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Subsonic and supersonic disturbances in the generation of surface waves in a liquid layer adjacent to a high-speed gas-stream

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Abstract. In an effort to shed further light upon the nature of " supersonic " disturbances as distinct from that of 'subsonic' disturbances in parallel compressible flows, this paper makes an investigation of the stability characteristics of the surface waves generated in a liquid layer adjacent to a high-speed gas-stream. It turns out that the nature of the surface waves generated in the liquid layer depends markedly upon the type of disturbances present in the high-speed gas-stream. For the case of 'subsonic' disturbances it is shown that the energy transfer from the gas stream to the surface waves is contributed predominantly by the Fourier component of the normal gas-pressure force-field in phase with the slope of the wavy surface. For the case of 'supersonic' disturbances, this energy transfer is shown to be predominantly due to the component of the pressure-field in phase with the surface-wave displacement and is related to the presence of travelling periodic waves in the gas-stream—this energy transfer is shown to promote always the growth of the surface waves.

Keywords. Subsonic disturbance;supersonic disturbance; surface waves.

1. Introduction

Contemporary literature shows that the precise role of what are defined as the 'supersonic' disturbances in the context of the stability of parallel compressible flows is not entirely clear, while they are supposed sometimes, as Miles [5] from the study of a perturbed vortex sheet in a compressible fluid found, to impose further restrictions upon the sufficient conditions for stability. The present author [6] carried out a phase-plane analysis of the linearised inviscid pressure disturbances in parallel compressible flows and found that while a given basic flow executes a unique set of 'subsonic' oscillations, no uniqueness exist in executing supersonic oscillations. In an effort to shed further light upon the nature of supersonic disturbances, this paper makes an investigation of the stability characteristics of the surface waves generated in a liquid layer adjacent to a high-speed gas-stream—a problem which is of interest, for instance, in the flight of reentry vehicles in earth's atmosphere. In this direction, we consider a simple wave train that is stationary relative to the surface waves in the liquid layer superposed on the incoming high-speed gas-stream, and then investigate the conditions under which the reaction of the disturbed flow is just sufficient to sustain the surface waves. This, in particular, involves the prediction of the energy transfer from the high-speed gas-stream to the surface waves. The liquid is assumed to be incompressible and inviscid, and the gas to be compressible and inviscid. A justification of this inviscid model for the determination of a first approximation to the disturbed motion of the high-speed gas-stream and the consequent energy transfer to the surface waves in the liquid layer comes from the fact that the predominant force field at the interface is due to the pressure field of the high-speed gas-stream which is not affected by the presence of the viscous boundary layer and which is considerably larger than the viscous frictional force at the interface. In the following we shall avoid the question of formulating the mechanism by which the surface-wave energy is dissipated.

It turns out that the nature of the surface waves generated in the liquid layer depends significantly on the type of disturbances present in the high-speed gas-stream. For the case of subsonic disturbances, it is shown that the energy transfer from the gas stream to the surface waves is contributed predominantly by the Fourier component of the normal gas-pressure force field in phase with the slope of the wavy surface. (This appears to be similar to the so-called " sheltering " mechanism proposed by Jefreys [1, 2] in the context of the generation of surface waves in water by a wind blowing over it, which is based on a flow-separation from leeward-side of each wave on the surface so that there is a pressure difference between the windward- and leeward side of each crest, and if the wind speed exceeds the wave speed, energy is thereby transferred from the air to the water). In calculating this energy transfer, use is made of the fact that this energy transfer is primarily determined by the conditions at the critical point where, $u(y)=c$, $u(y)$ being the gas speed, since it is known that the flexure of the interface causes the wave train to travel in the flow direction with a speed c, and the critical point may then occur away from the interface—an important discovery due to Miles [4] in the context of the generation of surface waves in water by a wind blowing over it. For the case of supersonic disturbances, this energy transfer is shown to be predominantly due to the component of the pressure-field in phase with the surface-wave displacement and is related to the presence of travelling periodic waves in the gas stream—this energy transfer is shown to promote always the growth of the surface waves.

2. Surface-wave generation by subsonic disturbances

Consider an initially stationary liquid of infinite depth in the region $y<0$, and a high-speed gas-stream in the x-direction past the liquid surface through the region $y>0$. The high-speed gas-stream is taken to be parallel, i.e., the flow properties are considered to vary negligibly over distances comparable with the wavelength, and in the direction of the surface waves in the liquid layer so that the disturbances due to the wavy interface given by

$$y_0 = a \exp [i\alpha (x-ct)] \tag{1}$$

are two-dimensional and strictly periodic in the x-coordinate. In the linearised theory for the disturbances that we develop here, one ignores the flow-separation from the wave-crests and requires the surface wave-amplitude to be small in comparison with the wavelength and also that the surface curvature be sufficiently small. In the present case, the motion of the liquid below the interface is neglected.

Then the perturbations in the flow-properties—longitudinal—and transverse-velocity components u and v, pressure p, density ρ, and temperature T are considered to be of the form exp $[i\alpha(x-ct)]$ times an amplitude function of y, where c is the complex wave velocity $c=c_r+ic_i$, t being the time. Under the approximations of parallel flows, the equations of continuity, motion, energy, and state yield on reduction five linear differential equations for the amplitudes of the linearised disturbances (see Lees and Lin [3]):

$$\rho\,(u-c)\,\gamma'+\rho\,(\phi'+if)+\rho'\phi=0 \tag{2}$$

$$\rho\,[i\,(u-c)\,f+u'\phi]=-\frac{i\pi}{\gamma M_1^2} \tag{3}$$

$$\alpha^2\rho\,[i\,(u-c)\,\phi]=-\frac{\pi'}{\gamma M_1^2} \tag{4}$$

$$\rho\,[i\,(u-c)\,\theta+T\phi']=-\,(\gamma-1)\,\rho T\,(\phi'+if) \tag{5}$$

$$\frac{\pi}{p}=\frac{\gamma'}{\rho}+\frac{\theta}{T} \tag{6}$$

where we have considered a perfect gas with constant ratio of specific heats γ. The disturbances are defined by

$$\hat{u},\ \hat{v},\ \hat{p},\ \hat{\rho},\ \hat{T}=(f,\ \alpha\phi,\ \pi,\ \gamma',\ \theta)\exp\,[i\alpha\,(x-ct)] \tag{7}$$

and the nondimensional parameters are

$$\gamma=\frac{C_p}{C_v},\ M_1^2=\frac{V_\infty^2}{\gamma RT_\infty} \tag{8}$$

where the subscript ∞ denotes the conditions on the gas stream at infinity, R is the gas constant, and the primes denote differentiation with respect to y.

The linearised boundary condition at the interface is

$$y=0:\phi=0 \tag{9a}$$

and the conditions of boundedness at infinity (valid for subsonic disturbances) are

$$y\to\infty:f,\ \phi,\ \pi,\ \gamma',\ \theta\to 0 \tag{9b}$$

One deduces from eqs (2)–(6),

$$\frac{d}{dy}\left[\frac{(u-c)\phi'-u'\phi}{T-M_1^2\,(u-c)^2}\right]-\frac{\alpha^2(u-c)}{T}\phi=0. \tag{10}$$

The disturbances in the gas stream are then classified as subsonic, sonic, or supersonic according as the phase velocity in the direction of the gas stream is, in magnitude, below, equal to, or above the local speed of sound, i.e.,

$$\begin{array}{lll} \text{subsonic disturbances} & : & M_1^2\,(u-c)^2<T \\ \text{sonic disturbances} & : & M_1^2\,(u-c)^2=T \\ \text{supersonic disturbances} & : & M_1^2\,(u-c)^2>T \end{array} \tag{11}$$

From eq. (4), one finds

$$\pi(0) = -\int_0^\infty \pi' dy = \gamma M_1^2 \int_0^\infty \frac{ia^2(u-c)}{T}\,\phi\,dy \tag{12}$$

which implies that the pressure distribution at the interface is governed by the cumulative action of the disturbance over the whole flow field, and is affected very little by the phenomena near the interface. Using eq. (10), one obtains

$$p_s = \pi(0) = -\frac{i\gamma M_1^2(u-c)}{T-M_1^2(u-c)^2}\,\phi'. \tag{13}$$

Since the work done by the pressure forces on the wavy interface is

$$c \int p_s \frac{\partial y_0}{\partial x}\,dx$$

it follows that in the absence of viscosity, the rate of energy transfer from the gas stream to unit area of the wavy interface is proportional to $-\mathrm{Im}\{p_s\}$. The growth of the surface waves in the liquid layer is then supposed to occur when the energy supply exceeds the rate of energy dissipation by viscosity in the fluid below the interface.

Note from eq. (13)

$$-\mathrm{Im}\{p_s\} \simeq \frac{\gamma M_1^2\,c_i}{T-M_1^2(u-c)^2}\,\mathrm{Im}\,\{\,\varphi^1\,\} \tag{14}$$

Next multiply eq. (10) by ϕ^*, the complex conjugate of ϕ, subtract from the resulting equation its complex conjugate, and then there follows

$$\frac{d}{dy}(\phi'\phi^*-\phi^{*\prime}\phi) - \left[\frac{T'-2M_1^2(u-c)u'}{T-M_1^2\,(u-c)^2}\right]\phi'\phi^* + \left[\frac{T'-2M_1^2(u-c)u'}{T-M_1^2(u-c)^2}\right]\phi^{*\prime}\phi$$

$$+\,(c-c^*)\left[\frac{\dfrac{T'-2M_1^2(u-c)u'}{T-M_1^2(u-c)^2}\,u'-u''}{|\,u-c\,|^2}\right]|\,\phi\,|^2=0 \tag{15}$$

where we have considered only slightly amplified surface waves.

Introducing the 'Wronskian',

$$W = \frac{i}{2}\,\frac{(\phi\phi^{*\prime}-\phi'\phi^*)}{T-M_1^2(u-c)^2} \tag{16}$$

eq. (15) becomes

$$\frac{dW}{dy} - c_i \left[\frac{\left\{ \frac{u'}{T - M_1^2 (u-c)^2} \right\}'}{|u-c|^2} \right] |\phi|^2 = 0. \tag{17}$$

Letting $c_i \to 0$, one obtains

$$W = \text{const}$$

except possibly at the critical point y_c, which now lies in the real interval $(0, \infty)$, and W has a discontinuity at that point. Integrate eq. (17) from y_c^- to y_c^+, then

$$W(y_c^+) - W(y_c^-) = [W]_c$$

$$= \int_{u_c^-}^{u_c^+} \frac{\left\{ \frac{u'}{T - M_1^2 (u-c)^2} \right\}' |\phi|^2 c_i \, du}{u' \left\{ (u-c_r)^2 + c_i^2 \right\}}. \tag{18}$$

A straightforward calculation gives (see Shivamoggi [7]),

$$W(y_c^+) - W(y_c^-) = [W]_c = \pi \frac{(u'/T)'_{u=c}}{u'_c} |\phi_c|^2 \tag{19}$$

provided in this limit, $c_i > 0$, $u_c^+ > u_c^-$. Since $\phi \to 0$ for $y \to \infty$, $W \equiv 0$, for $y > y_c$, so that

$$W(y) = -[W]_c, \quad y < y_c \tag{20}$$

Noting that $\phi = [u(y) - c]$ is a solution of eq. (10) in the limit $\alpha \to 0$, one finds for the neighbourhood of the interface

$$W(y) = \frac{i}{2} \frac{(u-c)}{T - M_1^2 (u-c)^2} [\phi^{*\prime} - \phi']$$

or

$$W(y) = \frac{(u-c)}{T - M_1^2 (u-c)^2} \operatorname{Im} \{\phi'\} \tag{21}$$

Therefore from eq. (14), (19)–(21), there follows as $c_i \to 0$, that

$$-\operatorname{Im} \{p_s\} \simeq -\pi \gamma M_1^2 \frac{(u'/T)'_{u=c}}{u'_c} |\phi_c|^2 \tag{22}$$

which shows that in the absence of viscosity the energy from the high-speed gas-stream through the subsonic disturbances is positive (such as to promote the growth of the surface waves in the liquid layer) when

$$(u'/T)'_{u=c} < 0 \tag{23}$$

3. Surface-wave generation by supersonic disturbances

In this case the gas flowing past the liquid surface is assumed to have a uniform velocity U and uniform temperature T. Noting that for the case of supersonic disturbances, one has periodic waves, it follows from eq. (13) that the pressure exerted by the gas at the interface is equal to

$$\triangle p_g(0) = \left[\frac{\rho_g \, i\gamma M_1^2 \, a \, \alpha (U-c)}{\left\{ \frac{M_1^2 (U-c)^2}{T} - 1 \right\}^{1/2}} \right] \exp [i\alpha \, x - ct]. \tag{24}$$

The linearised equations of motion of the liquid are

$$\rho_l \frac{\partial \mathbf{q}_l}{\partial t} = - \nabla (\triangle p_l) \tag{25}$$

$$\nabla \cdot \mathbf{q}_l = 0 \tag{26}$$

where $\mathbf{q}_l$ is the liquid velocity. The boundary conditions are,

(1) condition at infinity : $|\mathbf{q}_l| \to 0$ as $y \to -\infty$ (27)
(2) kinematic interfacial condition : $y = 0$: $\partial y_0 / \partial t = v_l$ (28)
(3) dynamic interfacial condition: $\triangle p_g = \triangle p_l$. (29)

The solutions to eqs (25), (26), subject to conditions (27), (28) are

$$u_l = a\alpha c \exp(\alpha y) \exp[i\alpha (x - ct)]$$

$$v_l = - i a\alpha c \exp(\alpha y) \exp[i\alpha (x - ct)]$$

$$\triangle p_l = \rho_l \, a\alpha c^2 \exp(\alpha y) \exp[i\alpha (x - ct)] \tag{30}$$

so that (29) gives

$$c^2 = \frac{\rho_g}{\rho_l} \frac{i\gamma M_1^2 (U-c)}{\{M_1^2 [(U-c^2)/T] - 1\}^{1/2}} \tag{31}$$

from which it is clear that for supersonic disturbances

$$\mathrm{Im}\{c\} > 0 \tag{32}$$

so that there always appears to be an energy transfer from the gas stream through the supersonic disturbances to the surface waves leading to their amplification—in significant contrast to the situation associated with the subsonic disturbances.

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