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Laminar compressible boundary layer flow at a three-dimensional stagnation point with vectored mass transfer

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Abstract. The effect of surface mass transfer velocities having normal, principal and transverse direction components ('vectored' suction and injection) on the steady, laminar, compressible boundary layer at a three-dimensional stagnation point has been investigated both for nodal and saddle points of attachment. The similarity solutions of the boundary layer equations were obtained numerically by the method of parametric differentiation. The principal and transverse direction surface mass transfer velocities significantly affect the skin friction (both in the principal and transverse directions) and the heat transfer. Also the inadequacy of assuming a linear viscosity-temperature relation at low-wall temperatures is shown.

Keywords. Skin friction; heat transfer; vectored mass transfer; noddle and saddle points; variable fluid properties: parametric differentiation.

List of Symbols

a, b	= velocity gradients in x and y directions, respectively
c	= ratio of velocity gradients, b/a
C_{f_x}, C_{f_y}	= skin-friction coefficients along x and y directions, respectively
f, s	= dimenionless stream functions such that $f'=u/u_e$ and $s'=v/v_e$
f_w	= mass transfer parameter, $-(\rho w)_w/(\rho_e\mu_e a)^{1/2}$
g	= dimensionless enthalpy, h/h_e
g_w	= cooling parameter for the wall, h_w/h_e
h	= enthalpy
Pr	= Prandtl number
q	= heat transfer rate
Re_x	= local Reynolds number, $u_c x/\nu_e$
St	= Stanton number
T	= temperature
u, v, w	= velocity components along x, y, z directions, respectively
x, y, z	= principal, transverse and normal directions, respectively
η	= similarity variable, $(\rho_e a/\mu_e)^{1/2} \int_0^z (\rho/\rho_e)dz$
μ	= coefficient of viscosity
ν	= kinematic viscosity
ρ	= density
τ_x, τ_y	= dimensional shear stress functions
ω	= exponent in the power-law variation of viscosity

Superscript

(prime) = differentiation with respect to η

Subscripts

e = condition at the edge of the boundary layer

w = condition at the surface $z=\eta=0$.

1. Introduction

With the advent of high-speed flight vehicles, an important aspect to be considered in the study of boundary layers are surface mass transfer effects involving normal and tangential components for two-dimensional or axisymmetric bodies and normal, principal and transverse direction components for three-dimensional bodies. These are called 'vectored' suction or injection and finds application in many situations, such as, (1) oblating thermal protection systems of lifting re-entry vehicles, (2) in studies on the effect of velocity slip and temperature jump on flow characteristics in rarefied flows, (3) film and transpiration cooling of turbine blades, etc. For two-dimensional and axisymmetric bodies vectored mass transfer, involving only the normal component, that is, suction and/or injection has been discussed extensively in the past. In recent years, the other case has also been studied as boundary layer flow over continuous moving solid surfaces (Sakiadis [11]; Bourne and Elliston [2]; Ackroyd [1]; Williams and Johnson [15]; Libby [6]). Inger and Swean [4] and Scala and Sutton [12] have studied the effects of vectored suction and injecton (i.e., taking into account both the normal and tangential components) on laminar compressible boundary layer flow of a gas with constant properties ($\rho\mu$=constant, Pr=1 and 0·7) at a constant pressure, and at an axisymmetric stagnation point, respectively. Nath and Muthanna [8] extended the analysis of Scala and Sutton [7] to include the effects of variable gas properties ($\rho \propto T^{-1}$, $\mu \propto T^{\omega}$, Pr=0·7). In particular, they have studied the effects of vectored suction and injection on the flow and heat transfer at the stagnation point of a two-dimensional body (a cylinder) and an axisymmetric body (a sphere).

In this paper, we have investigated the effect of surface mass transfer velocities having normal, principal and transverse direction components (vectored suction and injection) on the steady, laminar compressible boundary layer flow of a gas with variable properties at a three-dimensional stagnation point both for nodal and saddle points of attachment. The governing equations were solved by the method of parametric differentiation. The foregoing problem for a stationary wall ($f'_w = s'_w = 0$) has been studied by Libby [5] assuming constant gas properties and further by Wortman *et al* [16] and Vimala and Nath [14] who employed variable gas properties. It may be remarked that the studies made by Libby [5], Wortman *et al* [16] and Vimala and Nath [14] are only a particular case of the present problem.

2. Governing equations

The governing equations in dimensionless form for steady laminar hypersonic flow of a gas with variable properties (i.e., $\rho \propto T^{-1}$, $\mu \propto T^{\omega}$, Pr=0·7) in the neighbourhood of the stagnation point of a three-dimensional body, taking into account the effects of 'vectored' suction or injection are (Libby [5]; Wortman *et al* [16]; Vimala and Nath [14])

$$f''' + (\omega - 1) g' f'' g^{-1} + [(f + cs) f'' + g - f'^2] g^{1-\omega} = 0 \quad (1a)$$

$$s''' + (\omega - 1) g' s'' g^{-1} + [(f + cs) s'' + c (g - s'^2)] g^{1-\omega} = 0 \quad (1b)$$

$$g'' + (\omega - 1) g'^2 g^{-1} + Pr (f + cs) g' g^{1-\omega} = 0. \quad (1c)$$

The appropriate boundary conditions for vectored mass transfer are

$$f(0) = f_w, \; f'(0) = u_w/u_e = f'_w, \; f'(\infty) \to 1 \quad (2a)$$

$$s(0) = 0, \; s'(0) = v_w/v_e = s'_w, \; s'(\infty) \to 1 \quad (2b)$$

$$g(0) = g_w, \; g(\infty) \to 1. \quad (2c)$$

Here $f_w \gtrless 0$ according to whether there is suction or injection. We remark that $f_w > 0$, $f'_w > 0$, $s'_w \gtrless 0$ corresponds to downstream vectored suction, $f_w < 0$, $f'_w > 0$, $s'_w \gtrless 0$, to downstream vectored injection, $f_w > 0$, $f'_w < 0$, $s'_w \gtrless 0$ to upstream vectored suction and $f_w < 0$, $f'_w < 0$, $s'_w \gtrless 0$ to upstream vectored injection. It may be noted that $\omega = 0{\cdot}5$ corresponds to conditions encountered in hypersonic flight, $\omega = 0{\cdot}7$ corresponds to low-temperature flows and $\omega = 1$ represents the constant density viscosity product simplification which has been widely used in studying boundary layer flows (Gross and Dewey [3]). We have taken the Prandtl number to be constant, since its variation in boundary layer, for most atmospheric flight problems, is quite small (Wortman *et al* [16]). It is to be mentioned that most shapes of practical interest range from sphere ($c = 1$) to cylinder ($c = 0$) and the saddle shapes ($-1 < c < 0$) are included in the analysis for the sake of completeness (Wortman *et al* [16]).

The skin friction coefficients along x and y directions are given by (Libby[5]).

$$C_{f_x} = 2\tau_x/\rho_e u_e^2 = 2 (Re_x)^{-1/2} \bar{F}''(0) \quad (3a)$$

$$C_{f_y} = 2\tau_y/\rho_e u_e^2 = 2 (Re_x)^{-1/2} (v_e/u_e) \bar{S}''(0). \quad (3b)$$

Similarly, the heat transfer coefficient in terms of Stanton number can be expressed as (Libby [5]).

$$St = q_w/[(h_e - h_w) \rho_e u_e] = (Re_x)^{-1/2} \bar{G}'(0) \quad (4)$$

where

$$\bar{F}''(0) = g_w^{\omega-1} f''(0), \quad \bar{S}''(0) = g_w^{\omega-1} s''(0) \quad (5a)$$

$$\bar{G}'(0) = g_w^{\omega-1} g'(0)/[(Pr(1 - g_w)]. \quad (5b)$$

3. Results and discussion

The governing equations (1*a*–1*c*) under the boundary conditions (2*a*–2*c*) were solved by the method of parametric differentiation for various values of f_w, f'_w ($f'_w \geqslant 0$), s'_w ($s'_w \geqslant 0$), ω, c and g_w. We have not taken into account upstream vectoring because we have investigated the flow in the stagnation region. The detailed description of the method of parametric differentiation along with its application to various problems is given by Rubbert [9], Rubbert and Landahl [10]. Tan and DiBiano [13] and Na and Turski [7]; hence it is not described here. The starting solutions.for carrying out computations, corresponding to $f'_w = s'_w = 0$ for various values of f_w, ω, c and g_w, were obtained from Libby [5]. Runge-Kutta-Gill method was employed for numerical integration of the differential equations, taking step size 0·05 and the edge of the boundary layer to be between 4·0 and 6·0. It was observed that a further reduction in step size affected the results only in the fourth decimal place. Convergence was considered to have been achieved when the wall parameters $f''(0)$, $s''(0)$ and $g'(0)$ in two successive iterates agreed within 10^{-4}.

Though, numerical solutions were obtained for various values of the parameters, for the sake of brevity, only some representative results are presented in the following figures. The velocity, enthalpy, shear-stress and heat-transfer profiles for a saddle point of attachment ($c = -0{\cdot}5$), with vectored suction and injection are presented in figures 1–3. The profiles corresponding to a nodal point of attachment ($c \geqslant 0$) have not been shown for lack of space.* It is observed that for given f'_w and s'_w, the effect of suction ($f_w > 0$) is to make the velocity and enthalpy profiles more steep, whatever may be the value of ω and c. The velocity and enthalpy profiles when $\omega = 0{\cdot}5$ possess a point of inflexion as is evidenced by a maximum in $f''(\eta)$, $s''(\eta)$ and $g'(\eta)$ whatever may be the value of c or f_w. When $\omega = 1$, the profiles have a point of inflexion only for injection ($f_w < 0$). The effect of downstream vectored suction ($f_w > 0$, $f'_w > 0$, $s'_w > 0$) is to increase the skin friction profiles (both in the principal and transverse di-

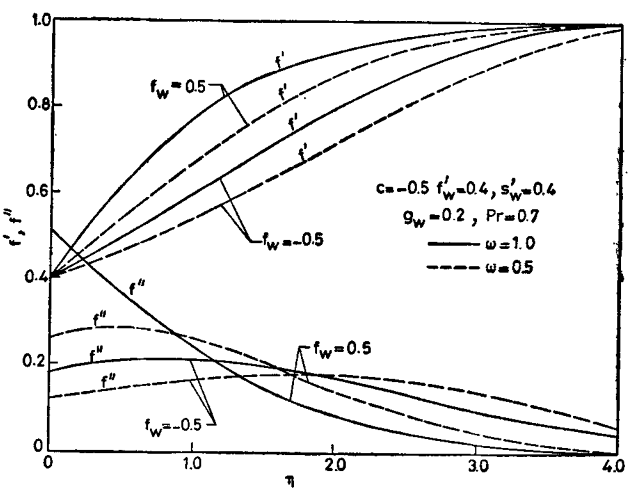

Figure 1. Variation of f' and f'' with η.

*The profiles may be obtained from the authors.

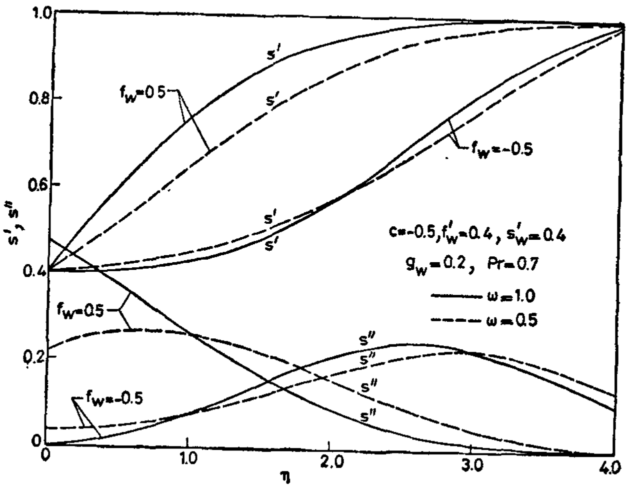

Figure 2. Variation of s' and s'' with η.

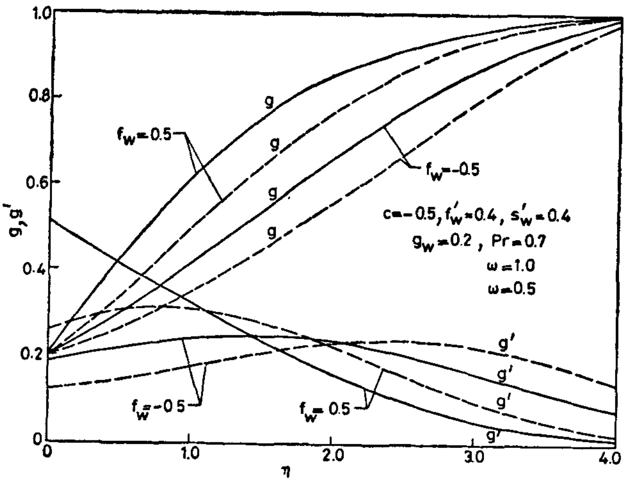

Figure 3. Variation of g and g' with η.

rections) and heat transverse profile while vectored injection ($f_w<0, f'_w>0, s'_w>0$) does the reverse. An interesting feature observed is that when $c=-0{\cdot}5$, the transverse velocity s' is influenced significantly by mass transfer. With injection, the transverse skin friction at the surface is very small and compared to suction, the difference between the $\omega=1$ and $\omega=0{\cdot}5$ profiles both for s' and s'' is comparatively less.

Figures 4–6 show the effect of f_w (the normal component of mass transfer) on the skin friction parameters $\bar{F}''(0)$ and $\bar{S}''(0)$ and heat transfer parameter $\bar{G}'(0)$ for various values of c. We see that as f_w increases $\bar{F}''(0)$ and $\bar{G}'(0)$ exhibit some interesting features. As c decreases from zero ($c<0$), the parameters $\bar{F}''(0)$ and $\bar{G}'(0)$ decrease, till a critical negative value of c is reached, after which these parameters start increasing. But this trend is not observed in the $\bar{S}''(0)$ profile. Figures 7–9 show the effect of the surface mass transfer velocity f'_w in the principal direction on the skin friction parameters $\bar{F}''(0)$ and $\bar{S}''(0)$ and heat transfer parameter $\bar{G}'(0)$. The effect of increasing f'_w results in a decrease in $\bar{F}''(0)$ while $\bar{S}''(0)$ and $\bar{G}'(0)$ increase for fixed values of c, f_w, s'_w and g_w. The decrease in $\bar{F}''(0)$ as f'_w increases is more at higher values of

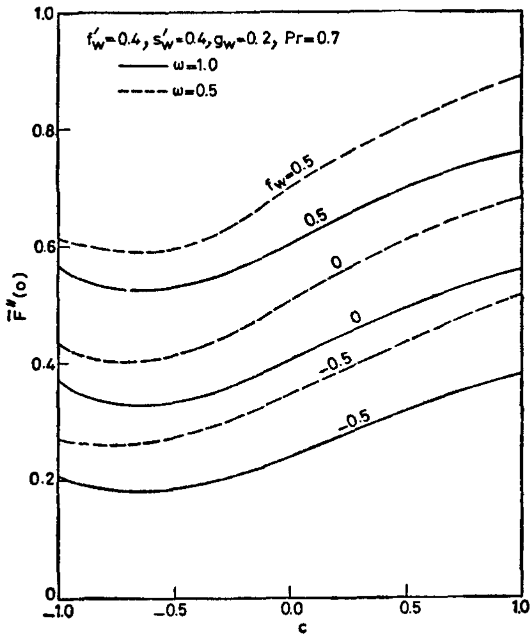

Figure 4. Effect of f_w on the variation of $\bar{F}''(0)$ with c.

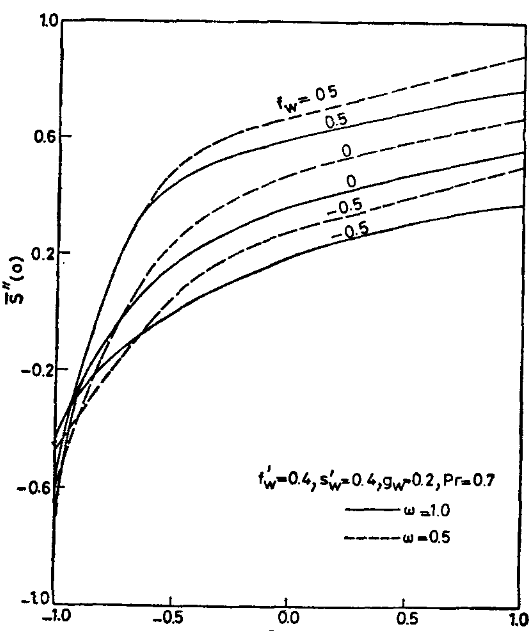

Figure 5. Effect of f_w on the variation of $\bar{S}''(0)$ with c.

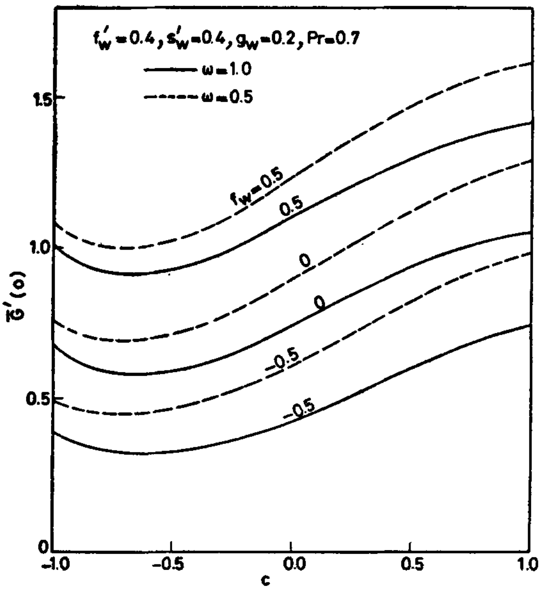

Figure 6. Effect of f_w on the variation of $\bar{G}'(0)$ with c.

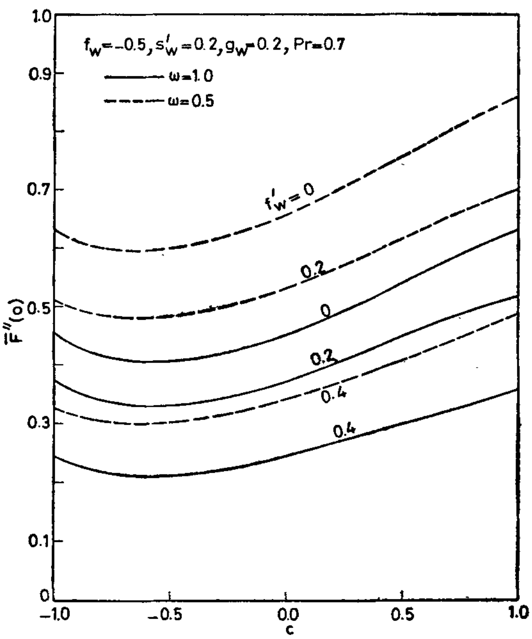

Figure 7. Effect of f'_w on the variation of $\bar{F}''(0)$ with c.

$\bar{F}''(0)$. As observed before, $\bar{F}''(0)$ and $\bar{G}'(0)$ decrease as c decreases till at some critical negative value of c, they start increasing. The physical reason for this behaviour is given by Wortman *et al* [16] hence it is not repeated here. Finally, figures 10–12 show the effect of s'_w, the surface mass transfer velocity in the transverse direction

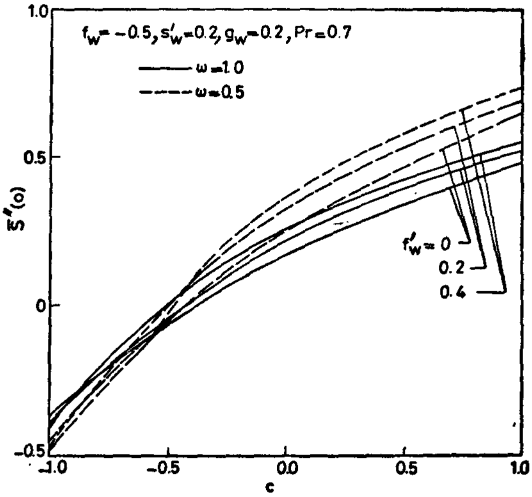

Figure 8. Effect of f'_w on the variation of $\bar{S}''(0)$ with c.

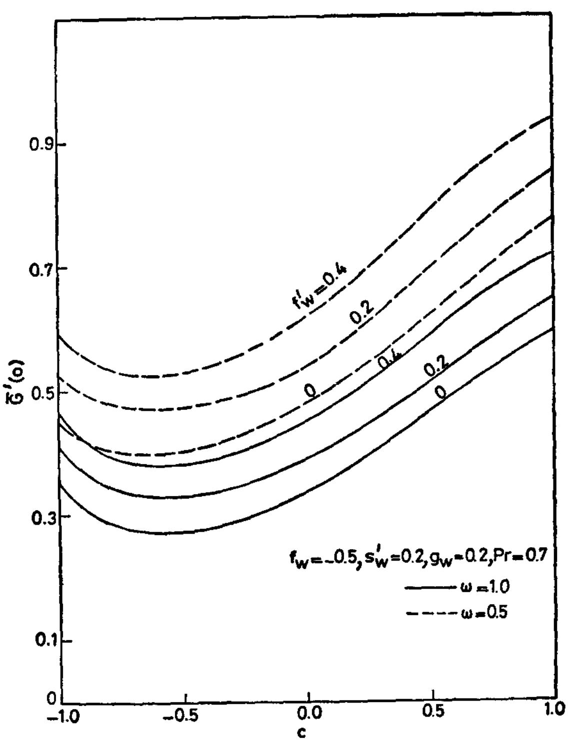

Figure 9. Effect of f'_w on the variation of $\bar{G}'(0)$ with c.

on the flow parameters. An increase in s'_w results in a decrease in $\bar{S}''(0)$ but $\bar{F}''(0)$ and $\bar{G}'(0)$ increase or decrease depending on the values of c. As expected, we observe that s'_w does not affect $\bar{F}''(0)$ and $\bar{G}'(0)$ at $c = 0$. Further, we find that for nodal points of attachment ($c > 0$) the effect of increasing s'_w results in an increase

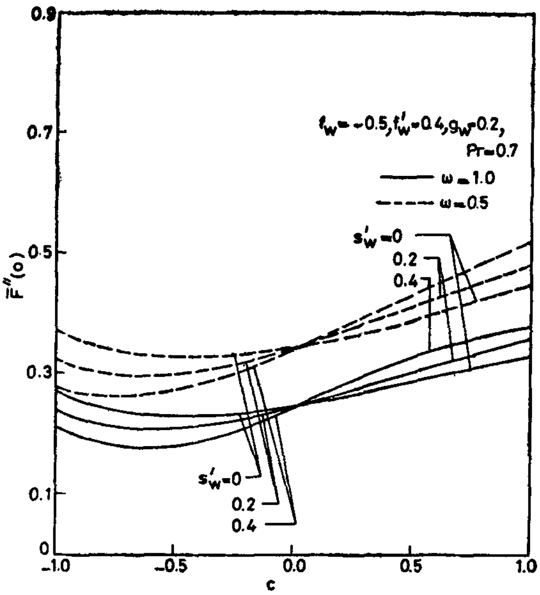

Figure 10. Effect of s'_w on the variation of $\bar{F}''(0)$ with c.

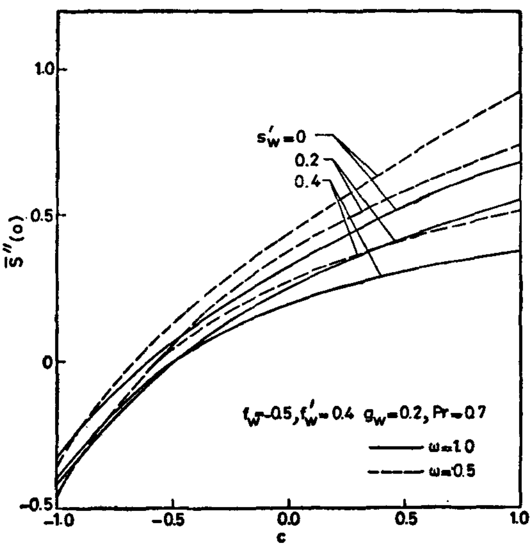

Figure 11. Effect of s'_w on the variation of $\bar{S}''(0)$ with c.

in $\bar{F}''(0)$ and $\bar{G}'(0)$, but at saddle points of attachment ($c < 0$) the reverse takes place. This is true, whatever may be the value of f_w, ω or f'_w.

On observing figures 4–12, we find that when $\omega = 0{\cdot}5$, we obtain higher values for $\bar{F}''(0)$, $\bar{S}''(0)$ and $\bar{G}'(0)$. In some cases (figures 4–9) the difference between $\omega = 1$ and $\omega = 0{\cdot}5$ profiles is quite significant. This further confirms the statement that the linear viscosity-temperature relation ($\omega = 1$) is not a good relation for obtaining an accurate estimation of the skin friction and heat transfer. We have compared

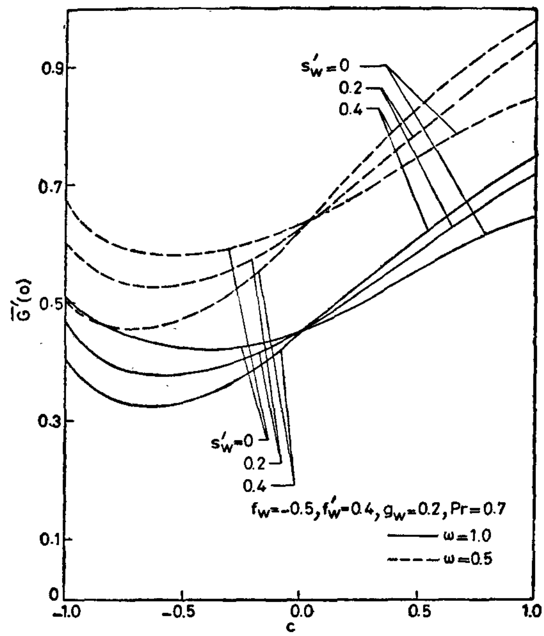

Figure 12. Effect of s'_w on the variation of $\overline{G}'$ (0) with c.

our results for $f'_w = s'_w = 0$ with those of Libby [5], Wortman *et al* [16] and Vimala and Nath [14] and found them in good agreement.

4. Concluding remarks

The effect of vectored suction or injection on the skin friction and heat transfer parameters is significant and they are increased due to vectored suction while vectored injection does the reverse. The skin-friction and heat-transfer parameters are strongly dependent on the nature of the stagnation point. The effect of the variation of the density-viscosity product across the boundary layer is to increase skin friction and heat transfer and this variation gives rise to a point of inflexion in velocity and enthalpy profiles.

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