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A large-scale heuristic modification of Newtonian gravity as an alternative approach to dark energy and dark matter

N. FALCON

Departamento de Física, FACYT, Universidad de Carabobo, Valencia, Carabobo, Venezuela.
E-mail: nelsonfalconv@gmail.com

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Abstract. The peculiarities of the inverse square law of Newtonian gravity in standard big bang cosmology are discussed. It is shown that the incorporation of an additive term to Newtonian gravitation, as the inverse Yukawa-like field, allows removing the incompatibility between the flatness of the Universe and the density of matter in the Friedmann equation, provides a new approach for dark energy, and enables theoretical deduction of the Hubble–Lemaître law. The source of this inverse Yukawa-like field is the ordinary baryonic matter and represents the large-scale contribution of gravity in accordance with the Mach principle. It is heuristically built from a specular reflection of the Yukawa potential, in agreement with astronomical and laboratory observables, resulting null in the inner solar system, weakly attractive in ranges of interstellar distances, very attractive in distance ranges comparable to the clusters of galaxies, and repulsive in cosmic scales. Its implications in the missing mass of Zwicky, virial theorem, Kepler's third law in globular clusters, rotation curves of galaxies, gravitational redshift, and Jean's mass are discussed. The inclusion of the inverse Yukawa-like field in Newtonian gravitation predicts a graviton mass of at least 10^{-64} kg and could be an alternative to the paradigm of non-baryonic dark matter concomitant with the observables of the big bang.

Keywords. Dark matter—dark energy—gravity—graviton mass.

1. Introduction

The uncritical acceptance in the cosmic scale of the gravitational inverse-square law, or Newtonian gravitation, entails serious difficulties to describe the dynamics of the Universe: the mass observed in the rich galaxy clusters is significantly less than that required to keep these systems stable gravitationally (the missing mass of Zwicky) and the rotation curves of the galaxies are incompatible with their virialized masses. In addition, the inconsistency between the observed average density of baryonic matter is much smaller than required by the Friedmann–Robertson–Walker models with a cosmological constant (ΛFRW model) and curvature null, a problem known as the missing mass. This has motivated the paradigm of non-baryonic dark matter. This hypothetical dark matter would have unknown properties, and only interacts gravitationally with ordinary matter. However, after more than a two decades of efforts, theoretical, astronomical observations, and experiments, its existence has been assumed conjecturally like the ancestral paradigms of ether, phlogiston, and caloric.

While it is true that the validity of inverse square law of Newton's gravity is verified with precisions $>10^{-8}$ for Eötvös-like experiments (Adelberger *et al.* 2003), there is no empirical evidence of their validity beyond the solar system (it is assumed true for estimating the mass of binary stars). The universal character of Newton's gravitation law was given by Kant in 1755, considering the deductive character of the planetary motion and Leverrier's prediction for the discovery of Neptune. Recall that galaxies acquire identity after the great debate by Shapley–Curtis in 1920. Also, Laplace and Seeliger theorized in the 18th century modifications to the law of gravitation (Laves 1898; Bondi 1951).

On the other hand, Newton's gravitation law postulates that the force of gravity has an infinite reach; even bigger than the radius of the observable Universe. Consequently, it implies that the rest mass of the graviton is null, in contradiction with the Grand Unification

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Theories and the detection of gravitational waves, which predict not-null rest mass (Chugreev 2017).

Another big problem in the hot big bang cosmology, closely linked to the expansion of the Universe, is the evidence of the accelerated expansion, commonly referred to as dark energy, whose understanding is still unfinished (Debono & Smoot 2016; Huterer & Shafer 2018; Genova–Santos 2020). Experiments to detect dark energy forces using atom interferometry (Sabulsky *et al.* 2019) show no evidence of new forces those results had placed stringent bounds on scalar field theories that modify general relativity on large scales, as in Chameleon and Symmetron theories of modified gravity.

It is justifiable to ask if there are theoretical alternatives to Newton's law of gravity valid on a large scale, concomitant with the astronomical observations and with the FRW models, in the standard cosmology of the hot big bang.

An alternative to solve both problems, dark energy and non-baryonic dark matter, is the theories of modified Newtonian dynamics (MoND) (Debono & Smoot 2016), such as non-local gravitation, establishing that the force of gravitation would be the result of two terms generated by ordinary matter: a first term as Newton's law and an additional term, representing the long-range contribution in the sense of the Mach principle (Falcon 2013).

Mach's principle postulates that local inertia is determined by the mass distribution of the rest of the Universe (Mach 1893). At any point in space, say, in the solar neighborhood, within our Galaxy or at other point in the Universe, the effective gravitational field must be the sum of the fields produced by all the celestial bodies on that point. Thus, each star must provide a small contribution to the total gravitational field. Although each term looks insignificant, its sum does not have to be null, and it will exert a total gravitational field not prescribed by Newton's gravitation law, when considering the gravitational field between two particles. Similarly, any point of space in the adjacencies of the local group of galaxies will be subjected to the gravitational interaction of the baryonic mass, corresponding to the galaxy clusters and the large-scale structures. As a result, there must be a global gravitational interaction that adds to the force that mediates between any pair of galaxies.

Clearly, it is not possible to calculate explicitly the global contribution to the gravitational force between two particles. Einstein tried it, through the cosmological term Λ, but it remained pending how to model its equivalent in stellar distance ranges inside a particular galaxy, and to the interior of the galaxy clusters. However, it is possible to propose alternatives for modeling the term of global inertia, heuristically constructed from observations, to represent the gravitational contribution of the large-scale distribution of baryonic matter in the local universe.

The model of an inverse Yukawa-like field to explicitly incorporate Mach's principle is presented in Section 2, showing that it includes as a particular case, the local Newtonian gravity, MoND-type Milgrom-like theories, and the massive graviton. Then in Section 3, the results of its inclusion in the ΛFRW cosmology are analyzed, highlighting the solution to the problem of dark matter and the theoretical deduction of Hubble–Lemaître's law and comprehensive interpretation of dark energy. The age of the universe is also presented. Later in Section 4, we discuss the virial theorem and then solve Zwicky's paradox; the large-scale variation of Kepler's third law in globular clusters, the gravitational redshift, and Jean's mass are discussed. Finally, the conclusions are shown in the last section.

2. The model: inverse Yukawa-like field

2.1 *Phenomenology and physical argument*

We assume that any particle with non-zero rest mass is subject to the gravitational inverse-square law, plus an additional force that varies with distance, caused by an inverse Yukawa-like field (U_{YF}). Thus, the net force of gravitation varies as the law of inverse square in negligibly small distance scales in comparison to the interstellar scale, but it varies in a very different way when the comoving distance is about of the order of kiloparsec or more. In this sense, our argument is a MoND theory or large-scale modification of the Newtonian gravity.

The origin of this field (U_{YF}) is the baryonic mass, like in the Newtonian gravity, and represents the contribution of the gravitational distribution at a great scale of the baryonic mass.

This potential per unit mass (in units of J kg^{-1}), as a function of the comoving distance, is heuristically built from a specular reflection of the Yukawa potential (Falcon 2011, 2013): null in the inner solar system, weakly attractive in ranges of interstellar distances, very attractive in distance ranges comparable to clusters of galaxies and repulsive to cosmic scales:

$$U_{\mathrm{YF}}(r) \equiv U_0(M)(r - r_0)\,\mathrm{e}^{-\alpha/r}, \qquad (1)$$

where r is the comoving distance in Mpc, $U_0(M) \equiv 4\pi l G M {r_0}^{-1}$ is a constant, and $l \equiv 1\ \mathrm{m}^{-1}$ is a dimensional parameter.

The coupling constants, selected from the astronomical data, are $r_0 \approx 50$ Mpc (the average distance between clusters of galaxies) and $\alpha \approx 2.5$ Mpc. We understand that an exact model could fit the precise values of the coupling constants without modifying the phenomenology. Figure 1 shows the variation of the $U_{\rm YF}$ for different ranges of the comoving distance r (in Mpc).

This heuristically constructed potential $U_{\rm YF}(r)$ is, from the conceptual perspective, similar to the one that is constructed phenomenologically to explain the restoring force of a spring (Hooke's law); thus, the potential explained the observations, even if it lacks a microscopic description.

On the other hand, notice that Birkhoff's theorem says that the gravitational field outside a spherically symmetric body behaves as if all the mass of the body were concentrated in the center. However, the theorem cannot be applied to the current discussion, because its application begins by constructing four spheres around the center of the mass distribution, in which the gravitational field is evaluated, and in the local universe there is no way to define the center of the four spheres. Therefore, Birkhoff's theorem could not be applied to the renormalized Newtonian theory of gravitation, such as in the $U_{\rm YF}$ field, unless we abandon the Copernican principle, suppose that the Earth is at the center of mass distribution of the Universe and then apply Birkhoff's theorem.

2.2 *The Eövos-like experiments and MoND-Milgrom theories*

Thus, the force per unit mass (acceleration), complement to the large scale of the Newtonian gravitation is:

$$F_{\rm IY}(r) \equiv -\frac{U_0(M)}{r^2}\, {\rm e}^{-\alpha/r}(r^2 + \alpha(r - r_0)). \quad (2)$$

That can be written in the dimensionless variable as:

$$F_{\rm IY}(x) \equiv -U_0(M)\, {\rm e}^{-\alpha_0/x}\left[\frac{x^2 + \alpha_0(x-1)}{x^2}\right], \quad (3)$$

where $x \equiv r/r_0$ and $\alpha_0 \equiv \alpha/r_0 = 0.05$.

Note that the maximum occurs by $x_m = 0.024$ as in the core of the Abell radius, for the typical clusters of galaxies, i.e., $r_m \approx 1.2$ Mpc (Figure 2). Also, note that the force is zero at $x = 0.2$; that is

$$r_c = \frac{\alpha}{2}\left(\sqrt{1 + \left(\frac{4r_0}{\alpha}\right)} - 1\right) = 10 \text{ Mpc}. \quad (4)$$

Notice that if r is negligibly small compared to $\alpha = 2.5$ Mpc then $F_{\rm IY}$ is null, and the gravity is only prescribed by the inverse-square law of Newtonian-gravitation, in accordance with Eövos-like experiments.

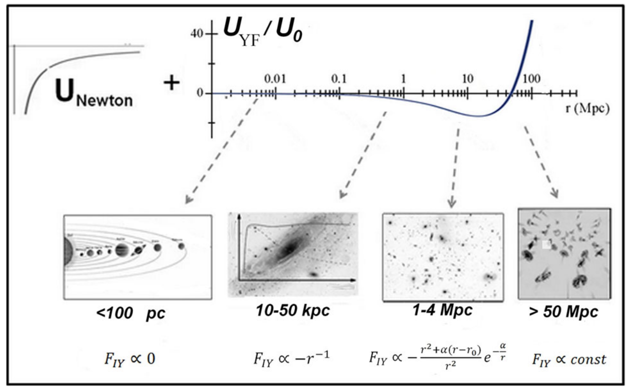

Figure 1. Modification of the Newtonian dynamics with $U_{\rm YF}$ in astronomical scale for different ranges of the comoving distance.

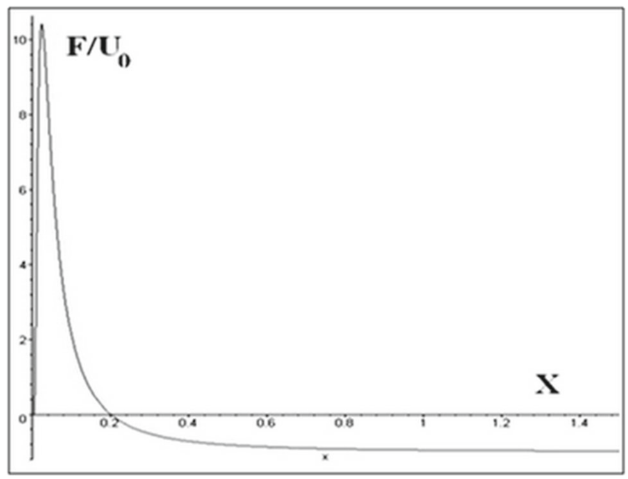

Figure 2. Force $F_{\rm IY}$ in dimensionless scale x.

For ranges of the comoving distances, between objects gravitationally bound, with r smaller, we obtain

$$|F_{\rm IY}(r \ll r_0)| \approx \frac{U_0(M)r_0}{2r + \alpha} \approx \left(\frac{U_0(M)r_0}{2}\right)r^{-1}. \quad (5)$$

Thus, $F_{\rm IY} \sim r^{-1}$ recovers the MoND-Milgrom results as a particular case. Milgrom (1983a,b) proposed a phenomenological modification of Newton's law which fits galaxy rotation curves solving the galaxy rotation curves problem and recovers the Tully–Fischer law. We get the result, the Milgrom, according to which the gravitational force depends on r^{-1} (Scarpa 2006; Debono & Smoot 2016). Details of this discussion will be presented in Section 4.4.

The maximum value of $F_{\rm IY}$ is $r_m \approx 1.2$ Mpc as in the core of the Abell radius for clusters of galaxies. For the average value of smooth transition to strong agglutination in galaxy's distribution ($r_c \approx 10$ Mpc) (Peebles & Rastra 2003), the $F_{\rm IY}$ is null. From

Figure 1 it is clear that $U(r)$ gives a constant repulsive force per unit mass, at cosmological scales providing an asymptotic cosmic acceleration. This cosmic acceleration, on a large scale, remains constant as it is observed when taking the limit of x very large, for ranges of distance like $x = r/r_0$ much greater than 50 Mpc, in (3), as shown in Figure 2.

2.3 *The mass of graviton*

The null value of $F_{\rm IY}$ in $x = 0.2$, together with the sign change in the $U_{\rm YF}$ (Figure 1), suggests that the range of the force of gravity is finite, i.e., in the order of 10 Mpc; and thus the graviton would have no null rest mass; then

$$m_g^0 \cong \frac{\hbar}{r_c c} \approx 10^{-64}\ \text{kg} \simeq 10^{-29}\ \text{eV c}^{-2}, \tag{6}$$

where c denotes the speed of light and $\hbar$ is the Planck constant.

Notice firstly the definition of massive graviton were origin in the 1930s when Wolfgang Pauli and Markus Fierz first developed a theory of a massive spin-2 field propagating on a flat space-time background (Bergshoeff *et al.* 2009), where the massive graviton could be decayed in two photons. In 2016, the LIGO and VIRGO projects reported the observational discovery of gravitational waves from a binary black hole merger event, GW150914. Their report specified a bound on the graviton mass $m_g^0 < 7.7 \times 10^{-23}$ eV c^{-2} (Bergshoeff *et al.* 2009; LIGO Scientific Collaboration and Virgo Collaboration 2017).

Also, we can calculate the Compton wavelength associated with massive gravitons as:

$$\lambda_g^0 \cong \frac{\hbar}{m_g^o c} \approx 3 \times 10^{18}\ \text{km}. \tag{7}$$

It is in good agreement with the lower bounds inferred from the observations of binary pulsars, cluster galaxy structure, and gravitational lensing (Gazeau & Novello 2011). Also, the LIGO Collaboration constrains the Compton wavelength of the graviton as $\lambda_g^0 > 10^{13}$ km (Bergshoeff *et al.* 2009).

The structures found on a large scale in the distribution of galaxies and gas, with characteristic dimensions much greater than 10 Mpc, e.g., Sloan Great Wall (Einasto *et al.* 2011) and Voids (Kovács *et al.* 2016), do not show symmetric axial distribution that would be expected if gravitation had infinite range. The hot gas in the superclusters of galaxies found by means of the Suyaev–Zeldovich effect (Planck Collaboration 2011) also does not have a spherical distribution that would be expected if gravitation were of infinite range. In large-scale structures with dimensions >10 Mpc, there is a gravitational bond between the galaxies, by a sequential chain of gravitational attractions between their neighboring components, but not by a common center. Assuming an infinite range for gravity would imply, among other things, to imagine colossal masses for the attractor center in the superclusters of galaxies, which are unobservable (Super Black Hole).

3. Results: ΛFRW cosmology

Let us now consider a usual ΛFRW model, with homogeneous and isotropic FRW metric together with energy–momentum tensor for a perfect fluid (Falcon 2013):

$$\Re^{\mu\nu} - \frac{g^{\mu\nu}}{2}\Re + \Lambda g^{\mu\nu} = \frac{-8\pi G}{c^2} T^{\mu\nu}, \tag{8}$$

where $\Re^{\mu\nu}$ is the Ricci tensor and $\Re$ is the Ricci scalar. Without loss of generality, we can write: $\Lambda \sim F_{\rm IY}(r)$. Note that the covariance is guaranteed because at cosmological scales, where it makes sense to consider galaxies as particles of a perfect fluid, the $F_{\rm IY}$ per unit mass is constant for ranges of the comoving distance $r > 50$ Mpc, as in Figure 2. Thus, the dark energy can be thought of as a "cosmic force" in the sense of the Mach principle, caused by ordinary matter, through the cosmological term.

The cosmological term leads to the usual Friedmann equations (Falcon 2013):

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 + \frac{kc^2}{R^2(t)} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}, \tag{9}$$

$$\frac{2\ddot{R}(t)}{R(t)} + \left(\frac{\dot{R}(t)}{R(t)}\right)^2 + \frac{kc^2}{R^2(t)} = -\frac{8\pi G}{c^2}P + \Lambda c^2. \tag{10}$$

Now, we assumed that $\Lambda \equiv \Lambda(r) \propto F_{\rm IY}(r)$, as a cosmic parameter (associated with a cosmic acceleration), or dynamic variable with respect to the comoving distance:

$$\Lambda \equiv \Lambda(r) = \Lambda_0 F_{\rm IY}(r) = -\Lambda_0 \frac{\mathrm{d}}{\mathrm{d}r} U(r), \tag{11}$$

where Λ_0 is the coupling constant

$$\Lambda_0 \equiv \frac{3H_0}{c^3} \cong 0.31 \times 10^{-42}\ \text{m}^{-3}\ \text{s}^2. \tag{12}$$

3.1 *New approach for cold dark matter*

When $r \to r_m$, i.e., when $x \to 0.024$, the comic parameter $\Lambda(r)$ in the intergalactic scale, applying

(2) to (11), is

$$\Lambda(r_m) = \Lambda_0 F_{IY}(r)|_{r\to r_m} = -\Lambda_0 U_0 \,\mathrm{e}^{-\alpha_0/x}\left[\frac{x^2+\alpha_0(x-1)}{x^2}\right]\Bigg|_{x\to 0.024}. \quad (13)$$

Replacing the previous constants, we have

$$\Lambda(r_m) \simeq 10.55(4\pi G\,\mathrm{kg\,m^{-2}})\frac{3H_0}{c^3}. \quad (14)$$

Now, the Friedmann equations are:

$$\left(\frac{\dot R(t)}{R(t)}\right)^2 + \frac{kc^2}{R^2(t)} = \frac{8\pi G}{3}\rho + \frac{\Lambda(r_m)c^2}{3}, \quad (15)$$

$$\frac{2\ddot R(t)}{R(t)} + \left(\frac{\dot R(t)}{R(t)}\right)^2 + \frac{kc^2}{R^2(t)} = -\frac{8\pi G}{c^2}P + \Lambda(r_m)c^2. \quad (16)$$

The definition of the critical density changes, because the potential U_{YF} now is not null when $k = 0$. The critical density (ρ_c), using (14), is now:

$$H_0^2 \equiv \left(\frac{\dot R(t)}{R(t)}\right)^2 = \frac{8\pi G}{3}\rho_c + \frac{\Lambda(r_m)c^2}{3}, \quad (17)$$

then

$$\rho_c = \frac{3H_0^2}{8\pi G}\left|1 - \frac{\Lambda(r_m)c^2}{3H_0^2}\right| \cong 9.5\frac{3H_0^2}{8\pi G} \approx 1.86\times 10^{12}\,M_\odot\,\mathrm{Mpc^{-3}}. \quad (18)$$

We can see that the value of the critical density increases, because the critical mass has been underestimated in the usual definition. Thus, it has to take into account the U_{YF}, in addition to baryonic mass.

Using the standard notation, $\Omega_b \equiv \rho/\rho_c$, $\Omega_\Lambda \equiv \Lambda c^2/3H_0^2$, $q_0 \equiv -(\ddot R/R)H_0^{-2}$ for the density of matter, cosmological, and deceleration parameters, respectively, and the definition

$$\Omega_{IY} \equiv \frac{\Lambda(r_c)c^2}{3H_0^2}. \quad (19)$$

The Friedmann Equations (9) and (19) are:

$$\frac{kc^2}{R^2(t)} = H_0^2[\Omega_b(1+\Omega_{IY}) + \Omega_\Lambda - 1], \quad (20)$$

$$q_0 = \frac{\Omega_b}{2}(1+\Omega_{IY})\left(1+\frac{3P}{c^2\rho}\right) - \Omega_\Lambda. \quad (21)$$

Also, (20) and (21) can be written as per the standard ΛFRW cosmology:

$$\frac{kc^2}{R^2(t)} = H_0^2[(\Omega_b + \Omega_b\Omega_{IY}) + \Omega_\Lambda - 1] = H_0^2[\Omega_m + \Omega_\Lambda - 1], \quad (22)$$

$$q_0 = \frac{(\Omega_b + \Omega_b\Omega_{IY})}{2}\left(1+\frac{3P}{c^2\rho}\right) - \Omega_\Lambda = \frac{\Omega_m}{2}\left(1+\frac{3P}{c^2\rho}\right) - \Omega_\Lambda. \quad (23)$$

Thus, in the present model, the Ω_c parameter of the cold dark matter would be the gravitational contribution caused by the large-scale distribution of the ordinary baryonic matter. This additional gravitational contribution incorporates Mach's principle through some heuristic representation, as the U_{YF} proposed.

Replacing (13) and (19) in (21), we obtain in flat universe model ($k = 0$):

$$1 = \Omega_b(1+\Omega_{IY}) + \Omega_\Lambda \Rightarrow \Omega_b 11.42 + \Omega_\Lambda = \Omega_m + \Omega_\Lambda = 1. \quad (24)$$

Now, the remarkable result is that: if $k = 0$ and $\Omega_{IY} \neq 0$ do not require the assumption of the non-baryonic dark matter, neither do they require exotic particles of cool dark matter, i.e., using $\Omega_b \approx 0.0223$ and $\Omega_\Lambda \approx 0.6911$ as in the cosmic microwaves background (CMB) measurements of the Planck Collaborations (2016, 2019), we obtain $\Omega_m \approx 0.255$ and $\Omega_m + \Omega_\Lambda = 0.255 + 0.6911 \approx 1$.

3.2 *Deduction of Hubble–Lemaître's law*

Consider the photons emitted from a remote galaxy with recession velocity v, and their observation in the reference local frame. Therefore, we should evaluate (1) at $r \gg 50\,\mathrm{Mpc}$, with initial condition $v = 0$ in $t = 0$. We find

$$v = \int a\,\mathrm{d}t = \int \left(\lim_{r\to\infty} F_{IY}(r)\right)\frac{\mathrm{d}r}{c} \Rightarrow v = \int \left(\lim_{x\to\infty} F_{IY}(x)\right)\frac{\mathrm{d}r}{c} \simeq \frac{U_0}{c}r. \quad (25)$$

Replacing as before, $U_0 = 4G$ kg m^{-2}, we obtain Hubble–Lemaître's law (Falcon 2013; Falcon & Aguirre 2014):

$$V = \left(\frac{4\pi G\ell}{c}\right)r \equiv H_0 r \cong \left(86.3\,\mathrm{km\,s^{-1}\,Mpc^{-1}}\right)r. \quad (26)$$

It is

$$H_0 = \frac{4\pi G\ell}{c}, \quad (27)$$

where $\ell \equiv 1$ kg m^{-2} is a dimensional parameter.

Notice that the value of H_0 is the theoretical upper limit, evaluated for most distant objects ($r \gg 50\,\mathrm{Mpc}$);

also Falcon & Aguirre (2014) found $H_0 \approx$ 83.56 km s^{-1} Mpc^{-1} in a selected sample of 392 galaxies, in the range of 50–1400 Mpc. Earlier, Falcón & Genova-Santos (2008) found H_0 values as high as 73 km s^{-1} Mpc^{-1} using the Sunyaev–Zeldovich effect and X-ray emission data in galaxy clusters, with the advantage that this method is independent of redshift.

The measurements of the anisotropies in the CMB constraints indirectly the set of cosmological parameters (k, H_0, Ω_m, Ω_Λ), by multiple statistics correlations over the acoustic peaks in the distribution of the radiation in the angular power spectrum. The results of the Planck Collaboration (2019) show lower values for H_0. In the Planck Collaboration, the estimates of the cosmological parameters show degeneration in the simultaneous estimates of H_0 and Ω_m, Figure 3 in Planck Collaborations (2016), because the CMB measurements are not a direct measure of the Hubble constant.

Most recent direct measurements of the constant of Hubble with the Space Telescope are $H_0 = 75.8^{+5.2}_{-4.9}$ and $78.5^{+6.3}_{-5.8}$ km s^{-1} Mpc^{-1} depending on the target calibration (de Jaege *et al.* 2020); furthermore, Riess (2019) had found that $H_0 =$ 74.22 $\pm$ 1.82 km s^{-1} Mpc^{-1} in the Large Magellanic Cloud.

In section 3.4 we will show a more extensive discussion on the validity of the theoretical limit of H_0, valid for galaxies much more distant than 50 Mpc.

3.3 *Comprehensive interpretation of dark energy*

When $r \rightarrow r_c$, i.e., when $x \rightarrow 0.2$, the comic parameter $\Lambda(r)$ in the intergalactic scale, applying (2) to (11), is

$$\Lambda(r_c) = \Lambda_0 F_{\mathrm{IY}}(r)|_{r\rightarrow r_c} = -\Lambda_0 U_0 r_0 \mathrm{e}^{-\alpha_0/x}\left[\frac{x^2+\alpha_0(x-1)}{x^2}\right]\Bigg|_{x\rightarrow 0.2}. \quad (28)$$

Replacing the previous constants Λ_0, U_0, r_0, and (27), we have

$$\Lambda(r_c) \cong 0.623(4\pi G\,\mathrm{kg\,m^{-2}})\frac{3H_0}{c^3} = 0.623\,\frac{3H_0^2}{c^2}. \quad (29)$$

Using (13), we obtain 0.623 h 10^{-52} m^{-2} as the lower limit because F_{IY} is evaluated in 10 Mpc; and we obtain 10^{-52} h m^{-2}$\approx$0.86 $\times$ 10^{-52} m^{-2} as the upper limit when F_{IY} is evaluated in $r \rightarrow \infty$.

Now, the Friedmann equations are:

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 + \frac{kc^2}{R^2(t)} = \frac{8\pi G}{3}\rho + \frac{\Lambda(r_c)c^2}{3}, \quad (30)$$

$$\frac{2\ddot{R}(t)}{R(t)} + \left(\frac{\dot{R}(t)}{R(t)}\right)^2 + \frac{kc^2}{R^2(t)} = -\frac{8\pi G}{c^2}P + \Lambda(r_c)c^2. \quad (31)$$

Thus, in the present model, the dark energy would be the cosmic acceleration in local frameworks, caused by the large-scale distribution of the ordinary baryonic matter, as prescribed by Mach's principle, through U_{YF} proposed. As before, replacing (29) in the cosmological density parameter, we get:

$$\Omega_\Lambda \equiv \frac{\Lambda(r_c)c^2}{3H_0^2} \cong 0.623\frac{(4\pi G\,\mathrm{kg\,m^{-2}})}{H_0 c} \approx 0.623\,h^{-1}. \quad (32)$$

Using (23) here as the upper limit for Hubble parameter (h = 0.863), we obtain $\Omega_\Lambda \approx 0.72$; in good agreement with the measurements of type Ia supernovae (Riess *et al.* 1998; Perlmutter *et al.* 1999; Peebles & Rastra 2003; Wondrak 2017).

3.4 *The age of the universe*

Using (19), the Friedman equation (for a flat universe) can be written in terms of the density cosmological parameters as

$$H^2 = H_0^2[\Omega_m(1+z)^3 + \Omega_\Lambda]. \quad (33)$$

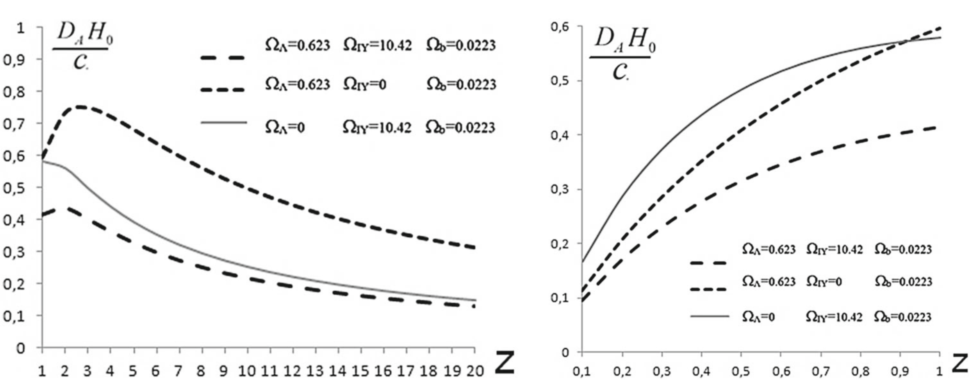

Figure 3. Angular diameter distance in function of the redshift for different sets of cosmological parameters.

The contribution of the radiation density has been omitted for simplicity, but it can be incorporated in the sum, multiplied by the factor $(1 + z)^4$, without loss of generality in the discussion. In a flat universe ($k = 0$), we can write the limit of the age at redshift z as

$$\tau = H_0^{-1} \int_0^{\infty} \left[(1+z)^3\Omega_b(1+\Omega_{\rm IY}) + \Omega_\Lambda\right]^{-1/2} \frac{dz}{z+1}. \tag{34}$$

Notice that it is the ΛFRW conventional equation for the age of the universe, because of arithmetic equality: $\Omega_m = \Omega_b(1+\Omega_{\rm IY}) = \Omega_b + \Omega_c$.

The numerical integration in the present model ($\Omega_b \approx 0.0223$, $\Omega_{\rm IY} \approx 10.42$, $\Omega_\Lambda \approx 0.623$ and upper limit of $H_0 \approx 86.3$ km s^{-1} Mpc^{-1}) indicates that the age of the universe is $\tau \sim 11.42$ Gyr, in good agreement with the age of the white dwarfs in the solar neighborhood (Tremblay *et al.* 2014; Kilic *et al.* 2017) and 11.2 Gyr for the globular clusters in the Milky Way (Krauss *et al.* 2003). The direct cosmochronology through white dwarfs offers an independent technique for the age of the Milky Way (Tremblay *et al.* 2014); in the inner halo, the estimated age is $12^{+1.4}_{-3.4}$ Gyr (Kilic *et al.* 2017). The Planck Collaboration (2019) reports that $\tau \sim 13.8$ Gyr. Using the Planck collaboration dataset ($\Omega_b \approx 0.0223$, $\Omega_\Lambda \approx 0.677$ and $H_0 \approx 67.74$) plus $\Omega_{\rm IY} \approx 10.42$, we obtain $\tau \sim 14.43$ Gyr.

4. Discussion

The proposed $U_{\rm YF}$ field falls within the finite range of gravitation and consequently a massive graviton. It is emphasized that theorizing a finite range of gravitation in a very short range of the order of 10 Mpc does not necessarily contradict the observations, since, as already mentioned, large-scale structures can be explained by the successive attraction between galaxies and the neighboring cluster; and they would not necessarily require an attractor center and forces of a longer range for their formation, balance, and dynamics.

On the other hand, we observe that Λ is constant in ranges of comoving distance much greater than 10 Mpc, i.e., limits taken in (25) and (28); therefore the covariance prescribed by the Theory of Relativity is fulfilled. The $U_{\rm YF}$ field is only locally covariant. Note further that no physical theory is globally covariant.

In the modification at the great scale of Newtonian gravity, as the proposed $U_{\rm YF}$ field, the dependence of density and pressure remains unchanged on the scale factor of expansion $R(t)$, and therefore, does not affect the Early Universe. Thus, the decoupling time matter radiation, the primordial fluctuations, and the anisotropies in the CMB, the nucleosynthesis (baryogenesis) should remain unchanged by a large-scale modification of Newtonian gravity. Note that $U_{\rm YF}$ is 40 orders of magnitude higher than the average distance per nucleon in the primordial plasma. Also, bear in mind that the Sachs–Wolfe effect does not change, because the size of the horizon at the time of recombination is $\sim$100 kpc, much less than the maximum range of the gravitational force with massive gravitons (10 Mpc) and at such ranges, the gravitons would travel the entire universe inside the horizon without decay.

4.1 *Angular diameter distance*

An observational test for the present cosmological assumption about $U_{\rm YF}$ is the source number count or redshift volume density distribution. The angular diameter distance is

$$D_A = \frac{cH_0^{-1}}{z+1} \int_0^{z} \left[(1+z)^3\Omega_b(1+\Omega_{\rm IY}) + \Omega_\Lambda\right]^{-1/2} dz, \tag{35}$$

where we used (34) and the redshift dependence of the universe-scale factor.

In Figure 3, we have shown the dimensionless angular diameter with and without $U_{\rm YF}$ (dashed lines), and only the parameter of the matter density (continuous line: $\Lambda = 0$). We can see that the $U_{\rm YF}$ reduces the angular diameter distance for all z-values and their maxima in $z \approx 2$.

4.2 *The virial theorem*

In 1937, Fritz Zwicky noticed using the virial theorem, that the Coma cluster masses are underestimated (Zwicky 1933), a problem now known as the missing mass problem. Let us consider Clausius's virial expression $G \equiv \sum_i \vec{p}_i \cdot \vec{r}_i$ when deriving with respect to time, and then averaging with respect to a complete period (τ), we obtain the well-known virialized expression between kinetic energies and power; that in the case of a particle subjected only to the Newtonian gravitational potential is (Falcon 2012):

$$\frac{1}{\tau}\int_0^{\tau} \frac{dG}{dt} dt' = \frac{1}{\tau}\int_0^{\tau} \sum_i -\vec{\nabla} U_i \cdot \vec{r}_i \, dt' + \frac{1}{\tau}\int_0^{\tau} \sum_i \frac{p_i^2}{2m_i} dt'$$

$$\frac{G(\tau) - G(0)}{\tau} = -\frac{1}{\tau}\int_0^{\tau} \sum_i \frac{GMm_i}{r_i} dt' + \frac{1}{\tau}\int_0^{\tau} \sum_i T_i \, dt'$$

$$0 = -\langle U \rangle + 2\langle T \rangle. \tag{36}$$

If now the particles are subjected to a gravitational potential that have an additional term, long-range, we have using (1) and (2) that:

$$0 = -\langle U\rangle + 2\langle T\rangle + \frac{4\pi GMr_0^{-1}l}{\tau} \times \int_0^{\tau}\sum_i m_i\,\mathrm{e}^{-\alpha/r}\left(\alpha + r_0 - \frac{\alpha r_0}{r}\right)\mathrm{d}t'. \quad (37)$$

See the proof in Appendix. As before, l is a dimensional factor with units of the kg m^{-2}. The integral of the last term can be evaluated using the mean value theorem, then

$$0 = -\langle U\rangle + 2\langle T\rangle + 4\pi GMlr_0^{-1}\alpha\sum_i m_i. \quad (38)$$

Thus, the energy balance (in joules by mass unit) results as:

$$\frac{v^2}{2} \equiv GM\left(-\frac{1}{r} + 4\pi lr_0^{-1}\,\mathrm{e}^{-\alpha/r}(r - r_0)\right) + 4\pi\alpha lr_0^{-1}GM. \quad (39)$$

In distance ranges of galaxy clusters, the U_{YF} potential dominates with respect to the Newtonian potential of the inverse-square law, and then the energy balance per unit mass is

$$\frac{v^2}{2} \cong 4\pi lr_0^{-1}GM[\mathrm{e}^{-\alpha/r}(r - r_0) + \alpha]. \quad (40)$$

In the particular case of the Coma cluster, with v = 1500 km s^{-1} and r = 4.25 Mpc (Shirmin 2016), (40) becomes

$$GM4\pi lr_0^{-1}[\mathrm{e}^{-\alpha/r}(r - r_0) + \alpha]\Big|_{r=4.71} \approx 7\times 10^{-19}\,\mathrm{m}^2\,\mathrm{s}^{-2} \gg \frac{v^2}{2} \simeq 1.12\times 10^{14}\,\mathrm{m}^2\,\mathrm{s}^{-2}.$$

An important result is that (38) solves Zwicky's paradox. The "missing mass" could be interpreted as the energy associated with the U_{YF} field.

4.3 *Kepler's third law in globular clusters and rotation curves*

The deviation of Kepler's third law in globular clusters would be a serious test for Newtonian gravity in the outer space farther than the solar system, and would check theoretical alternatives to the non-baryonic dark matter.

The introduction of the U_{YF} changes the movement equation of the astronomical bodies, and as a consequence Kepler's third law, using (2), then:

$$\frac{4\pi^2}{T^2} = \frac{GM}{r^3} - \frac{U_0(M)}{r^3}\,\mathrm{e}^{-\alpha/r}(r^2 + \alpha(r - r_0)), \quad (41)$$

where T denotes the orbital period.

The term of the Newtonian potential is greater than the term due to U_{YF} for ranges < 21.8 kpc, from which the latter becomes greater, as indicated in Figure 4. In Harris Catalogue (2010), there is at least 10% of globular clusters with distances greater than the indicated limit value.

Note that the dynamic relationship (24) and Figure 4 can also be used to reinterpret the rotation curves of the MoND-Milgrom theories (Milgrom 1983a,b; Falcon 2013) where the velocity dispersion increases monotonically with radial distance instead of the Keplerian behavior predicted by the inverse-square law of the distance.

In Figure 5 (left), the comparison between Newtonian forces and that associated with the U_{YF} term is shown on a logarithmic scale, for interstellar distance ranges. Both forces would be comparable in the 15–40 kpc intervals, being able to explain the missing mass in the rotation curves. In the range of distances >50 kpc, the Newtonian force is negligible compared to the inertia caused by the large-scale distribution of matter caused by the F_{IY} force.

On the other hand, in much greater distance ranges, for example for the interior of galaxy clusters, the force is proportional to the inverse of the comoving distance $F_{\mathrm{IY}}(r) \propto (r - 10)r^{-2}$, as shown in Figure 5 (right). The Newtonian force is 30 orders of magnitude smaller than F_{IY} on this scale. The null value of F_{IY} at r = 10 Mpc is in agreement with the previous discussion about the graviton's rest mass (Section 2). As said before, at cosmological distance scales the F_{IY} force is repulsive and manifests itself as the cosmic acceleration (dark energy). In ranges of comoving distances, F_{IY} is negligible and the gravitational force is prescribed by the law of the inverse square of the distance (Newtonian gravitation).

4.4 *Gravitational redshift*

The total astronomical redshift is the addition of the Doppler redshift due to the emitter–receiver movement, plus the one caused by the gravitational field (gravitational redshift) and cosmological redshift due to the cosmic expansion. The gravitational component of the redshift (z_g), in terms of the gravitational potential ϕ (Evans & Dunning-Davies 2004), is:

$$z_g = \frac{\Delta\lambda}{\lambda_e} = \frac{\lambda_{ob} - \lambda_e}{\lambda_e} = -\frac{\phi}{c^2}. \quad (42)$$

The photons emitted by the Newtonian potential source (GM/R) would also be affected by the local

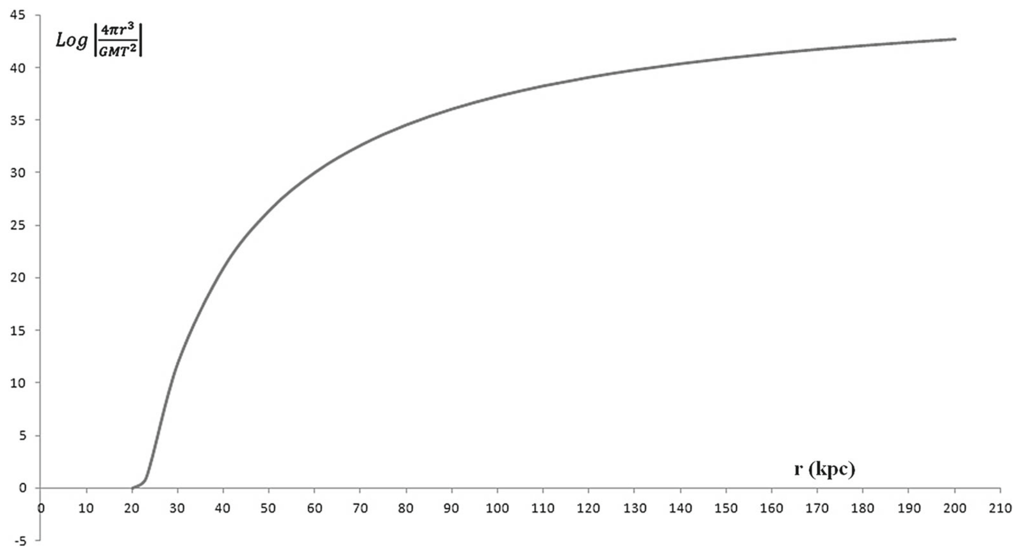

Figure 4. Expected deviation of Kepler's third law in globular clusters as a function of the comoving distance to the Milky Way center.

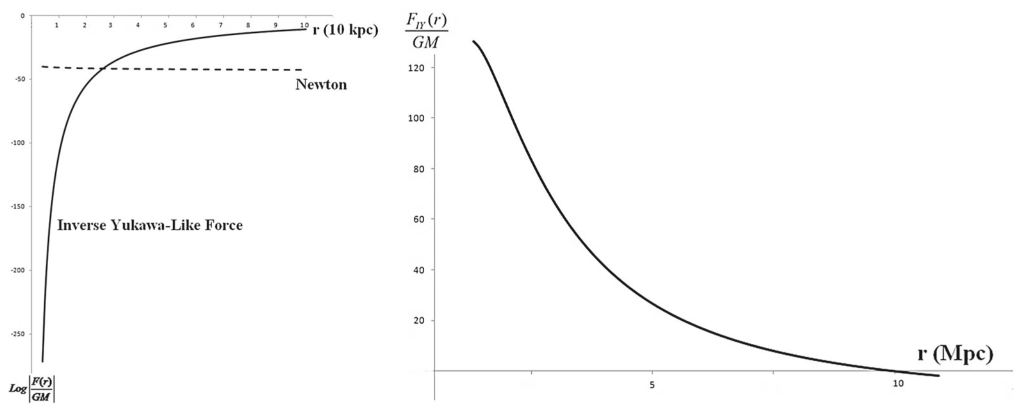

Figure 5. Relative intensity of the force due to Yukawa's field as a function of comoving distance, in the 10–100 kpc range, for interstellar scale (left), and in the 1–10 Mpc range, for intra-clusters of galaxies (right).

contribution of the gravitational field produced by the large-scale distribution of matter; then

$$z_g = -\frac{1}{c^2}\left(-\frac{GM}{R} + U_{\mathrm{YF}}\right), \tag{43}$$

where R is the radius of the emitting source and its comoving distance is r. Using (1) and (3), U_{YF} can be written as:

$$U_{\mathrm{YF}}(x) \equiv U_0(M) r_0 (x-1)\,\mathrm{e}^{-\alpha_0/x}. \tag{44}$$

Using the previous notation $x \equiv r/r_0$ and $\alpha_0 \equiv \alpha/r_0 = 0.05$, $U_0(M) r_0 = 4\pi GMl$, then

$$z_g = \left[\frac{R_s}{2R}\right]\left[-\frac{1}{R} + 4\pi l \mathrm{e}^{-\alpha_0/x}(x-1)\right], \tag{45}$$

where Schwarzschild's radius is used, as $R_s = 2GM/c^2$, and x is the comoving distance in Mpc.

Now the inertial frame provides, through U_{YF}, an additional contribution in gravitational redshift that increases with the comoving distance (x) (Figure 6). Note that the additional term is null when the distance to the source is 2.5 Mpc. An important point in (44) is that, two galaxies at the same distance could be present in different redshift depending on the particular R_s of each. These resolve Arp's controversy (Arp 2003).

Beyond 10 Mpc, the gravitational redshift remains constant in accordance with the previous assumption of the massive graviton. Then the maximum additional contribution (44) of the U_{YF} to the gravitational redshift is 7.83 ($x \geq 10\,\mathrm{Mpc}$) and z_g increases by a factor until 88.3. This result is interesting to understand the problem of AGN at high redshift and the nature of quasars. Also in Figure 6 (right), the

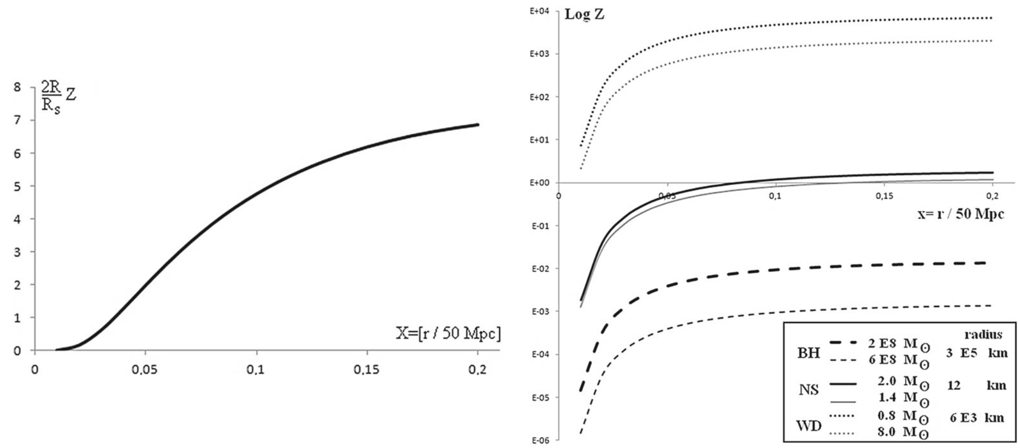

Figure 6. Gravitational redshift *vs.* comoving distance. For general objects in terms of the Schwarzschild radius (left) and compact objects in terms of dimensionless x (right).

variation (in logarithmic scale) of the gravitational redshift for different objects as a function of distance is shown; in particular, the observation of white dwarves in the interior of the globular clusters could be a test for U_{YF}. The extreme cases of high redshift are the SBH in the centers of galaxies, with the event horizon of the order of a light-year and masses in order of the core-mass in M87 (Event Horizon Telescope Collaboration 2019).

We have seen how the U_{YF} term associated with the large-scale distribution of matter causes a change in the effective gravitational potential in the gravitational redshift; it would be expected then that it also affects the deflection of light in the formalism of gravitational lenses; their discussion is very extensive and is beyond the scope of this communication; but the formalism consists of replacing the Schwarzschild radius incorporating the U_{YF}, analogously to (46).

4.5 *Jean's mass*

In the formalism of the gravitational collapse of protogalactic clouds, temporal evolution is considered through the gravitational amplification of density perturbation, that depends critically on the time needed for gravitational free-fall collapse (t_g) in comparison with the travel time of acoustic waves (t_s). Jean's length (λ_J^0) is the characteristic length for which pressure balances gravity; which in terms of the density (ρ), temperature (T_{e}), and hydrogen mass (m_{H}) is:

$$\lambda_J^0 \equiv \frac{c_s}{t_s} \simeq \frac{c_s}{t_g} = \sqrt{\frac{3\pi k_B T_e}{G\rho m_{\mathrm{H}}}}. \tag{46}$$

However, the effective gravitational free-fall collapse is now due to the Newtonian potential plus the contribution of U_{IY}. Replacing (41) into (46), given that $t_g \approx T$ then

$$\lambda_J = \lambda_J^0 [1 - A r^{-3} \mathrm{e}^{-\alpha/r} (r^2 + \alpha(r - r_0))]^{-1/2}. \tag{47}$$

where λ_{J} denotes the effective or "true" Jean's length due to the inclusion of the U_{IY} field and the A parameter (Perez *et al.* 2015), in terms of the density of the number of particles (n_0), it is given by

$$A \equiv \frac{3U_o(M)}{4\pi G\rho} \simeq \frac{3Mlr_0^{-1}}{\rho} \simeq \frac{5.8\,\mathrm{Mpc}}{n_0\ \mathrm{cm}^3}. \tag{48}$$

We observe in Figure 7 that the modification of the Newtonian gravitation on a large scale does not affect Jean's length in ranges of comoving distance <100 kpc, even for densities as low as that of the intergalactic medium. Increasing the density of the number of particles tends to smooth the difference between Jean's lengths with and without the large-scale correction for Newtonian gravity. For low densities, such as those in the intergalactic medium, the difference between the two models for Jean's length is significant. It is of the order of 40% less than λ_J^0 in the distance ranges of 0.5–2 Mpc, which coincides with the agglutination of hot gas observed in X-rays in galaxy clusters.

In the previous discussion, the free fall time was used; therefore, it is implicitly assumed that the protostellar or protogalactic cloud has dimensions of the order of r in the gravitational potential (both in Newtonian and in U_{IY}). Both lengths coincide when $x \ll 1\,\mathrm{Mpc}$; so, no changes are expected in the

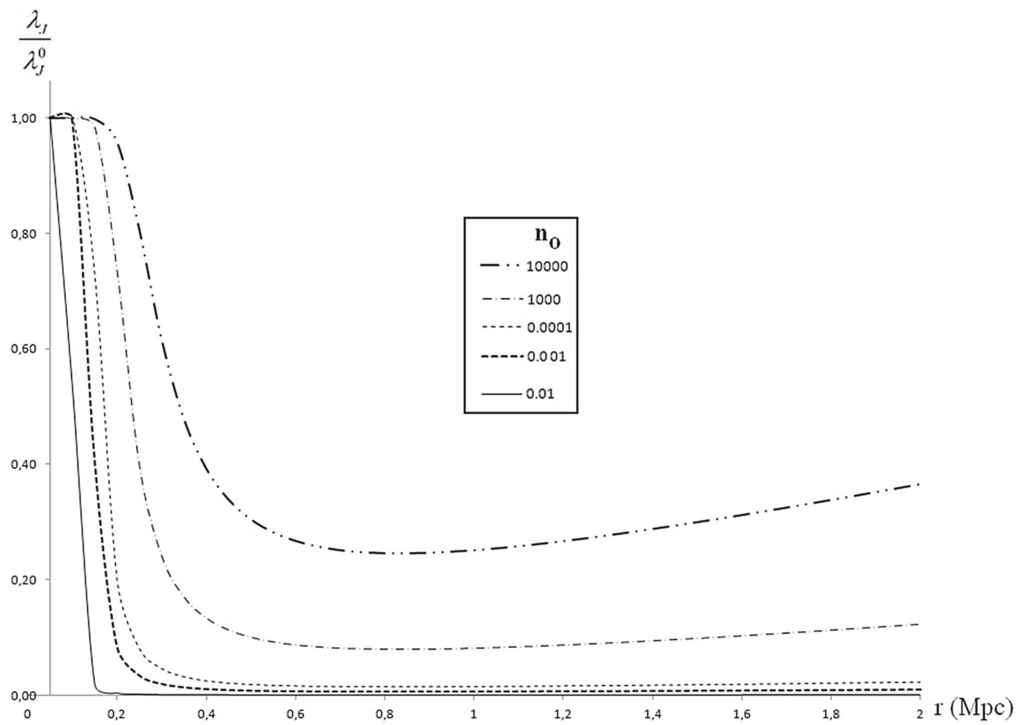

Figure 7. Variation of Jean's length at different scales.

fragmentation of protoplanetary or protostellar molecular clouds, for which $r \ll 1$ kpc. However, the inclusion of $U_{\rm IY}$ does modify, as seen in Figure 7, the fragmentation of protogalactic clouds, according to the initial phenomenological description (Figure 1) where the dynamics is prescribed differently for the different length scales.

5. Conclusions

The global description of the observable Universe that we have today motivates the review of gravitation (Trippe 2014; Debono & Smoot 2016; Huterer & Shafer 2018; Riess 2019; Genova–Santos 2020) and the postulation of new theoretical alternatives. Particularly interesting could be generalizations consistent with experimental data, astronomical observations, and the well-proven formalism of existing theories. This is the case of the proposed potential $U_{\rm YF}$, which is a heuristic construction useful to modify the Newtonian inverse-square law. The origin of this hypothetical $U_{\rm YF}$ potential is only the ordinary baryonic matter and represents the inertia due to the distribution of matter on a large scale, incorporating Mach's principle into the formalism. The proposed potential allows a local variation of gravity depending on the range of the distance as shown in Figure 1. It is consistent with the observations and leads to several important consequences:

1. All particles with no null rest mass are subject to gravitational inverse-square laws, plus an additional force term that varies with distance, called the $U_{\rm YF}$ field. At large distances from the sources, the reduction in the Newtonian field would be complemented by an additional interaction component that grows at much greater distances.
2. This $U_{\rm YF}$ field depends on comoving distance: null in the inner solar system, weakly attractive in ranges of interstellar distances, very attractive in distance ranges comparable to the clusters of galaxies and repulsive to cosmic scales.
3. The minimum of the potential energy of the $U_{\rm YF}$ field, located at the order of the comoving distance of 10 Mpc, implies the nullity of the force of gravity, and consequently predicts a graviton mass of at least 10^{-64} kg.
4. Also, the force ($F_{\rm IY}$) supplementary to Newtonian gravity would reach a maximum for distances of the order of 1.2 Mpc, favoring the agglutination of matter in galaxy clusters, and would be evidenced as a maximum in the redshift volume density distribution around $z = 2$ (Figure 3).

5. The inclusion of the large-scale term of the gravity removes the incompatibility between the flatness of the Universe and the density of matter in the Friedmann equation, without invoking the non-baryonic dark matter (22).
6. The usual cold dark matter parameter (Ω_c) would be the gravitational contribution caused by the large-scale distribution of the ordinary baryonic matter throughout the $U_{\rm YF}$ field (24).
7. The dark energy would result to be the cosmic acceleration in local frameworks, caused by the large-scale distribution of the ordinary baryonic matter, as prescribed by Mach's principle, through the $U_{\rm YF}$ proposed (32).
8. The Hubble–Lemaître's law would be the manifestation on a cosmic scale of the $U_{\rm IY}$ field, with theoretical expression as in (26). The theoretical upper limit, evaluated for the most distant objects ($r \gg 50$ Mpc) would be 86.3 km $\rm s^{-1}$ $\rm Mpc^{-1}$.
9. The age of the universe would not be much more than $\tau \sim 11.42$ Gyr, in good agreement with direct WD-cosmochronology.
10. The virial theorem is formally derived to show that the field $U_{\rm YF}$ implies an additional term in the energy balance due to the distribution of matter on a large scale. Therefore, the missing mass could be interpreted as the energy associated with the $U_{\rm YF}$ field and it solves Zwicky's paradox.
11. The comparison between Newtonian forces and that associated to the $U_{\rm YF}$ term results in: both forces would be comparable in the 15–40 kpc intervals, being able to explain the missing mass in the rotation curves. Inside galaxy clusters, the force is proportional to the inverse of the comoving distance (Figure 5).
12. $U_{\rm YF}$ provides an additional contribution for the gravitational redshift that increases with the comoving distance (45) and z_g increases by a factor until 88% ($r \gg 50$ Mpc).
13. The increase in gravitational redshift due to the $U_{\rm YF}$ field also implies that two galaxies located at the same distance could exhibit a different redshift based on their particular Schwarzschild radii. This solves Arp's controversy. This result is interesting to understand the problem of AGN at high redshift and the nature of quasars.
14. The modification of the Newtonian gravitation on a large scale does not affect Jean's length in ranges of comoving distance <100 kpc (Figure 7), but the Jean's length results 40% less at distance ranges of Mpc, which coincides with the agglutination of hot gas in X-ray galaxy clusters.
15. The energy density of the universe must increase due to the graviton's rest mass, which is incorporated through the $U_{\rm YF}$ field in Friedmann's Equation (18). Consequently, the critical density no longer corresponds to the Einstein-de Sitter model, which lacks a material basis.

Regardless of whether the expression for the so-called "Inverse Yukawa-like field" ($U_{\rm YF}$) proposed is exactly the proposal here, we see that its inclusion could be a viable alternative to the paradigm of non-baryonic dark matter and is concomitant with the Hot big bang model and the most recent astronomical observations; and would be thinks with the usual physics.

Appendix

Additional note about the virial theorem

Beginning with the Clasius's virial expression, we have:

$$\frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}G}{\mathrm{d}t}\mathrm{d}t' = \frac{1}{\tau}\int_0^\tau \sum_i -\vec{\nabla} U_i \cdot \vec{r}_i\,\mathrm{d}t' + \frac{1}{\tau}\int_0^\tau \sum_i \frac{p_i^2}{2m_i}\mathrm{d}t',$$

$$\frac{G(\tau)-G(0)}{\tau} = -\frac{1}{\tau}\int_0^\tau \sum_i \left(\frac{GMm_i}{r_i} + m_i\vec{F}_{\rm IY}(r)\cdot\vec{r}_i\right)\mathrm{d}t' + \frac{1}{\tau}\int_0^\tau \sum_i T_i\,\mathrm{d}t.$$

$$0 = -\frac{1}{\tau}\int_0^\tau \sum_i \left(\frac{GMm_i}{r_i} + \frac{m_i U_0(M)}{r}\,\mathrm{e}^{-\alpha/r}[r^2 + \alpha(r-r_0)]\right)\mathrm{d}t' + 2\langle T\rangle,$$

$$0 = -\left\langle \sum_i \frac{GMm_i}{r_i} + \sum_i m_i U_0(M)\,\mathrm{e}^{-\alpha/r}(r-r_0)\right\rangle + 2\langle T\rangle + \frac{1}{\tau}\int_0^\tau \sum_i m_i\,U_0(M)\,\mathrm{e}^{-\alpha/r}\left(\alpha + r_0 - \frac{\alpha r_0}{r}\right)\mathrm{d}t'.$$

The term within the integral is a continuous function, whose domains are all real numbers, differentiable on the interval (0, τ) and bounded by 1 and $f(\alpha)$, then follow with (38).

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