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Nonsimilar incompressible laminar boundary layers with magnetic field

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Abstract. The solution of the steady laminar incompressible nonsimilar magnetohydrodynamic boundary layer flow and heat transfer problem with viscous dissipation for electrically conducting fluids over two-dimensional and axisymmetric bodies with pressure gradient and magnetic field has been presented. The partial differential equations governing the flow have been solved numerically using an implicit finite-difference scheme. The computations have been carried out for flow over a cylinder and a sphere. The results indicate that the magnetic field tends to delay or prevent separation. The heat transfer strongly depends on the viscous dissipation parameter. When the dissipation parameter is positive (i.e. when the temperature of the wall is greater than the freestream temperature) and exceeds a certain value, the hot wall ceases to be cooled by the stream of cooler air because the 'heat cushion' provided by the frictional heat prevents cooling whereas the effect of the magnetic field is to remove the 'heat cushion' so that the wall continues to be cooled. The results are found to be in good agreement with those of the local similarity and local nonsimilarity methods except near the point of separation, but they are in excellent agreement with those of the difference-differential technique even near the point of separation.

Keywords. Nonsimilar; viscous dissipation; cylinder; sphere; finite difference scheme; skin friction; heat transfer.

1. Introduction

Most laminar boundary layer problems with or without magnetic field are nonsimilar. The nonsimilarity in these cases is due to various factors such as the freestream velocity distribution, transverse curvature etc. The nonsimilar problems are governed by partial differential equations. Since partial differential equations are comparatively difficult to solve, most investigators have solved the problem by employing approximate techniques such as similarity analysis, integral methods, series solutions, local similarity and local nonsimilarity methods (Dewey and Gross 1967; Sparrow *et al* 1970, Sparrow and Yu 1971; Davies 1963; Heiser and Bornhorst 1966). These techniques under suitable approximations allow the governing partial differential equations to be reduced to ordinary differential equations which are then solved numerically or analytically. Various aspects of the steady laminar incompressible boundary layer flow problem with magnetic field for two-dimensional bodies with or without pressure gradient have been considered by several authors (Davies 1963; Heiser and Bornhorst 1966; Rossow 1957; Cess 1960; Sherman 1961; Glauert 1961; Hildyard 1972) using one of the above-mentioned mathematical techniques. To the authors' knowledge, the solutions of the magnetohydrodynamic problems with viscous dissipation over two-dimensional and axisymmetric bodies using finite-difference schemes have not been reported in the literature.

In this paper, we have studied the nonsimilar laminar steady incompressible two-dimensional and axisymmetric magnetohydrodynamic boundary-layer flow and heat transfer problem with viscous dissipation for electrically conducting fluids with magnetic field and pressure gradient. The partial differential equations governing the flow were first transformed to a co-ordinate system having a finite range and then solved by using an implicit finite-difference scheme (Marvin and Sheaffer 1969; Vimala and Nath 1975). Computations were carried out for two problems, namely, cylinder in cross flow and the flow over a sphere. The results have been compared with those of local similarity, local nonsimilarity, and difference-differential methods (Sparrow *et al* 1970; Sparrow and Yu 1971; Terrill 1960; Smith and Clutter 1963).

2. Governing equations

We consider steady laminar incompressible electrically conducting fluid flow past a two-dimensional or an axisymmetric surface with an applied magnetic field. It is assumed that the fluid has constant physical properties. The magnetic Reynolds number is assumed to be small. Hence, the induced magnetic field is negligible in comparison to the applied magnetic field. We also assume that there is no external applied electric field. The Hall current and displacement current effects have also been neglected. For these conditions, the boundary layer equations with constant freestream temperature in dimensionless form governing the nonsimilar flow taking into account the effects of the viscous dissipation and Joule heating can be expressed as (Rossow 1957; Cess 1960).

$$f''' + ff'' + \beta(\xi)(1-f'^2) + \beta_1(\xi)M(1-f') = 2\xi(f'f_\xi' - f''f_\xi) \tag{1}$$

$$g'' + Pr\, fg' - Br\,(U/U_\infty)^2[f''^2 + \beta_1(\xi)M(f'^2-1)]$$
$$= 2\,Pr\xi(f'\,g_\xi - g'f_\xi) \tag{2}$$

with boundary conditions

$$f(\xi, 0) = f'(\xi, 0) = g(\xi, 0) = 0, f'(\xi, \infty) = g(\xi, \infty) = 1 \tag{3}$$

where

$$\xi = \int_0^x [U(x)/U_\infty)(r/L)^{2J}d(x/L),\ \eta = y(r/L)^J U(x)/(2\nu L U_\infty \xi)^{1/2} \tag{4a}$$

$$\psi(x, y) = (2\nu\, LU_\infty\xi)^{1/2} f(\xi, \eta),\ u = (L/r)^J\psi_y,\ v = -(L/r)^J\psi_x \tag{4b}$$

$$g = (T-T_w)/(T_\infty - T_w),\ M = Hm^2/Re = \sigma B_0^2 L/(\rho U_\infty) \tag{4c}$$

$$\beta(\xi) = 2(L/r)^{2J}[(dU/dx)/U^2] \int_0^x U(r/L)^{2J}\, dx \tag{4d}$$

$$\beta_1(\xi) = 2\xi(U_\infty/U)^2(L/r)^{2J},\ Br = \mu U_\infty^2/[K(T_w - T_\infty)]. \tag{4e}$$

Here ξ, η are the transformed co-ordinates; ψ and f are the dimensional and dimensionless stream functions, respectively; g is the dimensionless temperature; β is the pressure gradient parameter; β_1 is a function depending on the streamwise distance ξ; M is magnetic parameter; Hm is the Hartmann number, $Re(=U_\infty L/\nu)$ is the Reynold number; Br is the dissipation parameter (Brinkman number); T is the temperature; B_0 is the applied magnetic field along y direction; σ is the electrical conductivity; ρ is the density; L is the characteristic length; x and y are distances along and perpendicular to the surface, respectively; u and v are the velocity components along x and y directions, respectively; μ and ν are the coefficient of viscosity and kinematic viscosity respectively; K is the thermal conductivity; r is the radius of revolution of an axisymmetric body; U is the potential velocity at the edge of the boundary layer; Pr is the Prandtl number; $j=0$ for a two-dimensional body and $j=1$ for an axisymmetric body; the subscripts w and ∞ denote wall and freestream values, respectively; the subscripts x, y and ξ denote derivatives with respect to x, y and ξ, respectively and prime denotes differentiation with respect to η. It may be noted that eqs (1) and (2) reduce to classical boundary-layer equations when $M=0$.

The dimensionless skin-friction and heat-transfer coefficients are given by (Sparrow 1970).

$$C_f(\bar{Re})^{1/2}=2(r/L)^j\,[xU/(2\xi LU_\infty)]^{1/2}f''_w \tag{5a}$$

$$Nu(\bar{Re})^{-1/2}=(r/L)^j\,[xU/(2\xi LU_\infty)]^{1/2}\,g'_w \tag{5b}$$

with

$$C_f=2\tau_w/\rho U^2,\; Nu=x(\partial T/\partial y)_w/(T_\infty-T_w),\; \bar{Re}=Ux/\nu \tag{5c}$$

where C_f is the skin-friction coefficient, Nu is the Nusselt number, τ_w is the shear stress at the wall and $\bar{Re}$ is the local Reynolds number.

3. Transformation to finite co-ordinates

We transform eqs (1) and (2) to a new system of co-ordinates wherein the infinite range of integration (0, ∞) on η is replaced by a finite range (0, 1) on z which is more convenient from the viewpoint of computation. The transformation is given by (Marvin and Sheaffer 1969).

$$z=1-\exp(-\alpha\eta) \tag{6}$$

where the scaling factor α provides an optimum distribution of nodal points across the boundary layer. We define $f'=F$ and $\alpha(1-z)=\bar{\eta}$ and change the variable η to z. Consequently, eqs (1) and (2) reduce to

$$\begin{aligned}&\bar{\eta}^2F_{zz}+\bar{\eta}(f-\alpha)F_z+\beta(\xi)(1-F^2)+\beta_1(\xi)M(1-F)\\&\quad=2\xi(FF_\xi-\bar{\eta}f_\xi F_z)\end{aligned} \tag{7}$$

$$\bar{\eta}^2 g_{zz}+\bar{\eta}(Pr\, f-a)g_z-Br(U/U_\infty)^2\,[\bar{\eta}^2 F_z^2+\beta_1(\xi)M(F^2-1)]$$
$$=2Pr\;\xi\,(F\,g_\xi-\bar{\eta}\,f_\xi\,g_z). \tag{8}$$

The boundary conditions (3) can now be written as

$$F(\xi,0)=g(\xi,0)=0;\;F(\xi,1)=g(\xi,1)=1. \tag{9}$$

The solution of the partial differential eqs (7) and (8) under conditions (9), which govern the nonsimilar flow past an arbitrary two-dimensional or axisymmetric body can be obtained by using an implicit finite-difference scheme provided (U/U_∞), $\beta(\xi)$ and $\beta_1(\xi)$ (which depend on the shape of the body) are prescribed. Since the method is described in full detail in Marvin and Sheaffer (1969) and Vimala and Nath (1975), for the sake of brevity, it is not repeated here. In particular, we have solved the above equations for the case of a cylinder and a sphere.

4. Cylinder in cross flow

We consider the flow of an electrically conducting fluid over a circular cylinder with an applied magnetic field where the nonsimilarity arises due to the external velocity distribution at the edge of the boundary layer. This velocity is given by the potential theory

$$U/U_\infty=2\sin\bar{x},\;\bar{x}=x/R \tag{10a}$$

where the pressure is determined by

$$-\rho^{-1}(\partial p/\partial x)=U(dU/dx)+\sigma B_0^2(u-U)/\rho. \tag{10b}$$

Here p is the pressure; R and $\bar{x}$ are the radius of the cylinder and dimensionless distance, respectively. We now take R as the reference length instead of L and put $j=0$. Consequently, eqs (4a, d, e) and (5) using (10a) reduce to (Sparrow *et al* 1970)

$$\xi=2(1-\cos\bar{x}),\;\beta(\bar{x})=2\cos\bar{x}/(1+\cos\bar{x}),\;\beta_1(\bar{x})=1-\beta(\bar{x})/2 \tag{11a}$$

$$C_f(\bar{Re})^{1/2}=[2\,\bar{x}\sin\bar{x}/(1-\cos\bar{x})]^{1/2}f''_w \tag{11b}$$

$$\mathrm{Nu}(\bar{Re})^{-1/2}=[(\bar{x}/2)\sin\bar{x}/(1-\cos\bar{x})]^{1/2}\,g'_w. \tag{11c}$$

Equations (7) and (8) with (10a) and (11) under conditions (9) were solved for various values of M and Br using the implicit finite-difference scheme mentioned earlier. The step sizes $\triangle z=0{\cdot}005$ and $\triangle\bar{x}=0{\cdot}05$ were used for computational purposes and further reduction in them produced changes only in the fourth significant place. But in the neighbourhood of the point of separation $\triangle z=0{\cdot}0001$ and $\triangle\bar{x}=0{\cdot}01$ were used. The scaling factor a was taken as 0·5.

The velocity and temperature profiles are shown in figures 1 and 2. The velocity

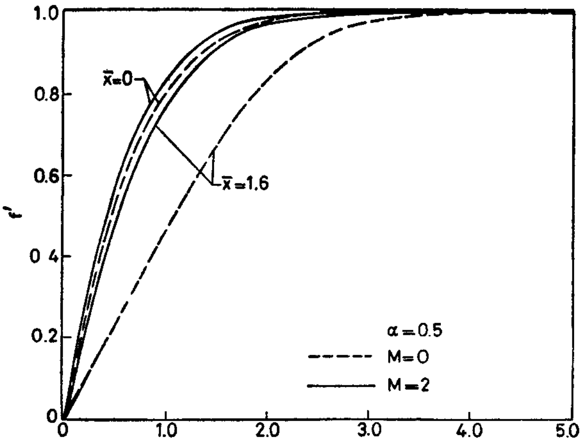

Figure 1. Velocity profiles (cylinder).

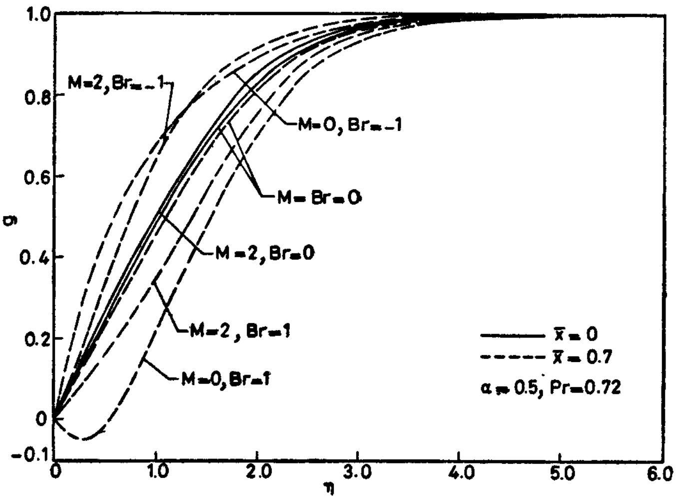

Figure 2. Temperature profiles (cylinder).

profiles become less steep as $\bar{x}$ increases, but the reverse is true when the magnetic parameter M increases. In the presence of the magnetic field, the profiles are comparatively less sensitive to the change in $\bar{x}$. It may be noted that the velocity profiles are independent of the dissipation parameter Br, but the temperature profiles strongly depend on Br. It may be remarked that in eq. (8), Br occurs in combination with U/U_∞ and $U/U_\infty=0$ at $\bar{x}=0$. Hence, eq. (8) at $\bar{x}=0$ becomes independent of Br whether M is zero or non-zero. Therefore for a given M, the temperature profiles for $Br=1$ or -1 at $\bar{x}=0$ are the same as those of $Br=0$. Further, when $Br=0$, the temperature profiles change little due to the change in $\bar{x}$ except near the point of separation. In fact, when the magnetic field is imposed, the change is still smaller and hence the temperature profiles for $M=2$ at $\bar{x}=0{\cdot}7$ could not be shown in figure 2. It is evident from figure 2 that for $M=0$ and $Br=1$ ($T_w>T_\infty$), the temperature $g(g=(T-T_w)/(T_\infty-T_w))$ for $\bar{x}\geqslant 0{\cdot}5$ (approximately) becomes negative near the wall which implies that the fluid near the wall is warmer than the wall itself owing to the generation of frictional heat. In such cases, the wall will not be cooled by the stream of air flowing past it. Similar effects have also been observed for the case of a flat plate without magnetic field (Schlichting 1968). It is also observed that as *Br* increases, the point at which $g<0$ moves upstream. On the other hand, the effect of the magnetic field is to prevent the generation of frictional heat and the fluid near the wall remains cooler than the wall and the wall continues to be further cooled. Thus, the magnetic field and the dissipation parameter play opposite roles as far as heating or cooling effect on the wall is concerned.

The variation of the skin-friction parameter f''_w with $\bar{x}$ is given in figure 3. Like

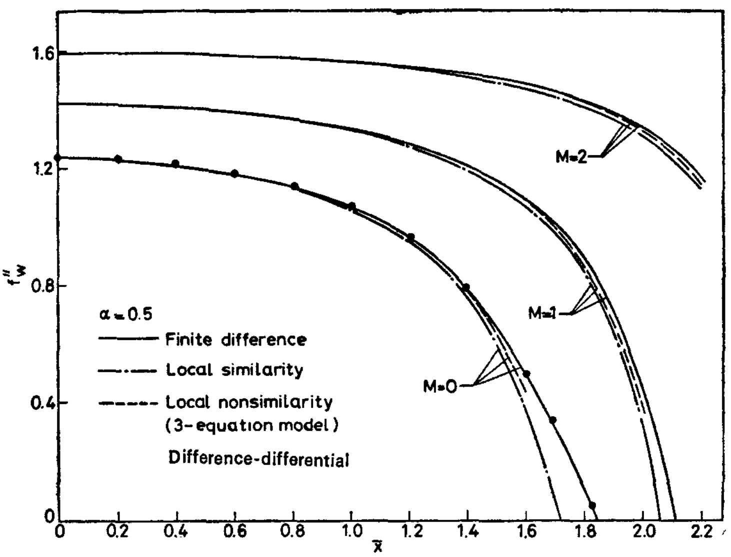

Figure 3. Variation of skin-friction with $\bar{x}$ (cylinder).

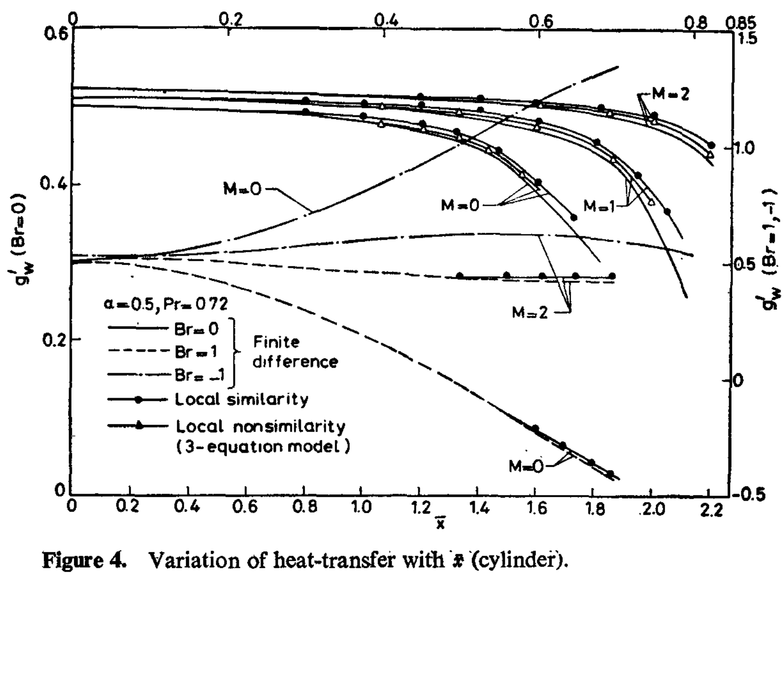

Figure 4. Variation of heat-transfer with $\bar{x}$ (cylinder).

velocity profiles, f''_w is also independent of Br. As expected f''_w decreases as $\bar{x}$ increases whatever may be the values of M. At a given $\bar{x}, f_w''$ increases as M increases. It is clearly seen from figure 3 that the effect of the magnetic field is to delay or prevent separation. The variation of the heat-transfer parameter g'_w with $\bar{x}$ is depicted in figure 4. For the sake of convenience, the results for $Br=0$ and $Br\neq 0$ are shown on a different scale. When $Br=0$, g'_w increases as M increases. However, as $\bar{x}$ increases, it remains nearly constant in the beginning and then decreases. When $Br=1$ and $M=0$, g'_w decreases as $\bar{x}$ increases and becomes negative at a certain $\bar{x}$ (the physical significance of g'_w becoming negative has been explained earlier). On the other hand when $Br=-1$ and $M=0$, g'_w increases as $\bar{x}$ increases. In the presence of the magnetic field, the change in g'_w with $\bar{x}$ is comparatively small. To check the accuracy of the present finite-difference method, we have compared our skin-friction results for $M=0$ with those of Terrill (1960) who used the difference-differential scheme and found them in excellent agreement (see figure 3). We have also compared our results with those of the local similarity and local nonsimilarity solutions and found them in good agreement except near the point of separation where these methods are not expected to give accurate results (Sparrow *et al* 1970). As expected, the local nonsimilarity solutions have been found to be more accurate than the local similarity solutions. It is further observed that the local similarity and local nonsimilarity results underestimate the skin friction but overestimate the heat transfer.

5. Flow over a sphere

We now consider the flow of an electrically conducting fluid over a sphere with an applied magnetic field, where the non-similarity is due both to external freestream velocity and to curvature of the body. In this case, the velocity at the edge of the boundary layer is given by

$$U(x)/U_\infty = 1{\cdot}5 \sin \bar{x},\ r/R = \sin \bar{x} \tag{12}$$

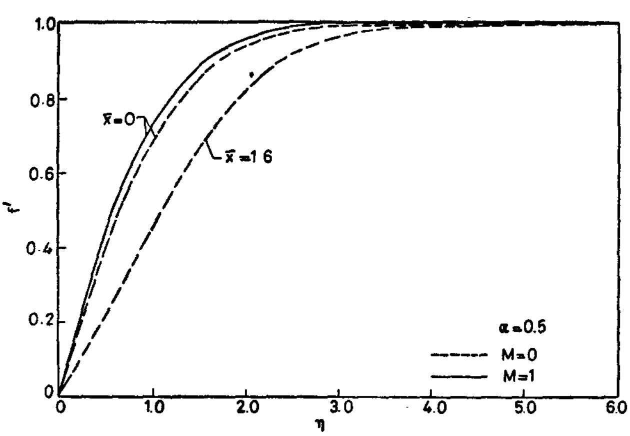

Figure 5. Velocity profiles (sphere).

and the pressure gradient is governed by (10b). Like for the cylinder, eqs (4a, d, e) and (5) taking $j = 1$ reduce to

$$\xi = (2-3\cos\bar{x}+\cos^3\bar{x})/2, \quad \beta(\bar{x})=2[\cos\bar{x}\,(\cos\bar{x}+2)/(1+\cos\bar{x})^2]/3 \quad (13a)$$

$$\beta_1(\bar{x}) = 4\,[(\cos\bar{x} + 2)/(1 + \cos\bar{x})^2]/9,$$

$$C_f\,(Re_x)^{1/2} = 2\beta_2(\bar{x})\,f''_w, \quad Nu\,(Re_x)^{-1/2} = \beta_2(\bar{x})\,g'_w \quad (13b)$$

$$\beta_2(\bar{x}) = \sin\bar{x}\,[(3\,\bar{x}\,\sin\bar{x})/\{2(2-\cos\bar{x} + \cos^3\bar{x})\}]^{1/2} \quad (13c)$$

The velocity and temperature profiles are shown in figures 5–6 and the skin-friction and heat-transfer parameters in figures 7 and 8. It may be remarked that qualitatively the results are similar to those of the cylinder, hence no detailed discussion of the

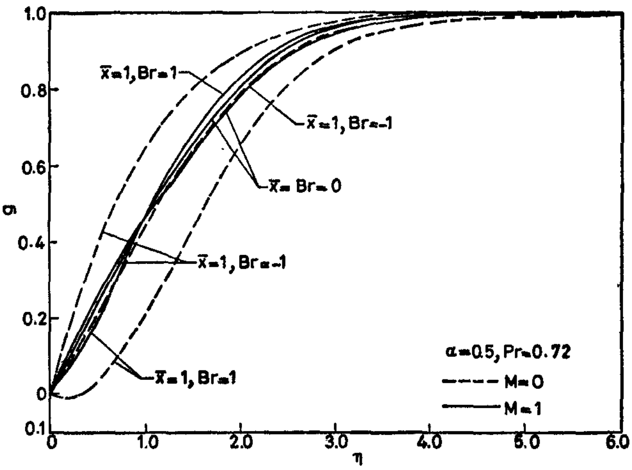

Figure 6. Temperature profiles (sphere).

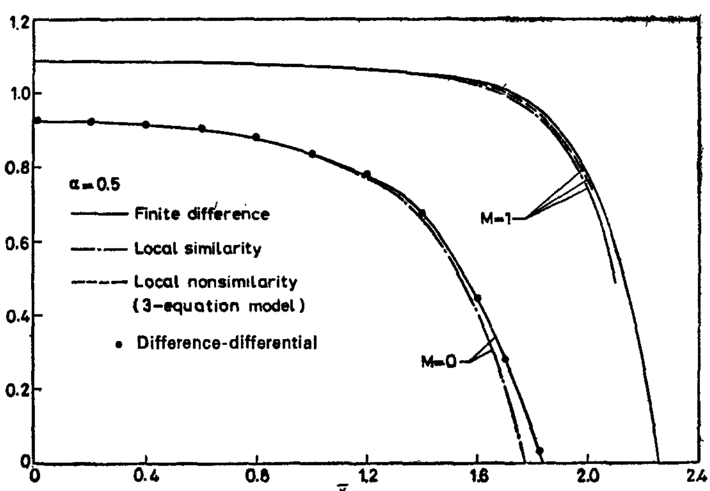

Figure 7. Variation of skin friction with $\bar{x}$ (sphere).

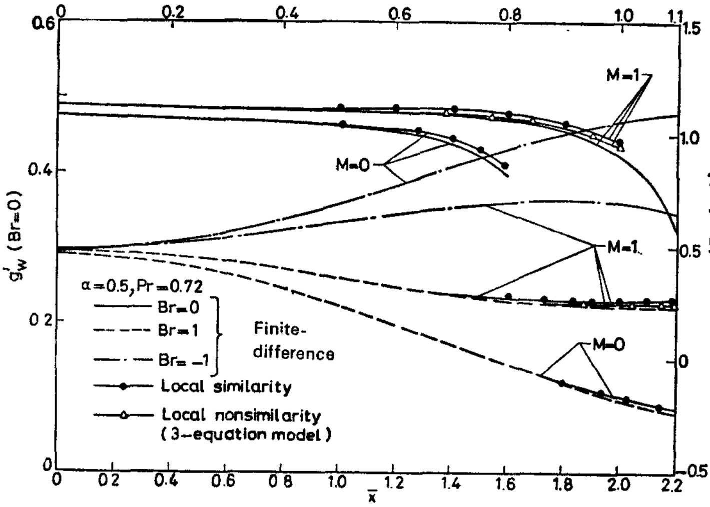

Figure 8. Variation of heat transfer with $\bar{x}$ (sphere).

results is given for this case. Our skin-friction results for $M=0$ are found to be almost identical with those of Smith and Clutter (1963) who used the difference-differential technique provided we divide their results by $2^{1/2}$ (see figure 7). This factor is due to slightly different sets of transformations used by Smith and Clutter (1963). Like cylinder, the results are found to be in good agreement with those of the local similarity and local nonsimilarity methods except near the point of separation.

6. Conclusions

The dissipation parameter for the case when the wall temperature is greater than the freestream temperature in the absence of a magnetic field generates frictional heat which makes the fluid near the wall warmer than the wall itself and thus prevents the wall being cooled. On the other hand, the magnetic field prevents the generation of frictional heat so that the fluid near the wall remains cooler than the wall. Further, the magnetic field tends to delay or prevent separation. The results are found to be in good agreement with those of the local similarity and local nonsimilarity methods except near the point of separation, but they are in excellent agreement with those of the difference-differential technique even near the point of separation.

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