivatives in (15), we get

$$\xi h'' + h' + \sigma(2f' h + f h') - \frac{\sigma}{16\,\xi}\left(f'' - \frac{f'}{\xi}\right)^2 = 0, \tag{19}$$

where a prime continues to denote differentiation with respect to ξ.

Now, changing independent variable ξ to s defined by

$$s = f(\xi)/4 \tag{20}$$

equation (19) transforms to

$$s(1-s)\frac{d^2h}{ds^2} + \{1 - (1 - 2\sigma + 1)s\}\frac{dh}{ds} + 4\sigma h = \sigma s(1-s)^3, \tag{21}$$

with the boundary conditions:

$$s = 0 : h \text{ is finite}, \quad s = 1 : h = 0. \tag{22}$$

Equation (21) is the known nonhomogeneous Gauss' hypergeometric equation, whose solution is given by (Robert [4])

$$h = A_0\, {}_2F_1(\alpha, \beta; 1; s) + B_0\, [{}_2F_1(\alpha, \beta; 1; s) \log s + \sum_1^\infty K_n s^n] + h_p \tag{23}$$

where

$$K_n = \frac{(\alpha)_n(\beta)_n}{(n!)^2}\left\{\sum_{m=0}^{n-1}\left(\frac{1}{\alpha+m} + \frac{1}{\beta+m}\right) - \sum_{m=1}^{n}\frac{2}{m}\right\}, \tag{24}$$

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2), \ldots, (\alpha+n-1), \tag{25}$$

$$\alpha + \beta = 1 - 2\sigma, \quad \alpha\beta = -4\sigma.$$

A_0, B_0 are arbitrary constants and h_p is the particular integral to be determined.

The first boundary condition of (22) requires that coefficient of log s must be zero i.e. B_0=0. Hence

$$h = A_0\, {}_2F_1(\alpha, \beta; 1; s) + h_p. \tag{26}$$

(i) When Prandtl number σ has any value, except (1/3), (3/4), (5/3) and (6/5), the particular integral can easily be determined by the method of undetermined multipliers (Morris and Brown [3])

$$h_p = As^4 + Bs^3 + Cs^2 + Ds + E, \tag{27}$$

where

$$A = \frac{\sigma}{4(5-3\sigma)}, \qquad B = -\frac{\sigma(11-9\sigma)}{2(5-3\sigma)(6-5\sigma)},$$

$$C = \frac{3\sigma(30\sigma^2-59\sigma+27)}{4(3-4\sigma)(6-5\sigma)(5-3\sigma)}, \quad D = \frac{4C-\sigma}{2(1-3\sigma)}, \text{ and } E = (D/4\sigma). \qquad (28)$$

The second boundary condition gives the value of A_0 as:

$$A_0 = -(A+B+C+D+E)\frac{1}{{}_2F_1(\alpha, \beta; 1; 1)}. \qquad (29)$$

Hence the required solution is

$$h = (As^4+Bs^3+Cs^2+Ds+E)-(A+B+C+D+E)\frac{{}_2F_1(\alpha, \beta; 1; s)}{{}_2F_1(\alpha, \beta; 1; 1)}, \qquad (30)$$

where the constants A, B, C, D and E are given by (28).

(ii) When $\sigma = (1/3)$, by series solution method it can easily be found that

$$h = \frac{1}{48}s^4 - \frac{1}{13}s^3 + \frac{6}{65}s^2 + \left(1-\frac{4}{3}s\right)\left\{\frac{113}{1040} - \frac{21}{780}\,{}_3F_2\left(1, 1, \frac{10}{3}; 3, 3; 1\right)\right\}$$

$$-\frac{7}{780}s_3^2 F_2\left(1, 1, \frac{10}{3}; 3, 3; s\right). \qquad (31)$$

(iii) When $\sigma = (3/4)$, the solution comes out to be

$$h = \frac{3}{44}s^4 - \frac{17}{66}s^3 + \frac{3}{16}s^2 + \left(1 - 3s + \frac{15}{8}s^2\right)\left\{-\frac{1}{66}\right.$$

$$\left.+\frac{8}{132}\,{}_3F_2\left(1, 1, \frac{9}{2}; 4, 4; 1\right)\right\} + \frac{1}{132}s^3{}_3 F_2\left(1, 1, \frac{9}{2}; 4, 4; s\right). \qquad (32)$$

In a similar manner the solutions for $\sigma = (5/3)$ and $(6/5)$ can easily be obtained.

5. Numerical discussion

In figures 1 and 2 the function h, which is proportional to the temperature difference due to frictional heat, is plotted against the similarity variable ξ for various values of Prantdl number σ. The curve corresponding to $\sigma=1$ is the same as that from Crocco's integral (17).

It can be concluded from the analysis and the graphs that as the Prandtl number decreases from unity the overall temperature difference, which is proportional to $(\theta_1+\theta_2)$, near the axis of the jet increases but as we move away from the axis it goes on decreasing. The reverse phenomenon happens in the case of increasing Prandtl, number.

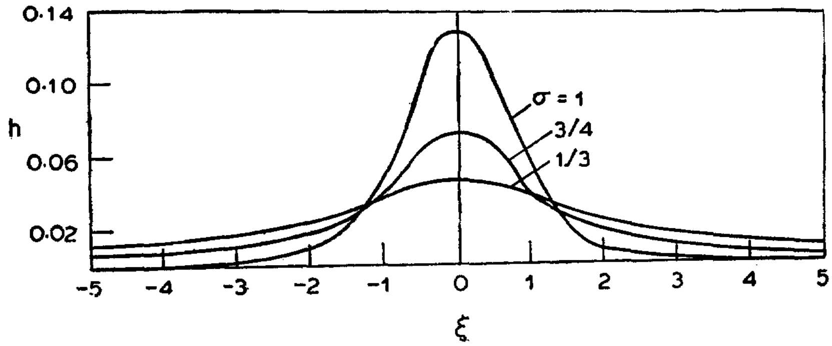

Figure 1. The function h, which is proportional to the temperature difference due to frictional heat, plotted against the similarity variable ξ ($\sigma < 1$)

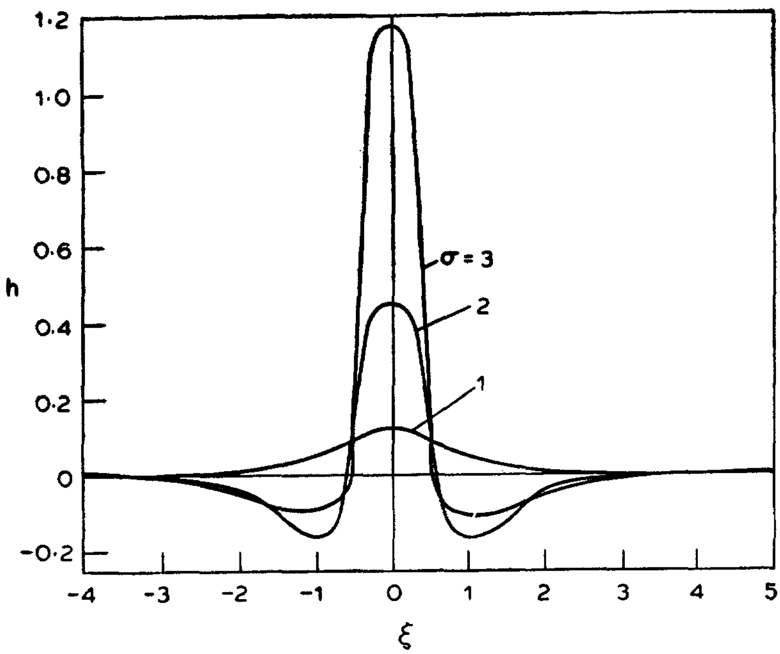

Figure 2. The function h, which is proportional to the temperature difference due to frictional heat, plotted against the similarity variable ξ ($\sigma > 1$)

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