Proc. Indian Acad. Sci., Vol. 87A (Mathematical Sciences-2), No. 5, May 1978, pp. 137–145,

Magneto-elastic torsional waves in a bar under initial stress

SURYA NARAIN
Department of Mathematics, Harish Chandra (P.G.) College, Varanasi 221 001

MS received 27 January 1976

Abstract. The magneto-elastic torsional wave in a bar under initial stress is studied. Most of the equations have been formulated on Biot's incremental deformation theory and the initial stress has been taken in the form of uniform tension. Two cases have been considered: first when the material of the rod is homogeneous and second when it is non-homogeneous. In both the cases frequency equations for the wave have been calculated.

Keywords. Stress components; strain components; displacement component.

1. Introduction

Kolsky [7] studied the lower mode of Pochhammer's frequency equation for torsional wave and the result was that it is non-dispersive and coincides with shear wave velocity. Davies and Owen [4] investigated the higher modes of torsional wave. Jones [6] pointed out the importance of higher modes near the wave front of a disturbance and in the velocity of the source. Clark [2] discussed the torsional wave propagation in a hollow cylindrical bar. Chakravorti [3] investigated the propagation of torsional wave in a perfectly conducting elastic cylinder and a tube under the influence of a uniform axial magnetic field. In this paper magneto-elastic torsional wave propagation in a solid rod of circular cross-section under initial stress has been discussed. Most of the equations have been formulated on Biot's incremental deformation theory. The initial stress is taken in the form of a uniform tension. Frequency equation for the phase velocity of torsional wave have been calculated. Two cases have been discussed: first when material of the rod is homogeneous and second when it is non-homogeneous. The non-homogeneity of the rod is due to the variable shear modulus μ and variable density ρ. Lastly frequency equations for the said wave have been calculated. Thus several papers come out as a particular case of our paper.

2. Basic equations

The problem being one of magnetoelasticity the basic equations are those of electromagnetism and of elasticity. The Maxwell equations governing the electro-magnetic field are:

$$\text{Curl } \mathbf{H} = 4\pi \mathbf{J} \quad , \quad \text{Curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\text{div } \mathbf{B} = 0 \quad , \quad \mathbf{B} = \mu_e \mathbf{H} \tag{1}$$

where the displacement current is neglected and Gaussian units have been used. Also, by Ohm's law,

$$\mathbf{J} = \sigma \left\{ \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} \right\}. \tag{2}$$

In eqs (1) and (2) **H**, **B**, **E**, **J** respectively denote the magnetic intensity, magnetic induction, electric intensity and current density vectors; μ_e and σ respectively denote magnetic permeability and electrical conductivity of the body, **u** represents displacement vector in the strained state and c is the velocity of light.

In the cartesian coordinate system if a body is under initial stress $\bar{\sigma}_{33}$ along z-direction only, the equilibrium equations satisfied by incremental stresses S_{ij} and initial stresses are (Biot [1]).

$$\begin{aligned}
&\frac{\partial}{\partial x} S_{11} + \frac{\partial}{\partial y} S_{12} + \frac{\partial}{\partial z} S_{31} + \bar{\sigma}_{33} \frac{\partial \omega_y}{\partial z} + F_x = \rho \frac{\partial^2 u}{\partial t^2} \\
&\frac{\partial}{\partial x} S_{12} + \frac{\partial}{\partial y} S_{22} + \frac{\partial}{\partial z} S_{23} - \bar{\sigma}_{33} \frac{\partial \omega_x}{\partial z} + F_y = \rho \frac{\partial^2 v}{\partial t^2} \\
&\frac{\partial}{\partial x} S_{31} + \frac{\partial}{\partial y} S_{23} + \frac{\partial}{\partial z} S_{33} + \bar{\sigma}_{33} \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) + F_z = \rho \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{3}$$

where u, v, w are displacement components in incremental stage and

$$\begin{aligned}
\omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \\
\omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right).
\end{aligned} \tag{4}$$

These equilibrium equations in cylindrical polar coordinates are given by,

$$\begin{aligned}
&\frac{\partial}{\partial r} \sigma_{rr} + \frac{1}{r} \frac{\partial}{\partial \theta} \sigma_{r\theta} + \frac{\partial}{\partial z} \sigma_{rz} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \bar{\sigma}_{33} \frac{\partial \omega_\theta}{\partial z} + F_r = \rho \frac{\partial^2 u_r}{\partial t^2} \\
&\frac{\partial}{\partial r} \sigma_{r\theta} + \frac{1}{r} \frac{\partial}{\partial \theta} \sigma_{\theta\theta} + \frac{\partial}{\partial z} \sigma_{\theta z} + \frac{2\sigma_{r\theta}}{r} - \bar{\sigma}_{33} \frac{\partial \omega_r}{\partial z} + F_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2} \\
&\frac{\partial}{\partial r} \sigma_{rz} + \frac{1}{r} \frac{\partial}{\partial \theta} \sigma_{\theta z} + \frac{\partial}{\partial z} \sigma_{zz} + \frac{\sigma_{rz}}{r} + \bar{\sigma}_{33} \left(\frac{\partial \omega_\theta}{\partial r} - \frac{1}{r} \frac{\partial \omega_r}{\partial \theta} \right) + F_z = \rho \frac{\partial^2 u_z}{\partial t^2}
\end{aligned} \tag{5}$$

where σ_{ij} are incremental stress components along rotated directions of the axes and

$$\begin{aligned}
\omega_r &= \frac{1}{2}\left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \\
\omega_\theta &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right)
\end{aligned} \tag{6}$$

where $\mathbf{u} = (u_r, u_\theta, u_z)$ and $\bar{\sigma}_{33}$ is the initial stress along original z-directions. Assuming that the stresses and displacements are independent of θ, the equilibrium equation takes the form,

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{\partial}{\partial z}\sigma_{rz} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\bar{\sigma}_{33}}{2}\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{\partial^2 u_z}{\partial r \partial z}\right) + F_r = \rho\frac{\partial^2 u_r}{\partial t^2}$$

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \frac{2\sigma_{r\theta}}{r} + \bar{\sigma}_{33}\frac{\partial^2 v_\theta}{\partial z^2} + F_\theta = \rho\frac{\partial^2 u_\theta}{\partial t^2} \tag{7}$$

$$\frac{\partial}{\partial r}\sigma_{rz} + \frac{\partial}{\partial z}\sigma_{zz} + \frac{\sigma_{rz}}{r} + \frac{\bar{\sigma}_{33}}{2}\left(\frac{\partial^2 u_r}{\partial r \partial z} - \frac{\partial^2 u_z}{\partial r^2}\right) + F_z = \rho\frac{\partial^2 u_z}{\partial t^2}$$

where $\mathbf{F} = [F_r, F_\theta, F_z]$ is the Lorentz force per unit volume due to the axial magnetic field and is given by,

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \tag{8}$$

The electromagnetic field equations in vacuum are,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}^* = 0 \tag{9}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{h}^* = 0 \tag{10}$$

$$\text{Curl } \mathbf{E}^* = -\frac{1}{c}\frac{\partial \mathbf{h}^*}{\partial t} \tag{11}$$

$$\text{Curl } \mathbf{h}^* = \frac{1}{c}\frac{\partial \mathbf{E}^*}{\partial t} \tag{12}$$

where $\mathbf{h}^*$ is the perturbation in the magnetic field in vacuum and $\mathbf{E}^*$ is the electric field in vacuum. Moreover, the incremental boundary forces per unit area may be written as

$$\begin{aligned}
\triangle fr &= \sigma_{rr} n_r + \sigma_{r\theta} n_\theta + \sigma_{zr} n_z + \bar{\sigma}_{33}\,\omega_\theta\, n_r \\
\triangle f\theta &= \sigma_{\theta r} n_r + \sigma_{\theta\theta} n_\theta + \sigma_{\theta z} n_z - \bar{\sigma}_{33}\,\omega_r\, n_z \\
\triangle fz &= \sigma_{zr} n_r + \sigma_{\theta z} n_\theta + \sigma_{zz} n_z + \bar{\sigma}_{33}(e_{rr} + e_{\theta\theta}) n_z \\
&\quad - \bar{\sigma}_{33}\, e_{zr}\, n_r - \bar{\sigma}_{33}\, e_{\theta z}\, n_\theta
\end{aligned} \tag{13}$$

where n_i is the cosine of the angle between normal to the deformed boundary and ith direction before incremental deformation. In case of torsional vibration, $u_r = u_z = 0$ and $u_\theta = v$.

Hence

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{zr} = 0. \tag{14}$$

Consequently, the incremental boundary forces may be written as,

$$\begin{aligned} \triangle fr &= 0 \\ \triangle f\theta &= \sigma_{rz} + \frac{\bar{\sigma}_{33}}{2}\frac{\partial \omega_\theta}{\partial z} \\ \triangle fz &= 0 \end{aligned} \tag{15}$$

3. Consideration of the problem and method of solution

Let us consider a semi infinite solid cylindrical rod of radius under initial stress which is a tension P along the axis of the rod and suppose that the elastic properties of the rod are symmetrical about z-axis. The torsional wave is then characterized by the incremental displacement.

$$u_r = 0,\ u_z = 0 \text{ and } u_\theta = v(r, z) \tag{16}$$

and hence,

$$\begin{aligned} &e_{rr} = 0,\ e_{\theta\theta} = 0,\ e_{zz} = 0,\ e_{zr} = 0 \\ &2e_{\theta z} = -\frac{\partial v}{\partial z},\ 2e_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} \\ &2\omega_r = -\frac{\partial v}{\partial z},\ 2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r} \end{aligned} \tag{17}$$

where e_{ij} are incremental strain components and ω_i are rotational components respectively. Now, let us suppose that

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h} \tag{18}$$

where $\mathbf{H}_0$ is the initial magnetic field acting parallel to z-axis and $\mathbf{h}$ is the small perturbation in the field. If the rod considered is a perfect conductor of electricity (i.e. $\sigma \to \infty$), eq. (2) gives,

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} = \left[-\frac{H}{c}\frac{\partial v}{\partial t},\ 0,\ 0\right] \tag{19}$$

where $H = |\mathbf{H}_0|$.

Eliminating **E** from (1) and (19) we get,

$$\mathbf{h} = \left[0,\ H\frac{\partial v}{\partial z},\ 0\right]. \tag{20}$$

From eqs (1) and (20) we have

$$\mathbf{J} \times \mathbf{B} = \mathbf{J} \times \mu_e \mathbf{H}_0 = \left[0,\ -\frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2},\ 0\right]. \tag{21}$$

Considering eqs (14), (16) and (17) we see that two equations of motion are identically satisfied and the remaining one takes the form,

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \frac{2}{r}\sigma_{r\theta} + \frac{P}{2}\frac{\partial^2 v}{\partial z^2} - \frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{22}$$

Also, the stress-strain relations are,

$$\sigma_{r\theta} = 2\,\mu e_{r\theta},\ \sigma_{\theta z} = \mu\, e_{\theta z}. \tag{23}$$

Case I—When the material of the rod is homogeneous

In this case eq. (22) with the help of (17) and (23) takes the form,

$$\mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \left(\mu + \frac{P}{2} - \frac{H^2}{4\pi}\right)\frac{\partial^2 v}{\partial z^2} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{24}$$

Considering the harmonic wave along z-axis, we take the solution of (24) as,

$$v = V(r)\exp\left[i\,(qz + pt)\right] \tag{25}$$

then eq. (24) reduces to,

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \left(K^2 - \frac{1}{r^2}\right)V = 0 \tag{26}$$

where

$$K^2 = \frac{\rho p^2}{\mu} - \frac{q^2\left(\mu + \frac{P}{2} - \frac{H^2}{4\pi}\right)}{\mu}. \tag{27}$$

Putting $R = Kr$ in eq. (26) we have,

$$\frac{d^2 V}{dR^2} + \frac{1}{R}\frac{dV}{dR} + \left(1 - \frac{1}{R^2}\right)V = 0. \tag{28}$$

The solution (28) is given by

$$V = A\,J_1\,(Kr) \tag{29}$$

where A is constant and J_1 is Bessel's function of first kind and first order. Therefore the solution of (24) is

$$v = A\,J_1\,(Kr)\exp\left[i\,(qz + pt)\right]. \tag{30}$$

Suppose the boundary $r = a$ of the rod which separates the rod from vacuum be free from stresses. Then the required boundary conditions are

$$\sigma_{r\theta} + T_{r\theta} = T^*_{r\theta} \text{ on } r = a$$

$$E = E^* \text{ on } r = a \tag{31}$$

where $T_{r\theta}$ and $T^*_{r\theta}$ are Maxwell stresses in the body and in vacuum respectively. We can easily verify that (Chakravorty [3])

$$T_{r\theta} = T^*_{r\theta} = 0. \tag{32}$$

Equation (31) together with (23) yields $e_{r_\theta} = 0$ which with the help of (17) and (25) gives,

$$\frac{\partial}{\partial r}\left\{\frac{J_1\,(Kr)}{r}\right\} = 0 \text{ at } r = a. \tag{33}$$

On simplification this gives

$$\frac{J_0\,(Ka)}{J_1\,(Ka)} = \frac{2}{Ka}. \tag{34}$$

Equation (34) has multiple roots; the first three being $Ka = 0,\ 5{\cdot}136,\ 8{\cdot}418$. If R_1 stands for any one of the roots then from eq. (27) and (34) we have

$$\frac{C_1}{\beta_1} = \left\{\frac{R_1^2\left(\dfrac{\lambda}{a}\right)^2}{4\pi^2\left(1 + G - \dfrac{H^2}{4\pi\mu}\right)} + 1\right\} \tag{35}$$

where $G = P/2\mu$, λ the wavelength $= 2\pi/q$, c_1 the phase velocity $= p/q$ and

$$\beta_1 = \left\{\frac{\mu + \dfrac{P}{2} - \dfrac{H^2}{4\pi}}{\rho}\right\}^{1/2} \tag{36}$$

is the velocity of transverse wave under initial tension P along the direction of propagation under the influence of magnetic field of intensity **H**. Equation (35) gives the relation between phase velocity and wavelength of torsional wave for various values of R_1, G and H. The presence of G and H in eq. (35) shows the effect of initial stress and magnetic field respectively on phase velocity. If there were no magnetic field then,

$$\frac{C_2}{\beta_2} = \left\{ \frac{R_1^2\left(\frac{\lambda}{a}\right)^2}{4\,\pi^2\,(1 + G)} + 1 \right\}^{1/2} \tag{37}$$

where

$$\beta_2 = \left\{ \frac{\mu + \frac{P}{2}}{P} \right\}^{1/2} \tag{38}$$

which coincides with the results of Dey (5). In eqs (37) and (38) if $G = 0$ and $H = 0$, then the result coincides with the classical case of Kolsky [7].

The first root $R_1 = 0$ gives $c_1 = \beta_1$ i.e. the velocity of torsional wave coincides with shear wave velocity and propagation is non-dispersive. It is easy to verify that for other values of R_1 dispersion takes place and c_1 increases along with R_1 for fixed values of λ, a, G and H. Writing eq. (35) in the form

$$\frac{C_1}{\beta_1} = \left\{ \frac{R_1^2\left(\frac{\lambda}{a}\right)^2}{4\pi^2} + 1 + G - \frac{H^2}{4\,\pi} \right\}^{1/2} \tag{39}$$

where $\beta_1 = \sqrt{(\mu/\rho)}$. This shows that c_1 increases with P. Also, from eq. (35) we see that c_1 is zero if,

$$G = -1 - \frac{R_1^2\left(\frac{\lambda}{a}\right)^2}{4\pi^2} + \frac{H^2}{4\pi\mu}. \tag{40}$$

This indicates that torsional wave does not propagate in the medium when takes the value given by eq. (40). This happens when the material is unstable or at the point of collapse.

Case II—When the material of the rod is non-homogeneous.

We assume

$$\mu = \mu_0 r^2$$

$$\rho = \rho_0 r^2 \tag{41}$$

where μ_0, ρ_0 are constants and r is the radius vector. Thus eq. (22) with the help of (17), (23) and (41) takes the form,

$$\frac{\partial^2 v}{\partial r^2}+\frac{3}{r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)+\left\{1+\left(\frac{P}{2\mu_0}-\frac{H^2}{4\pi\mu_0}\right)\frac{1}{r^2}\right\}\frac{\partial^2 v}{\partial z^2}=\frac{\rho_0}{\mu_0}\frac{\partial^2 v}{\partial t^2}. \tag{42}$$

Considering the harmonic wave along z-axis, we take the solution of (42) as,

$$v=V_1(r)\exp\left[i\left(q_1 z+p_1 t\right)\right] \tag{43}$$

then eq. (42) reduces to,

$$\frac{d^2 V_1}{dr^2}\,\frac{3}{r}\frac{dV_1}{dr}+\left(\lambda-\frac{\mu^2}{r^2}\right)V_1=0 \tag{44}$$

where

$$\lambda^2=\frac{\rho_0^2\, p_1^2}{\mu_0}-q_1^2$$

$$\mu^2=\frac{Pq_1^2}{2\,\mu_0}-\frac{H^2\, q_1^2}{4\pi\mu_0}+3.$$

Putting

$$V_1=\frac{1}{r}\psi(r) \tag{45}$$

in (44) we obtain

$$\frac{d^2\psi}{d\,r^2}-\frac{1}{r}\frac{d\psi}{dr}+\left(\lambda^2-\frac{\nu^2}{r^2}\right)\psi=0 \tag{46}$$

where

$$\nu^2=\mu_0^2+1. \tag{47}$$

The solution of eq. (46) may be given by

$$\psi(r)=A_1\, J_\nu(\lambda r)+B_1\, Y_\nu(\lambda r) \tag{48}$$

where A_1, B_1 are constants and J_ν, Y_ν are Bessel functions of first and second kind. Using eqs (43) and (45) we obtain

$$v=\frac{1}{r}\left\{A_1\, J_\nu(\lambda r)+B_1\, Y_\nu(\lambda r)\ \exp\left[i\left(q_1 z+p_1 t\right)\right]\right\}. \tag{49}$$

Equation (31) together with (23) yields $e_{r\theta} = 0$ which with the help of (17) and (49) gives

$$\lambda r\, J_{\nu-1}(\lambda r) - (\lambda\nu + 2)\, J_\nu(\lambda r) = 0 \text{ at } r = a. \tag{50}$$

On simplification we obtain

$$\frac{J_{\nu-1}(\lambda a)}{J_\nu(\lambda a)} = \frac{\lambda\nu + 2}{\lambda a}. \tag{51}$$

Equation (51) has multiple roots. If R_2 stands for any one of the roots then from eqs (45) and (51) we have,

$$\frac{C_2}{\beta_2} = \left[\mu \left\{ 1 + \frac{\lambda \left(\left(\frac{R_2}{a}\right)^2 - \mu^2 - 3 \right)}{4\pi^2 \left(1 - G_1 - \frac{H^2 a^2}{4\pi\mu} \right)} \right\} \right]^{1/2} \tag{52}$$

where $G = P/2\mu$, λ the wave length $= 2\pi/q$, $c_2 =$ the phase velocity $= p_1/q_1$, and

$$\beta_2 = \frac{a\left(1 - G - \frac{H^2 a^2}{4\pi\mu}\right)^{1/2}}{\rho} \tag{53}$$

is the velocity of the transverse wave under initial tension P along the direction of propagation under the influence of magnetic field of intensity **H**. Equation (52) gives the relation between phase velocity and wavelength of torsional wave for various values of R_2, G_2 and H. The rest of the discussion regarding this equation are exactly similar to the previous case.

References

[1] Biot M A 1965 *Mechanics of Incremental Deformation* (New York: John Wiley)

[2] Clark S K 1956 *J. Acoust. Soc. Am.* **6** 28

[3] Chakravorti *A* 1969 *Z. Angew. Math. Mech.* **49** 717

[4] Davies R M and Owen J D 1950 *Proc. R. Soc.* **A17** 204

[5] Dey S 1972 *Pure Appl. Geophys.* **94** 53

[6] Jones R P N 1959 *Q. J. Mech. Appl. Math.* **12** 8

[7] Kolsky H 1953 *Stress Waves in Solids* (London: Oxford Univ. Press)