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Cosmic future of universe inferred from the horizon behaviours in $\Lambda \propto a^{-2}$, $\Lambda \propto H^2$, $\Lambda \propto \rho$ cosmological constant models

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Abstract. To investigate the cosmic future of the universe, first, we obtain the particle and event horizons, and their evolution in time for three well-known varying cosmological constant models. We investigate the implications of these varying cosmological constant models on the particle and event horizons, and their time evolution during the universe history and future. We study the behaviours around the origin and at the far future of the universe for all cases. Finally, we obtained the general behaviours of the horizons and time evolution functions with respect to the general scale factor a for each of the three cases of varying cosmological constant. The results show that for two of our three cases, a big bounce scenario is inevitable for the universe starting from the big bang and ending up with a bouncing re-collapse to its initial state.

Keywords. Particle horizon—event horizon—cosmology—big bounce—cosmological constant—dark energy.

1. Introduction

Supernovae Type 1A observations have proved that the universe experiences an accelerated expansion from the early exponential inflationary expansion phase to present phase (Guth 1982; Albrecht & Steinhardt 1982; Linde 1982; Riess *et al.* 1998; Perlmutter *et al.* 1999). To explain this expansion, cosmologists have proposed the cosmological constant which was mistakenly first proposed and revoked by Einstein, and ΛCDM model has been born (Kirshner 2004; Bass 2011; Sola 2015; Arun 2017; Sola *et al.* 2017). There is still no consensus about what cosmological constant really is, and no exact field theory that gives the observed cosmological constant value. Observational results show a 120-order magnitude smaller actual value than the predicted value by quantum field theories. To fix this cosmological constant problem, the expansion of universe has been connected to dark energy, modified gravity theories and finally, varying cosmological constant models (Lima & Carvalho 1994; Vishwakarma 2001; Ryden 2002; Melia 2007; Miao *et al.* 2011; Bamba *et al.* 2012; Oztas & Smith 2015; Des 2016; Riess *et al.* 2016; Abbott *et al.* 2017a, b; Oztas 2018; Oztas *et al.* 2018; Dil *et al.* 2019).

Here, we consider three previously proposed varying cosmological constant models by us and others (Lima & Carvalho 1994; Vishwakarma 2001; Melia 2007; Oztas & Smith 2015; Oztas 2018; Oztas *et al.* 2018; Dil *et al.* 2019) to calculate the proper particle and event horizons and their evolution in time. The boundary between the past and future of an observer who is observing the observable universe as his past, is called the particle horizon. While the particle horizon defines only the past events for the observer, another horizon, called as event horizon defines the future event for the observer. The event horizon is the boundary beyond which is not observable future of the observer (Rindler 1956).

In the literature, the particle and event horizons and the evolution of them have been widely studied (Harisson 1991; Margalef *et al.* 2012, 2013) for ΛCDM model containing different constituents of

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universe, such as matter and cosmological constant, or matter, radiation and cosmological constant, or any other possible combinations of non-varying cosmological constant.

We investigate the possible implications and results of the three different varying cosmological constant models on the proper particle and event horizons and their evolution in time, which means the velocity of the change of particle and event horizons during the universe history and future. We begin with the varying cosmological constant $\Lambda \propto 1/a(t)^2$ model, where $a(t)$ is the increasing scale factor of the expanding universe. Then, we consider the $\Lambda \propto H^2$ model as a second case, where H is the Hubble parameter, which decreases with the age and expansion of universe. Later on, we take $\Lambda \propto \rho(t)$ model as case 3, where $\rho(t)$ is the decreasing matter density with the expansion of universe. According to these three cases, our varying cosmological constant decreases with time.

In this paper, we first obtain the proper particle and event horizons for these three cases of the cosmological constant. Later on, we investigate the time evolution of these horizons which give the recession velocity of the horizons for these three cases. Moreover, we study the behaviours of the proper particle and event horizons, and their velocities around the origin of the universe $a \to 0$ and at the far future of the universe $a \to \infty$ for all cases. Finally, we investigate the general behaviours of our 12 functions of particle and event horizons and velocities with respect to the general scale factor a for each of the three cases of varying cosmological constant, and interpret the implications of our results.

2. Particle horizon

The Friedmann equations for a FRW universe are given by

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda(t)c^2}{3}, \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda(t)c^2}{3}, \tag{2}$$

where $\Lambda(t)$ is the varying cosmological constant in time. Then, the particle horizon from the beginning of the universe to any time now is given by

$$P_H = \int_0^t \frac{c\mathrm{d}t}{a}. \tag{3}$$

For more general purposes, this equation can be transformed as

$$P_H = c\int_0^t \frac{1}{a}\frac{\mathrm{d}t}{\mathrm{d}a}\mathrm{d}a = c\int_0^a \frac{1}{a}\frac{\mathrm{d}a}{\dot{a}} = c\int_0^a \frac{1}{a}\frac{\mathrm{d}a}{aH}, \tag{4}$$

where $\dot{a}/a = H$ is used. The particle horizon in (3) or (4) is the co-moving particle horizon at any time t, by multiplying it by the scale factor a, we obtain the proper particle horizon at any time t, such that

$$P_H(a) = ac\int_0^a \frac{1}{a^2}\frac{\mathrm{d}a}{H} = aP_H. \tag{5}$$

Time evolution of the particle horizon is obtained from the time derivative of (5) by using the chain rule, as follows

$$\frac{\mathrm{d}P_H(a)}{\mathrm{d}t} = \frac{\mathrm{d}P_H(a)}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}(aP_H)}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t} = \left(P_H + a\frac{\mathrm{d}P_H}{\mathrm{d}a}\right)aH, \tag{6}$$

where we use $\mathrm{d}a/\mathrm{d}t = aH$. It is also clear from (4) that $\mathrm{d}P_H/\mathrm{d}a = c/a^2H$, then using it in (6) yields

$$\frac{\mathrm{d}P_H(a)}{\mathrm{d}t} = aP_HH + c. \tag{7}$$

Moreover, the term aP_HH in (7) is the recession velocity of the particle horizon due to the expansion of the universe. The distance covered by light from the origin of the universe plus the distance due to this expansion effect is already equal to the proper particle horizon.

To obtain H in Equations (5) and (7), we obtain ρ from (1) as

$$\frac{4\pi G}{3}\rho = \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{2}\frac{kc^2}{a^2} + \frac{1}{2}\frac{\Lambda(t)c^2}{3} \tag{8}$$

and insert into (2), as

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{2}\frac{kc^2}{a^2} + \frac{1}{2}\Lambda(t)c^2. \tag{9}$$

Here, we consider the matter dominant case of the universe as $p = 0$. Now, to investigate the particle horizon and its time evolution by obtaining H, we need to consider the universe models consisting of three different varying cosmological constant cases proposed by us (Oztas *et al.* 2018).

2.1 $\Lambda(t) \propto \frac{1}{a(t)^2}$ *case for particle horizon*

In this model, we assume that the cosmological constant decreases inversely proportional to the square of the expansion parameter of the universe

$$\Lambda(t) = A\frac{\Lambda_0}{a^2(t)}, \tag{10}$$

where A is the proportionality constant. Using this in (9), we obtain

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{2}\frac{kc^2}{a^2} + \frac{\mathrm{A}}{2}\frac{\Lambda_0 c^2}{a^2}. \tag{11}$$

We can now introduce the density parameters for the matter, varying cosmological constant and curvature terms, respectively, as

$$\Omega = \frac{\rho}{\rho_c}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda c^2/8\pi G}{\rho_c},$$

$$\Omega_k = \frac{\rho_k}{\rho_c} = \frac{3kc^2/8\pi G a^2}{\rho_c}, \tag{12}$$

where $\rho_c = 3H^2/8\pi G$ is the critical density. Consequently, the current values of these density parameters at $t = t_0$ are obtained as

$$\Omega_0 = \frac{8\pi G\rho_0}{3H_0^2}, \quad \Omega_{\Lambda,0} = \frac{\Lambda_0 c^2}{3H_0^2}, \quad \Omega_{k,0} = -\frac{kc^2}{H_0^2}. \tag{13}$$

Using these density parameters in (11), we find

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{2}\frac{\Omega_{k,0}H_0^2}{a^2} + \frac{\mathrm{A}}{2}\frac{3\Omega_{\Lambda,0}H_0^2}{a^2}. \tag{14}$$

Here, we will use $\dot{a}/a = H$ to obtain

$$\frac{\mathrm{d}a}{\mathrm{d}t} = aH,$$

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}(aH) = \frac{\mathrm{d}a}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}a}(aH)$$
$$= aH\left(H + a\frac{\mathrm{d}H}{\mathrm{d}a}\right) = aH^2 + a^2H\frac{\mathrm{d}H}{\mathrm{d}a}. \tag{15}$$

Then, we use the obtained result in (15) instead of $\ddot{a}$ in (14), and obtain

$$H^2 + aH\frac{\mathrm{d}H}{\mathrm{d}a} = -\frac{1}{2}H^2 + \frac{1}{2}\frac{\Omega_{k,0}H_0^2}{a^2} + \frac{3}{2}\mathrm{A}\frac{\Omega_{\Lambda,0}H_0^2}{a^2}. \tag{16}$$

Arranging this equation yields

$$aH\frac{\mathrm{d}H}{\mathrm{d}a} + \frac{3}{2}H^2 = \frac{1}{2}\frac{3A\Omega_{\Lambda,0}H_0^2 + \Omega_{k,0}H_0^2}{a^2} \tag{17}$$

and from the partial differentiation, it gives

$$\frac{\mathrm{d}}{\mathrm{d}a}(H^2) + \frac{3}{a}H^2 = \frac{3A\Omega_{\Lambda,0}H_0^2 + \Omega_{k,0}H_0^2}{a^3}. \tag{18}$$

Solution of the differential equation for H^2 gives

$$H^2 = \frac{3A\Omega_{\Lambda,0}H_0^2 + \Omega_{k,0}H_0^2}{a^2} + \frac{C}{a^3}, \tag{19}$$

where the integration constant C is obtained from the present day value of scale factor $a(t_0) = 1$ and Hubble constant H_0 as

$$H_0^2 = 3A\Omega_{\Lambda,0}H_0^2 + \Omega_{k,0}H_0^2 + C,$$
$$C = H_0^2(1 - 3A\Omega_{\Lambda,0} - \Omega_{k,0}). \tag{20}$$

Finally, by using C in (19), H is found to be

$$H = H_0\sqrt{\frac{3\Omega_{\Lambda,0} + \Omega_{k,0}}{a^2} + \frac{1 - 3\Omega_{\Lambda,0} - \Omega_{k,0}}{a^3}}, \tag{21}$$

where we take $A = 1$. Using this H in (5), we obtain the particle horizon as

$$P_H = c\int_0^a \frac{1}{a}\frac{\mathrm{d}a}{aH_0\sqrt{\frac{3\Omega_{\Lambda,0}+\Omega_{k,0}}{a^2} + \frac{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}{a^3}}}$$
$$= D_H\int_0^a \frac{1}{a}\frac{\mathrm{d}a}{\sqrt{3\Omega_{\Lambda,0} + \Omega_{k,0} + \frac{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}{a}}}. \tag{22}$$

where $D_H = c/H_0$. The integral yields

$$P_H = D_H\frac{2}{\sqrt{|3\Omega_{\Lambda,0} + \Omega_{k,0}|}} \times \mathrm{arcsinh}\left(\sqrt{\left|\frac{3\Omega_{\Lambda,0} + \Omega_{k,0}}{1 - 3\Omega_{\Lambda,0} - \Omega_{k,0}}\right|a}\right), \tag{23}$$

the co-moving particle horizon at any time. By multiplying it scale factor a, we obtain the proper particle horizon $P_H(a) = aP_H$ as in (5). Moreover, using (21) and (23) in Equation (7), the time evolution of the particle horizon is obtained as

$$\frac{dP_H(a)}{dt} = 2c\sqrt{1+\frac{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}{(3\Omega_{\Lambda,0}+\Omega_{k,0})a}} \times \operatorname{arcsinh}\left(\sqrt{\left|\frac{3\Omega_{\Lambda,0}+\Omega_{k,0}}{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}\right|a}\right) + c. \tag{24}$$

Here, first term on the right hand side of the equation represents the recession velocity of the particle horizon.

2.2 $\Lambda \propto H^2$ *case for particle horizon*

This model assumes the cosmological constant decreases with the age of the universe and is given by

$$\Lambda(t) = 3A\Omega_{\Lambda,0}H^2. \tag{25}$$

Inserting this Λ in Equation (9) gives

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{2}\frac{\Omega_{k,0}H_0^2}{a^2} + \frac{3}{2}A\Omega_{\Lambda,0}H^2. \tag{26}$$

Using (15) instead of $\ddot{a}$ above yields

$$H^2 + aH\frac{dH}{da} = -\frac{1}{2}H^2 + \frac{1}{2}\frac{\Omega_{k,0}H_0^2}{a^2} + \frac{3}{2}A\Omega_{\Lambda,0}H^2 \tag{27}$$

and

$$\frac{d}{da}(H^2) + \frac{3(1-A\Omega_{\Lambda,0})}{a}H^2 = \frac{\Omega_{k,0}H_0^2}{a^3}. \tag{28}$$

Solution of this differential equation is

$$H^2 = \frac{\Omega_{k,0}H_0^2}{1-3A\Omega_{\Lambda,0}}\frac{1}{a^2} + \frac{C}{a^{3(1-A\Omega_{\Lambda,0})}}. \tag{29}$$

Using the present day conditions, the integral constant C is found to be

$$C = H_0^2\frac{1-3A\Omega_{\Lambda,0}-\Omega_{k,0}}{1-3A\Omega_{\Lambda,0}}. \tag{30}$$

After it is used in H^2, we find

$$H = H_0\sqrt{\frac{\Omega_{k,0}}{(1-3\Omega_{\Lambda,0})a^2} + \frac{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}{(1-3\Omega_{\Lambda,0})a^{3(1-\Omega_{\Lambda,0})}}}, \tag{31}$$

where we used $A = 1$. Using this H in (5), we find the particle horizon as

$$\begin{aligned} P_H &= c\int_0^a \frac{1}{a}\frac{da}{aH_0\sqrt{\frac{\Omega_{k,0}}{(1-3\Omega_{\Lambda,0})a^2} + \frac{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}{(1-3\Omega_{\Lambda,0})a^{3(1-\Omega_{\Lambda,0})}}}}, \\ &= D_H\int_0^a \frac{1}{a}\frac{da}{\sqrt{\frac{\Omega_{k,0}}{(1-3\Omega_{\Lambda,0})} + \frac{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}{(1-3\Omega_{\Lambda,0})}a^{(3\Omega_{\Lambda,0}-1)}}}, \end{aligned} \tag{32}$$

and after the integration, it is

$$P_H = D_H\frac{2}{\sqrt{|\Omega_{k,0}(1-3\Omega_{\Lambda,0})|}} \times \operatorname{arcsinh}\left(\sqrt{\left|\frac{\Omega_{k,0}}{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}\right|\frac{1}{a^{3\Omega_{\Lambda,0}-1}}}\right). \tag{33}$$

By multiplying this co-moving particle horizon by the scale factor a, we again obtain the proper particle horizon $P_H(a) = aP_H$ as in (5). Also, by using (31) and (33) in Equation (7), the time evolution can be obtained as

$$\frac{dP_H(a)}{dt} = \frac{2c}{|1-3\Omega_{\Lambda,0}|} \times \sqrt{1+\frac{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}{\Omega_{k,0}}a^{3\Omega_{\Lambda,0}-1}} \times \operatorname{arcsinh}\left(\sqrt{\left|\frac{\Omega_{k,0}}{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}\right|\frac{1}{a^{3\Omega_{\Lambda,0}-1}}}\right) + c. \tag{34}$$

The first term on the right hand side again represents the recession velocity of the particle horizon.

2.3 $\Lambda \propto \rho(t)$ *case for particle horizon*

In this model, the particle horizon and its time evolution will be calculated for a varying cosmological constant, which has decreased by decreasing matter content of the universe in time, and it is given by

$$\Lambda(t) = \mathrm{A}8\pi G\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\rho(t). \tag{35}$$

For this case, we insert the cosmological constant into Equations (1) and (2), and arrange to obtain

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1+\mathrm{A}\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right) + \frac{\Omega_{k,0}H_0^2}{a^2}, \tag{36}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho\left(1 - 2A\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right). \quad (37)$$

By inserting ρ from Equation (36) into (37), we reach

$$\frac{\ddot{a}}{a} = \frac{K_3}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{K_3}{2}\frac{\Omega_{k,0}H_0^2}{a^2}, \quad (38)$$

where

$$K_3 = \frac{2\Omega_{\Lambda,0} - \Omega_{m,0}}{\Omega_{\Lambda,0} + \Omega_{m,0}}, \quad (39)$$

where $A = 1$ is taken. Substituting $\ddot{a}$ (15) into (39) leads to

$$H^2 + aH\frac{dH}{da} = \frac{K_3}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{K_3}{2}\frac{\Omega_{k,0}H_0^2}{a^2}. \quad (40)$$

After arranging it to give the differential equation, we obtain

$$\frac{d}{da}(H^2) + \frac{2 - K_3}{a}H^2 = -K_3\frac{\Omega_{k,0}H_0^2}{a^3}, \quad (41)$$

whose solution is

$$H^2 = \frac{\Omega_{k,0}H_0^2}{a^2} + Ca^{K_3-2}. \quad (42)$$

From the present day values, C turns out to be

$$C = H_0^2(1 - \Omega_{k,0}), \quad (43)$$

which is to be used in H^2 to give

$$H = H_0\sqrt{\frac{\Omega_{k,0}}{a^2} + (1 - \Omega_{k,0})a^{K_3-2}}. \quad (44)$$

We insert H into the particle horizon integral in (5), then

$$P_H = c\int_0^a \frac{1}{a}\frac{da}{aH_0\sqrt{\frac{\Omega_{k,0}}{a^2} + (1 - \Omega_{k,0})a^{K_3-2}}} = D_H\int_0^a \frac{1}{a}\frac{da}{\sqrt{\Omega_{k,0} + (1 - \Omega_{k,0})a^{K_3}}}, \quad (45)$$

the integration yields

$$P_H = D_H\frac{2}{K_3\sqrt{|\Omega_{k,0}|}} \times \operatorname{arcsinh}\left(\sqrt{\left|\frac{\Omega_{k,0}}{1 - \Omega_{k,0}}\right|\frac{1}{a^{K_3}}}\right). \quad (46)$$

By multiplying this by the scale factor a, we again obtain the proper particle horizon $P_H(a) = aP_H$ as in (5). Also, by using (44) and (46) in Equation (7), the time evolution can be obtained as

$$\frac{dP_H(a)}{dt} = \frac{2c}{K_3}\sqrt{1 + \frac{1 - \Omega_{k,0}}{\Omega_{k,0}}a^{K_3}} \times \operatorname{arcsinh}\left(\sqrt{\left|\frac{\Omega_{k,0}}{1 - \Omega_{k,0}}\right|\frac{1}{a^{K_3}}}\right) + c. \quad (47)$$

The first term on the right hand side is the recession velocity of the particle horizon. In the following sections, we will analyse the behaviour of the particle horizon and its time evolution, recession velocity, around the origin of the universe ($a \to 0$) and the far future of the universe ($a \to \infty$).

3. Event horizon

It is the distance travelled by the light from the present time to the maximum possible future time. Normally, for the event horizon, the integral limits in Equation (3) is from the present time $a = 1$, to possible far future $a = \infty$. However, for the event horizon at any time, it is given by the same integral with different limits as

$$E_H = c\int_t^\infty \frac{dt}{a} = c\int_a^\infty \frac{1}{a}\frac{da}{\dot{a}} = c\int_a^\infty \frac{1}{a}\frac{da}{aH}. \quad (48)$$

This is the co-moving event horizon from which we obtain the proper event horizon $E_H(a)$ at any time by multiplying it by the scale factor a as in (5)

$$E_H(a) = ac\int_a^\infty \frac{1}{a^2}\frac{da}{H} = aE_H. \quad (49)$$

We can obtain the time evolution of the event horizon from the time derivative of (49) with the chain rule, such as

$$\frac{dE_H(a)}{dt} = \frac{dE_H(a)}{da}\frac{da}{dt} = \frac{d(aE_H)}{da}\frac{da}{dt} = \left(E_H + a\frac{dE_H}{da}\right)aH. \quad (50)$$

From the integral (48), we infer that $dE_H/da = -c/a^2H$, and use it in (50), then

$$\frac{dE_H(a)}{dt} = aE_HH - c. \quad (51)$$

Moreover, the term aE_HH in (51) is the recession velocity of the event horizon due to the expansion of the universe. The integrands in (5) and (49) are same, but the integral limits are different. We will investigate the effect of this difference on event horizon and its time evolution for three different varying cosmological constant cases.

3.1 $\Lambda(t) \propto \frac{1}{a(t)^2}$ *case for event horizon*

The Hubble parameter of the $\Lambda(t) \propto 1/a(t)^2$ case is obtained in (21), using this in the event horizon (49) yields

$$E_H = D_H \int_a^{\infty} \frac{1}{a} \frac{\mathrm{d}a}{\sqrt{3\Omega_{\Lambda,0} + \Omega_{k,0} + \frac{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}{a}}} \tag{52}$$

and the integration gives

$$E_H = D_H \frac{2}{\sqrt{|3\Omega_{\Lambda,0} + \Omega_{k,0}|}} \times \operatorname{arcsinh}\left(\sqrt{\left| \frac{3\Omega_{\Lambda,0} + \Omega_{k,0}}{1 - 3\Omega_{\Lambda,0} - \Omega_{k,0}} \right| a} \right) \Bigg|_a^{\infty}. \tag{53}$$

After substituting the integral limits, the event horizon diverges for all possible conditions for $\Lambda(t) \propto 1/a(t)^2$ case as

$$E_H = \infty. \tag{54}$$

Therefore, the proper event horizon $E_H(a) = aE_H$ in (50) and its time evolution (51) also diverge.

3.2 $\Lambda \propto H^2$ *case for event horizon*

The Hubble parameter of the $\Lambda(t) \propto H^2$ case is found in (31), and when we insert it into the event horizon (49), we obtain

$$E_H = D_H \int_a^{\infty} \frac{1}{a} \frac{\mathrm{d}a}{\sqrt{\frac{\Omega_{k,0}}{(1-3\Omega_{\Lambda,0})} + \frac{1-3\Omega_{\Lambda,0}-\Omega_{k,0}}{(1-3\Omega_{\Lambda,0})} (a^{3\Omega_{\Lambda,0}-1})}} \tag{55}$$

and after the integration, it is

$$E_H = D_H \frac{2}{\sqrt{|\Omega_{k,0}(1 - 3\Omega_{\Lambda,0})|}} \times \operatorname{arcsinh}\left(\sqrt{\left| \frac{\Omega_{k,0}}{1 - 3\Omega_{\Lambda,0} - \Omega_{k,0}} \right| \frac{1}{a^{3\Omega_{\Lambda,0}-1}}} \right) \Bigg|_a^{\infty}. \tag{56}$$

Substituting the integral limits yields

$$E_H = \begin{cases} \infty, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 < 0, \\ \infty, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 = 0, \\ \frac{-2D_H}{\sqrt{|\Omega_{k,0}(1 - 3\Omega_{\Lambda,0})|}} \times \operatorname{arcsinh}\left(\sqrt{\left| \frac{\Omega_{k,0}}{1 - 3\Omega_{\Lambda,0} - \Omega_{k,0}} \right| \frac{1}{a^{3\Omega_{\Lambda,0}-1}}} \right), & \text{if} \quad 3\Omega_{\Lambda,0} - 1 > 0. \end{cases} \tag{57}$$

By multiplying it by the scale factor a, we find the proper event horizon $E_H(a) = aE_H$ as in (49). As it is clear that the event horizon converges only for the $3\Omega_{\Lambda,0} - 1 > 0$ case. Moreover, by using (31) and (57) in Equation (51), the time evolution of the event horizon is obtained as

$$\frac{\mathrm{d}E_H(a)}{\mathrm{d}t} = \frac{-2c}{|1 - 3\Omega_{\Lambda,0}|} \sqrt{1 + \frac{1 - 3\Omega_{\Lambda,0} - \Omega_{k,0}}{\Omega_{k,0}} a^{3\Omega_{\Lambda,0}-1}} \times \operatorname{arcsinh}\left(\sqrt{\left| \frac{\Omega_{k,0}}{1 - 3\Omega_{\Lambda,0} - \Omega_{k,0}} \right| \frac{1}{a^{3\Omega_{\Lambda,0}-1}}} \right) - c \tag{58}$$

for $3\Omega_{\Lambda,0} - 1 > 0$ case. The first term on the right hand side denotes the recession velocity of the event horizon.

3.3 $\Lambda \propto \rho(t)$ *case for event horizon*

The Hubble parameter of the $\Lambda(t) \propto \rho(t)$ case is obtained in (44), using this in the event horizon (49) gives

$$E_H = D_H \int_a^{\infty} \frac{1}{a} \frac{\mathrm{d}a}{\sqrt{\Omega_{k,0} + (1 - \Omega_{k,0}) a^{K_3}}} \tag{59}$$

and the integration yields

$$E_H = D_H \frac{2}{K_3 \sqrt{|\Omega_{k,0}|}} \operatorname{arcsinh}\left(\sqrt{\left| \frac{\Omega_{k,0}}{1 - \Omega_{k,0}} \right| \frac{1}{a^{K_3}}} \right) \Bigg|_a^{\infty}. \tag{60}$$

Substituting the integral limits yields

$$E_H = \begin{cases} \infty, & \text{if} \quad K_3 < 0, \\ \infty, & \text{if} \quad K_3 = 0, \\ \frac{-2D_H}{K_3 \sqrt{|\Omega_{k,0}|}} \operatorname{arcsinh}\left(\sqrt{\left| \frac{\Omega_{k,0}}{1 - \Omega_{k,0}} \right| \frac{1}{a^{K_3}}} \right), & \text{if} \quad K_3 > 0. \end{cases} \tag{61}$$

If we multiply it by the scale factor a, we obtain the proper event horizon $E_H(a) = aE_H$ in (49). It is obvious that the event horizon converges only for the $K_3 > 0$

case. Also, using (44) and (61) in Equation (51), the time evolution of the event horizon is obtained as

$$\frac{\mathrm{d}E_H(a)}{\mathrm{d}t} = \frac{-2c}{K_3}\sqrt{1+\frac{1-\Omega_{k,0}}{\Omega_{k,0}}a^{K_3}} \times \operatorname{arcsinh}\left(\sqrt{\left|\frac{\Omega_{k,0}}{1-\Omega_{k,0}}\right|\frac{1}{a^{K_3}}}\right) - c. \quad (62)$$

The first term on the right hand side is the recession velocity of the event horizon. In the following sections, we will analyze the behaviour of the event horizon and its time evolution, recession velocity, around the origin of the universe ($a \to 0$) and the far future of the universe ($a \to \infty$).

4. Horizons and velocities at origin and far future of universe

4.1 *Origin of universe*

4.1.1 $\Lambda(t) \propto 1/a(t)^2$ *case* For the $\Lambda(t) \propto 1/a(t)^2$ case, the horizons and velocities behave for the $a \to 0$, such as: The proper particle horizon

$$\lim_{a\to 0} P_H(a) = 0, \quad (63)$$

from (5) $P_H(a) = aP_H$ and (23), and its time derivative

$$\lim_{a\to 0} \frac{\mathrm{d}P_H(a)}{\mathrm{d}t} = \frac{2}{|3\Omega_{\Lambda,0} + \Omega_{k,0}|}, \quad (64)$$

from (24). Now, when we investigate the behaviour of the event horizon for same limit, because it is $E_H = \infty$, we see

$$\lim_{a\to 0} E_H(a) = \infty, \quad (65)$$

from (49) and (54), and its time evolution

$$\lim_{a\to 0} \frac{\mathrm{d}E_H(a)}{\mathrm{d}t} = \infty, \quad (66)$$

from (51) and (54).

4.1.2 $\Lambda \propto H^2$ *case* For the $\Lambda(t) \propto H^2$ case, the horizons and velocities behave for the $a \to 0$, such as: The proper particle horizon

$$\lim_{a\to 0} P_H(a) = \begin{cases} 0, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 < 0 \\ \infty, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 = 0\,, \\ 0, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 > 0 \end{cases} \quad (67)$$

from (5) $P_H(a) = aP_H$ and (33), and its time evolution

$$\lim_{a\to 0} \frac{\mathrm{d}P_H(a)}{\mathrm{d}t} = \begin{cases} \emptyset, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 < 0 \\ \infty, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 = 0\,, \\ \infty, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 > 0 \end{cases} \quad (68)$$

from (34). Now, we investigate the behaviour of the proper event horizon for same limit as

$$\lim_{a\to 0} E_H(a) = \begin{cases} \infty, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 < 0 \\ \infty, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 = 0 \\ 0, & \text{if} \quad 3\Omega_{\Lambda,0} - 1 > 0 \end{cases} \quad (69)$$

from (49) $E_H(a) = aE_H$ and (57), and for the only non-diverging $3\Omega_{\Lambda,0} - 1 > 0$ case the velocity behaves as

$$\lim_{a\to 0} \frac{\mathrm{d}E_H(a)}{\mathrm{d}t} = -\infty, \quad (70)$$

from (58).

4.1.3 $\Lambda \propto \rho(t)$ *case* For the $\Lambda(t) \propto \rho(t)$ case, the horizons and velocities behave for the $a \to 0$, such as: The proper particle horizon

$$\lim_{a\to 0} P_H(a) = \begin{cases} 0, & \text{if} \quad K_3 < 0 \\ \infty, & \text{if} \quad K_3 = 0\,, \\ 0, & \text{if} \quad K_3 > 0 \end{cases} \quad (71)$$

from (5) $P_H(a) = aP_H$ and (46), and its time evolution

$$\lim_{a\to 0} \frac{\mathrm{d}P_H(a)}{\mathrm{d}t} = \begin{cases} \emptyset, & \text{if} \quad K_3 < 0 \\ \infty, & \text{if} \quad K_3 = 0\,, \\ \infty, & \text{if} \quad K_3 > 0 \end{cases} \quad (72)$$

from (47). This behaviour of $P_H(a)$ and its time derivative are similar to ones in $\Lambda \propto H^2$ case in Section 4.1.2, but the physical meaning of the conditions on $3\Omega_{\Lambda,0} - 1$ and K_3 differ. Now, we investigate the behaviour of the proper event horizon as

$$\lim_{a\to 0} E_H(a) = \begin{cases} \infty, & \text{if } K_3 < 0 \\ \infty, & \text{if } K_3 = 0 \\ 0, & \text{if } K_3 > 0 \end{cases} \quad (73)$$

from (49) $E_H(a) = aE_H$ and (61), and for the only non-diverging $K_3 > 0$ case, the velocity behaves as

$$\lim_{a\to 0} \frac{\mathrm{d}E_H(a)}{\mathrm{d}t} = -\infty, \quad (74)$$

from (62). This behaviour of $E_H(a)$ and velocity are similar to ones in $\Lambda \propto H^2$ case in Section 4.1.2, but the physical meaning of the conditions on $3\Omega_{\Lambda,0} - 1$ and K_3 differ.

These limits are for the behaviours of the particle and event horizons, and their velocities at the origin of the universe, $a \to 0$. In the following section, we will investigate behaviours of same quantities in the far future of the universe, $a \to \infty$.

4.2 *Far future of universe*

4.2.1 $\Lambda(t) \propto 1/a(t)^2$ *case* For the $\Lambda(t) \propto 1/a(t)^2$ case, the horizons and velocities behave for the $a \to \infty$, such as: The proper particle horizon

$$\lim_{a\to\infty} P_H(a) = \infty \tag{75}$$

from (5) $P_H(a) = aP_H$ and (23), and its time derivative

$$\lim_{a\to\infty} \frac{\mathrm{d}P_H(a)}{\mathrm{d}t} = \infty \tag{76}$$

from (24). Now, when we investigate the behaviour of the event horizon for same limit, because it is $E_H = \infty$, we see

$$\lim_{a\to\infty} E_H(a) = \infty \tag{77}$$

from (49) and (54), and its time evolution

$$\lim_{a\to\infty} \frac{\mathrm{d}E_H(a)}{\mathrm{d}t} = \infty \tag{78}$$

from (51) and (54).

4.2.2 $\Lambda \propto H^2$ *case* For the $\Lambda(t) \propto H^2$ case, the horizons and velocities behave for the $a \to \infty$, such as: The proper particle horizon

$$\lim_{a\to\infty} P_H(a) = \begin{cases} \infty, & \text{if } 3\Omega_{\Lambda,0} - 1 < 0 \\ \infty, & \text{if } 3\Omega_{\Lambda,0} - 1 = 0\,, \\ \infty, & \text{if } 3\Omega_{\Lambda,0} - 1 > 0 \end{cases} \tag{79}$$

from (5) $P_H(a) = aP_H$ and (33), and its time evolution

$$\lim_{a\to\infty} \frac{\mathrm{d}P_H(a)}{\mathrm{d}t} = \begin{cases} \infty, & \text{if } 3\Omega_{\Lambda,0} - 1 < 0 \\ \infty, & \text{if } 3\Omega_{\Lambda,0} - 1 = 0 \\ \dfrac{2c}{|1 - 3\Omega_{\Lambda,0}|} + c, & \text{if } 3\Omega_{\Lambda,0} - 1 > 0 \end{cases}, \tag{80}$$

from (34). Now, we investigate the behaviour of the proper event horizon for same limit as

$$\lim_{a\to\infty} E_H(a) = \begin{cases} \infty, & \text{if } 3\Omega_{\Lambda,0} - 1 < 0 \\ \infty, & \text{if } 3\Omega_{\Lambda,0} - 1 = 0\,, \\ -\infty, & \text{if } 3\Omega_{\Lambda,0} - 1 > 0 \end{cases} \tag{81}$$

from (49) $E_H(a) = aE_H$ and (57), and for the only non-diverging $3\Omega_{\Lambda,0} - 1 > 0$ case, the velocity behaves as

$$\lim_{a\to\infty} \frac{\mathrm{d}E_H(a)}{\mathrm{d}t} = -\frac{2c}{|1 - 3\Omega_{\Lambda,0}|} - c, \tag{82}$$

from (58).

4.2.3 $\Lambda \propto \rho(t)$ *case* For the $\Lambda(t) \propto \rho(t)$ case, the horizons and velocities behave for the $a \to \infty$, such as: The proper particle horizon

$$\lim_{a\to\infty} P_H(a) = \begin{cases} \infty, & \text{if } K_3 < 0 \\ \infty, & \text{if } K_3 = 0\,, \\ \infty, & \text{if } K_3 > 0 \end{cases} \tag{83}$$

from (5) $P_H(a) = aP_H$ and (46), and its time evolution

$$\lim_{a\to\infty} \frac{\mathrm{d}P_H(a)}{\mathrm{d}t} = \begin{cases} \infty, & \text{if } K_3 < 0 \\ \infty, & \text{if } K_3 = 0 \\ \dfrac{2c}{|K_3|} + c, & \text{if } K_3 > 0 \end{cases}, \tag{84}$$

from (47). This behaviour of $P_H(a)$ and its time derivative are similar to ones in $\Lambda \propto H^2$ case in Section 4.2.2, but the physical meaning of the conditions on $3\Omega_{\Lambda,0} - 1$ and K_3 differ. Now, we investigate the behaviour of the proper event horizon as

$$\lim_{a\to\infty} E_H(a) = \begin{cases} \infty, & \text{if } K_3 < 0 \\ \infty, & \text{if } K_3 = 0\,, \\ -\infty, & \text{if } K_3 > 0 \end{cases} \tag{85}$$

from (49) $E_H(a) = aE_H$ and (61), and for the only non-diverging $K_3 > 0$ case, the velocity behaves as

$$\lim_{a\to\infty} \frac{\mathrm{d}E_H(a)}{\mathrm{d}t} = -\frac{2c}{|K_3|} - c, \tag{86}$$

from (62). This behaviour of $E_H(a)$ and velocity are similar to ones in $\Lambda \propto H^2$ case in Section 4.2.2, but the physical meaning of the conditions on $|1 - 3\Omega_{\Lambda,0}|$ and $|K_3|$ differ.

These limits are for the behaviours of the particle and event horizons, and their velocities in the far future of the universe, $a \to \infty$.

We illustrate all the behaviours of proper particle and event horizons, and their time evolution at the origin and far future of the universe in Table 1.

5. Conclusions

We consider a universe containing a varying cosmological constant in three different cases with different dependencies. For these cases, we first introduce the calculation of the co-moving particle horizons. Later on, we obtain the proper particle horizons and their time evolution for three different varying cosmological constant cases in Equations (24), (34) and (47). Then, we calculate the co-moving and proper event horizons, and time evolution of the proper horizons for three varying cosmological constant cases in Equations (54), (57) and (61).

Table 1. Behaviours of horizons and their time evolution at the origin and far future of the universe for three cases.

$$\left(\gamma = \frac{2}{|3\Omega_{\Lambda,0} + \Omega_{k,0}|},\ \kappa_1 = \frac{2c}{|1 - 3\Omega_{\Lambda,0}|} + c,\ \kappa_2 = \frac{2c}{|K_3|} + c\right)$$

Cases: At the origin $a \to 0$; $P_H(a) = 0$			
$\Lambda \propto \frac{1}{a^2}$	$\frac{dP_H(a)}{dt} = \gamma$	$E_H(a) = \infty$	$\frac{dE_H(a)}{dt} = \infty$
$\Lambda \propto H^2$	$\frac{dP_H(a)}{dt} = \infty$	$E_H(a) = 0$	$\frac{dE_H(a)}{dt} = -\infty$
$\Lambda \propto \rho(t)$	$\frac{dP_H(a)}{dt} = \infty$	$E_H(a) = 0$	$\frac{dE_H(a)}{dt} = -\infty$
Cases: At the far future $a \to \infty$; $P_H(a) = \infty$			
$\Lambda \propto \frac{1}{a^2}$	$\frac{dP_H(a)}{dt} = \infty$	$E_H(a) = \infty$	$\frac{dE_H(a)}{dt} = \infty$
$\Lambda \propto H^2$	$\frac{dP_H(a)}{dt} = \kappa_1$	$E_H(a) = -\infty$	$\frac{dE_H(a)}{dt} = -\kappa_1$
$\Lambda \propto \rho(t)$	$\frac{dP_H(a)}{dt} = \kappa_2$	$E_H(a) = -\infty$	$\frac{dE_H(a)}{dt} = -\kappa_2$

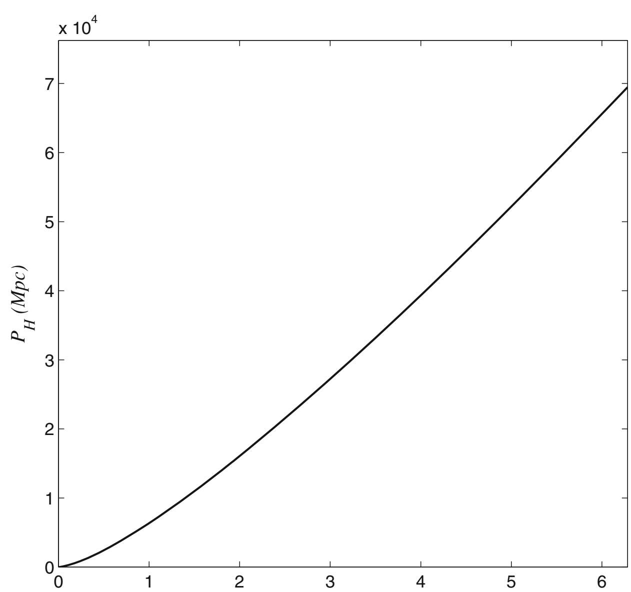

Figure 1. Behaviour of particle horizon for case 1.

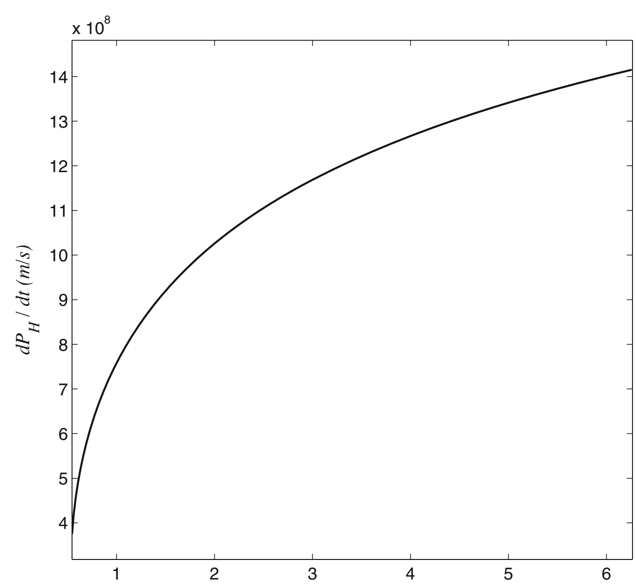

Figure 2. Behaviour of particle horizon velocity for case 1.

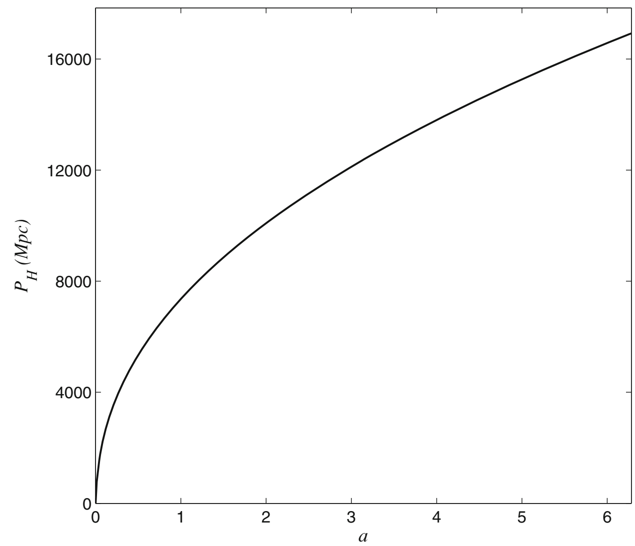

Figure 3. Behaviour of particle horizon for case 2.

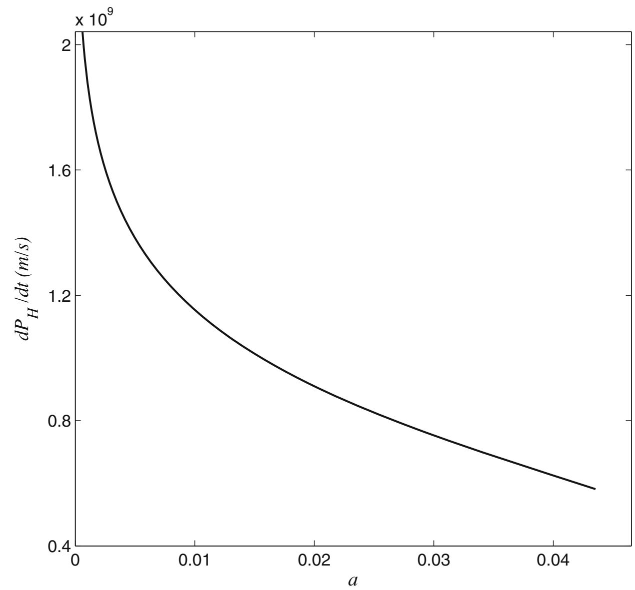

Figure 4. Behaviour of particle horizon velocity for case 2.

Because the particle horizon is defined as the distance covered by the light from the origin of universe to present time, our particle horizon results in Equations (23), (33) and (46) imply an expanding universe from a zero size via the positive particle horizons in three cases of varying cosmological constant. Moreover, the event horizon is defined as the possible maximum distance covered by the light from the present time to possible maximum future time, and therefore, our event horizons results in Equations (57) and (61) of second $\Lambda \propto H^2$ and third $\Lambda \propto \rho(t)$ cases, which imply a crunching and re-collapsing of universe via the negative event horizons.

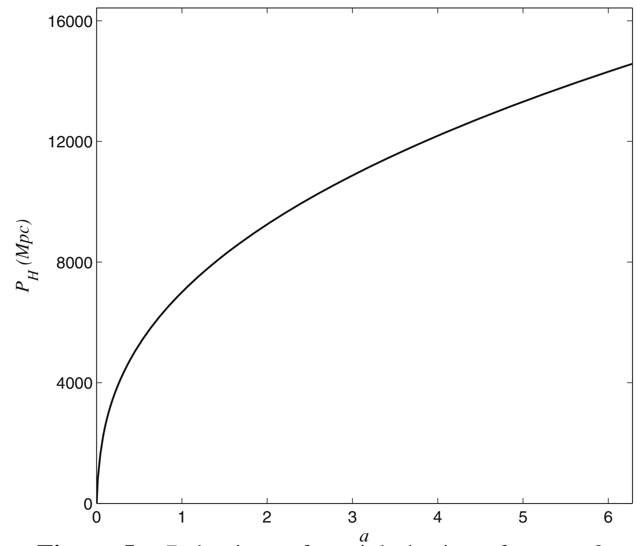

Figure 5. Behaviour of particle horizon for case 3.

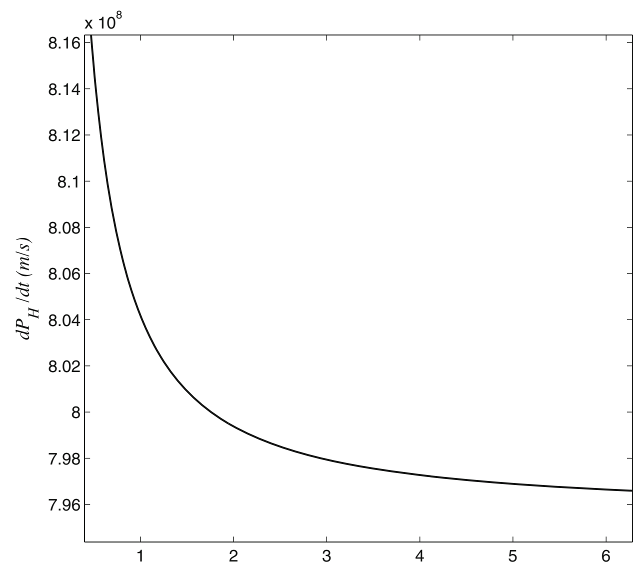

Figure 6. Behaviour of particle horizon velocity for case 3.

After obtaining the proper particle and event horizons and their time evolution, we study the asymptotic behaviours of these functions at the origin of the universe $a \to 0$, and at the far future of the universe $a \to \infty$ for three cases of the cosmological constant in Section 4. Also, We present all the behaviours of the particle and event horizons, and their velocities at these limits in Table 1. Accordingly, we find that particle horizons start from 0 and end up at ∞ for all three cases. Then, velocities of the particle horizons start from ∞, and end up at constant values κ_1 and κ_2, except from the case 1, where the velocity starts from a constant value γ, and ends up at ∞. The results of

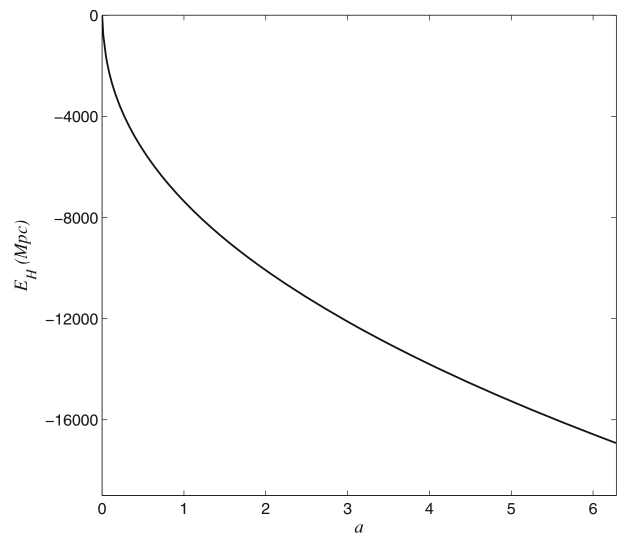

Figure 7. Behaviour of event horizon for case 2.

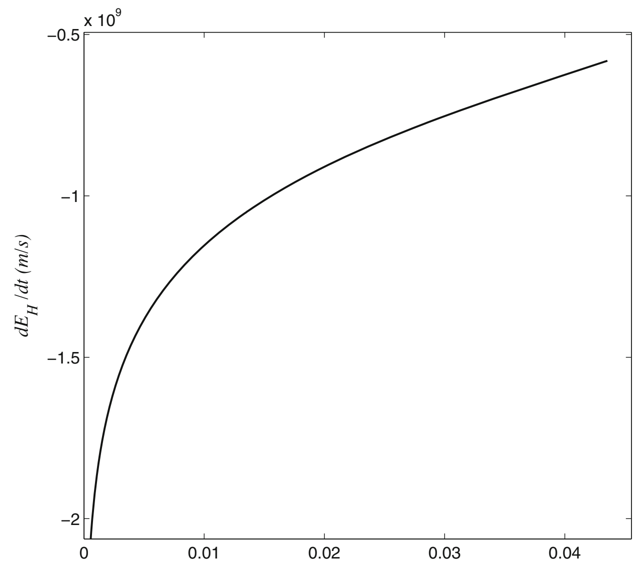

Figure 8. Behaviour of event horizon velocity for case 2.

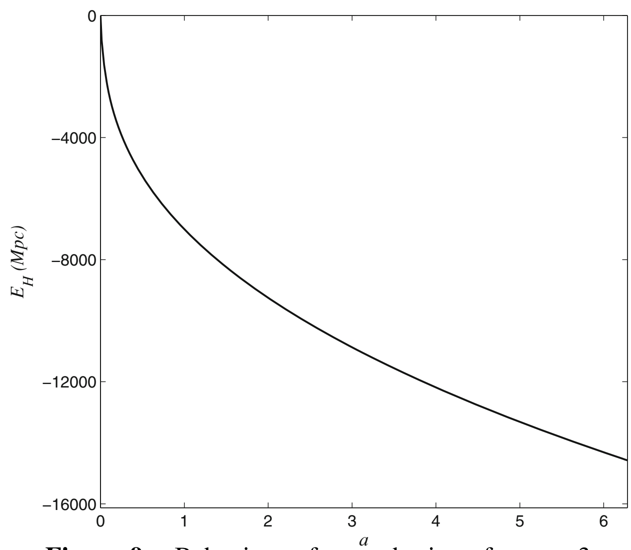

Figure 9. Behaviour of event horizon for case 3.

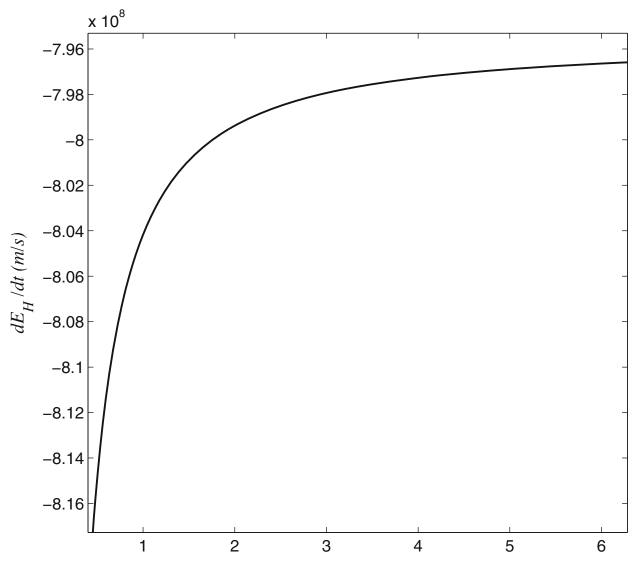

Figure 10. Behaviour of event horizon velocity for case 3.

cases 2 and 3 are in accordance with the observations, such that the universe inflates at a very high speed, while its radius increases from 0 to a finite value today. Moreover, the case 1, $\Lambda \propto 1/a^2$ reveals an inconsistency for the velocity of the particle horizon, such that it diverges at the far future of the universe.

On the other hand, the event horizons start from 0, and end up at $-\infty$ for the last two cases. Then, for these cases, velocities of the event horizons start from $-\infty$, and end up at constant values $-\kappa_1$ and $-\kappa_2$. In the first case, event horizon is obtained divergent before investigating the $a \to 0$ and $a \to \infty$ behaviours, therefore, its velocity is also obtained divergent. The negativity of the horizons and velocities can be interpreted as the universe experiences a re-collapse at the far future.

Except the case 1, all our findings are consistent with predictions for big bounce models of the universe for the certain conditions of cases 2 and 3. The existence of an event horizon, and the behaviours of the horizons and their velocities, implying the big bounce cosmologies, are valid for $3\Omega_{\Lambda,0} - 1 > 0$ condition in case 2, and for $K_3 > 0$ condition in case 3. The condition of case 2 implies that $\Omega_{\Lambda,0} > 1/3$ and it is validated because the observed value is about 0.70 for the density parameter of the cosmological constant. Similarly, the condition of case 3 implies that

$\Omega_{\Lambda,0} > \Omega_{m,0}/2$ from Equation (39), which is also validated due to the value of the cosmological constant.

For all a values, the general behaviours of the proper and event horizons, and their velocities are presented in Figures 1–10. Although the behaviours of particle horizons at $a \rightarrow 0$ and $a \rightarrow \infty$ are same, we observe a difference on the general behaviours of particle horizon cases 1, 2 and 3. In case 1, the particle horizon increases linearly, but in cases 2 and 3, it increases exponentially till $a = 1$ and linearly after $a = 1$. However, the general behaviour of the velocity of the particle horizon for case 1 is completely opposite to the cases 2 and 3. While it increases to infinity in case 1, it decreases to a constant value in cases 2 and 3. On the other hand, the general behaviour of event horizon and its velocity are not presented in figures for case 1 because of the divergent nature of them, but for the cases 2 and 3, they are presented. From Figures 7–10, we observe that the general behaviours of event horizons in cases 2 and 3 are very close to each other, and represent an increase toward the $-\infty$, at the origin of the big bang, implying a re-collapsing scenario of the big bounce cosmologies. The velocity of this re-collapse changes from a very fast speed to constant value in the negative direction for cases 2 and 3.

As concluding remark, the varying cosmological constant models, $\Lambda \propto H^2$ (case 2) and $\Lambda \propto \rho(t)$ (case 3) imply a big bounce cosmology, where the universe starts from a zero radius, and proceeds with a finite value, then re-collapses back to the original zero value. During the expansion phase, it starts from an infinite velocity and proceeds to a finite value, and then re-collapses back with a finite negative velocity.

References

Abbott T. M. C., Abdalla F. B., Alarcon A., Aleksic J., Allam S. 2017a, arXiV:1708.01530v1

Abbott T. M. C., Abdalla F. B., Annis J., Betchtol K., Benson A. 2017b, arXiV:1711.00403

Albrecht A., Steinhardt P. S. 1982, Phys. Rev. Lett., 48, 1220

Arun K. 2017, Adv. Space Res., 60, 166

Bamba K., Capozziello S., Nojiri S., Odintsov S. D. 2012, Astrophys. Space Sci. 342, 155

Bass S. D. 2011, J. Phys. G, 38, 043201

Dark Energy Survey (DES), 2016

Dil E., Oztas A. M., Dil E. 2019, Astron. Comput., 28, 100295

Guth A. H. 1982, Phys. Rev. D, 23, 347

Harisson E. 1991, ApJ, 383, 60

Kirshner R. P. 2004, Proc. Natl. Acad. Sci. USA, 101, 8

Lima J. A. S., Carvalho J. C. 1994, Gen. Rel. Gravit. 26, 909

Linde A. D. 1982, Phys. Lett. B, 108, 389

Margalef-Bentabol B., *et al.* 2012, JCAP, 2012, 035

Margalef-Bentabol B., *et al.* 2013, JCAP, 2012, 015

Melia F. 2007, MNRAS, 382, 1917

Miao L., Xiao-Dong L., Shuang W., Yi W. 2011, Commun. Theor. Phys. 56, 525

Oztas A. M., Smith M. L. 2015, MNRAS, 449, 1270

Oztas A. M. 2018, MNRAS, 481, 2238

Oztas A. M., Dil E., Smith M. L. 2018, MNRAS, 476, 458

Perlmutter S., Aldering G., Goldhaber G., Knop R. G., Nugent P. 1999, ApJ, 517, 565

Riess A. G., Filippenko A. V., Challis P., Clocchiatti A., Dierks A. 1998, AJ, 116, 1009

Riess A. G., Macri L. M., Hoffmann S. L., Scolnic D., Casertano S. 2016, ApJ, 826, 56

Rindler W. 1956, MNRAS, 116, 662

Ryden B. 2002, *Introduction to Cosmology*, Addison-Wesley, Reading, MA

Sola J. 2015, J. Phys. Conf. Ser., 453, 012015

Sola J., Gomez-Valent A., de Cruz Perez J. 2017, arXiv:1709.07451v1

Vishwakarma R. G. 2001, Class. Quantum Grav., 18, 1159