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Trapped region in Kerr–Vaidya space–time

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Abstract. We review the basic definitions and properties of trapped surfaces and discuss them in the context of Kerr–Vaidya line element. Our study shows that the apparent horizon does not exist in general for axisymmetric space–times. The reason being the surface at which the null tangent vectors are geodesics and the surface at which the expansion of such vectors vanishes do not coincide. The calculation of an approximate apparent horizon for space–times that ensure its existence seems to be the only way to get away with this problem. The approximate apparent horizon, however, turned out to be non-unique. The choice of the shear-free null geodesics, at least in the leading order, seems to remove this non-uniqueness. We also propose a new definition of the black hole boundary.

Keywords. Trapped surface—Kerr–Vaidya metric—black hole boundary—apparent horizon—geometric trapped surface—geometric horizon.

1. Introduction

The notion of the trapped surface was first introduced by Penrose (1965), and the concept later emerged as a quasi-local technique to characterize a black hole. They are seen as a replacement to the event horizon to define the black hole boundary, which depends on the global knowledge of space–time. The teleological nature (Ashtekar & Krishnan 2004; Ashtekar & Galloway 2005) and the unobservability (Visser 2014) of the event horizon have been discussed numerously in the past. It has thus been regarded as unsuitable for characterizing astrophysical black holes, which are dynamic in nature and have to be studied by a finite-sized laboratory for the finite time. However, even though the trapped surfaces are quasi-local making them observable in principle, they are not a well-defined quantity to characterize black holes. The principal reason behind this is the foliation dependence of the trapped surface. As the choice of the foliation depends on the choice of null vectors, the trapped surface depends on the choice of null vectors (Faraoni *et al.* 2017).

Defining a black hole boundary has been like a reverse engineering process. We somehow know the answer for some simple cases based on which we want to define a black hole boundary such that the definition reproduces that answer. Such simple cases are stationary space–times of spherical and axial symmetry for which all the horizons (event, apparent, killing, and trapping horizons) coincide. The notion of trapped surfaces introduced in this way to define a black hole boundary poses no problem at all for stationary space–times, even though trapped surfaces depend on the choice of foliations for these space–times (Krishnan 2014). Then comes the second stage of making use of the definition of the trapped surface to identify the black hole boundary of the space–time for which the concept of an event/killing horizon fails. The trapped surface as a black hole boundary has celebrated its success in application to the general spherically symmetric space–times. The problem of foliation dependence, in that case, can be eliminated by making the particular choice of foliation that respects the symmetry of space–time (Faraoni *et al.* 2017). All spherically symmetric foliations give the same trapped surface for such space–times. This problem, however, is more visible and profound if we move to the more general case of axially symmetric space–times, and there do not exist simple solutions to this problem in axial symmetry.

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This article is the introduction of the trapped surface and its application to the axially symmetric space–time by taking the reference of the Kerr–Vaidya metric. We introduce trapped surfaces, define various types of trapped surfaces, and discuss, in brief, the notion of the geometric horizon as a black hole boundary in Section 1.1. Recently, the geometric horizon has been considered as an alternative over the trapped surface in locating the black hole boundary because of its invariant characterization, quasi-local nature, and foliation independence. However, its applicability has been demonstrated only for limited classes of space–times until now. In Section 1.2, we present some of the properties of the trapped surfaces based on their metric, and in Section 1.3, we introduce the slowly evolving horizon. The concept of a slowly evolving horizon is useful in the sense that it does not mingle with the problem of foliation dependence, provided that the condition for its existence satisfies. Section 2 describes the application of the trapped surface to the axisymmetric space–time, and for this, we chose one of the simplest examples of Kerr–Vaidya geometry in advanced coordinates. We also discuss our results in the context of the geometric horizon. Finally, in Section 3, we propose a new black hole boundary.

1.1 *Trapped surfaces and geometrical horizons*

We begin here with the notion of an event horizon, which is believed to be the true boundary of a black hole by some authors despite its teleological nature.

Asymptotically flat space–time containing black holes can be separated into two regions: (1) the region from which all null curves can reach the future null infinity (where future-directed null curves end in an infinite time in the Minkowskian space–time) and (2) the region from which no null curve reaches the future null infinity. A boundary separating these two regions of space–time is defined as an event horizon. This definition of the event horizon depends on the complete knowledge of future-directed null curves, which is impossible to achieve in a finite time making it physically unobservable (Visser 2014). This global nature of the event horizon also makes it a less useful entity for dynamical space–times. Similarly, the definition of the event horizon could not be applied to space–times that are not asymptotically flat. Thus, the notion of the trapped surfaces that depends on the quasi-local nature of space–time is handy in describing black holes evolving in time. In principle, it is not possible to define a horizon that can be detected by completely local measurements. The reason behind this is rooted in the equivalence principle, which implies an impossibility of measuring space–time curvature by local measurements.

In the orientable space–time manifold $\mathcal{M}$, let us take a future-directed field of null vectors l^α and n^α satisfying $l^\alpha n_\alpha = -1$. We assume that l^α is outgoing in the sense it is pointed away from the trapped region, and n^α is ingoing. The divergence (or the convergence) of the field of null congruence l^μ is given by its covariant derivative $l^\mu_{;\mu}$. If the vector field l^μ is arbitrarily parameterized, then we affinely parameterize it before the calculation of divergence, and this introduces the additional correction term $\kappa = -n_\mu l^\nu l^\mu_{;\nu}$, called the surface gravity. Thus, the divergence (or convergence) of the affinely parameterized null congruence is given as (Faraoni 2015)

$$\theta_l = l^\mu_{;\mu} + n_\mu l^\nu l^\mu_{;\nu} \quad \text{and} \quad \theta_n = n^\mu_{;\mu} + n_\mu l^\nu n^\mu_{;\nu}, \tag{1}$$

where θ_l denotes the outgoing expansion and θ_n denotes the ingoing expansion (for details on the expansion of congruence see textbook by Poisson (2004)). Among these, the condition for the vanishing of outgoing expansion $\theta_l = 0$ is pivotal for the definition of trapped surfaces. A surface is said to be trapped if $\theta_l < 0$ and $\theta_n < 0$. Thus, on the trapped surface, both the ingoing and outgoing null geodesics are converging. A marginally outer trapped surface (MOTS) has $\theta_l = 0$ and $\theta_n < 0$. This surface outlines the boundary from which the outgoing null geodesics start to converge. The three-dimensional surface, which can be foliated entirely by the MOTS, is called a marginally outer trapped tube (MOTT). This MOTT is defined as the future apparent horizon in recent literature (Barbado *et al.* 2016; Levi & Ori 2016), and we will adopt this definition here. The definition of an apparent horizon and, in general, of the trapped surfaces varies over the literature. The Hayward's trapping horizon is defined as an apparent horizon satisfying $n^\mu(\theta_l)_{;\mu} \neq 0$ (Hayward 1994). This additional condition imposed implies that the space–time surface is trapped inside the future outer trapping horizon and normal outside it. Similarly, a space like apparent horizon is defined as a dynamical horizon (Ashtekar & Krishnan 2004).

The definitions of trapped surfaces given here provide an easy way of identifying black holes and these are just by the calculation of null expansions θ_l and θ_n. We thus do not have to wait until the end of time to know the formation of an event horizon to identify black holes. Trapped surfaces indeed are convenient and unique for the numerical and analytical study of spherically symmetric space–times (for spherically symmetric foliation, which is an obvious choice of foliation because of the symmetry of space–time). However, even for the trapped surfaces not being entirely local entities, their identification depends on the closed hypersurface Σ of the space–time whose foliation spans the whole space–time. Expansions should be calculated over all the points of the closed hypersurface Σ, not just at a single point for their identification. Hence, trapped surfaces are quasi-local in nature and depend on the complete knowledge of the hypersurface Σ, if not of space–time.

As mentioned above, with any choice of spherically symmetric foliations, a trapped surface in spherically symmetric space–time is unique. However, the complication starts to appear when we consider foliations that are not spherically symmetric, even to calculate the trapped surface in spherically symmetric space–times. The foliation dependence of the trapped surface in Vaidya space–time has been demonstrated by numerical calculations with some special choice of non-spherically symmetric foliation in Schnetter & Krishnan (2006) and Nielsen *et al.* (2011). The discussion of the foliation dependence of the trapped surface in Schwarzschild and Vaidya space–time can also be found in the review by Krishnan (2014). We will describe the trapped surface in Kerr–Vaidya space–time and its foliation dependence.

Another type of horizon that is commonly used to characterize the black hole boundary is the notion of geometric horizon (Coley *et al.* 2017; Coley & McNutt 2018). The existence of the geometric horizon is assured by the conjecture that the non-stationary black holes contain the hypersurface, which is more algebraically special (Coley & McNutt 2018; McNutt & Coley 2018) and such a hypersurface is identified as the geometric horizon. This method of locating the black hole does not suffer the problem of foliation dependence. The property of being more special can be quantified invariantly by making particular combinations of curvature invariants zero on the horizon. Three different sets of invariants derived from curvature tensors are commonly used in the literature to calculate the geometric horizon in general. The first approach relies on finding the suitable Killing vector field that becomes the null generator on the horizon. Then, the hypersurface where the square norm of the Killing vector vanishes is the event horizon. If it is not possible to find such a field of Killing vector, then scalar polynomial curvature invariants could be calculated for stationary space–times and space–times conformal to stationary space–times (McNutt 2017). At the event horizon of such space–times, it has been shown that the squared norm of the wedge products of the n-linearly independent gradients of scalar polynomial curvature invariants vanishes, n being the local co-homogeneity of space–time (Page & Shoom 2015; McNutt & Page 2017). This method of locating the black hole boundary is restricted to the stationary space–times as this method, in principle, is looking for the null surface where the magnitude of some particular combinations of invariants vanishes. However, the horizons of dynamical space–times are not null in general. The extension of this method to the dynamic space–times conformal to the stationary space–times lies in the fact that the event horizon is invariant under conformal transformation.

Another procedure for calculating the geometric horizon is by finding the zeros of the certain combinations of Cartan invariants (McNutt *et al.* 2018; Coley *et al.* 2019). This method is exactly similar, in principle, to the scalar polynomial invariants for finding the geometric horizon. Thus, this method is also limited to stationary space–times and any space–times conformal to them. However, Cartan invariants are regarded as the improvement over the scalar polynomial invariants as this method involves the linear combinations of the components of a curvature tensor, and they are possible to construct from Cartan invariants.

The third procedure for the calculation of the geometric horizon is similar to the calculation of the trapped surfaces, and thus the black hole boundary calculated using this procedure can be called a geometric trapped surface. The calculation is based on the conjecture that the hypersurface constituting geometric trapped surface is more algebraically special (Page & Shoom 2015; McNutt & Page 2017). The procedure is to look for the geometrically preferred outgoing null vectors and then find the hypersurface where the expansion of such vectors vanishes. The null vector is called the geometrically preferred null vector because the congruence of the null vectors thus chosen is such that the covariant derivatives of the curvature tensor are more algebraically special there (Coley *et al.* 2017; McNutt & Coley 2018). The applicability of this method extends beyond the above two procedures of finding the geometric horizon as this method can be extended to the dynamical space–times also. However, until now, it has not been used to locate the black hole boundary of any axisymmetric dynamical space–times. The equivalence/correspondence between these three procedures for calculating the geometrical horizon lies in the fact that they all involve the calculation of invariants derived from the curvature tensor (see, e.g., McNutt & Coley 2018; Coley *et al.* 2019 for details).

1.2 *Properties of trapped surfaces*

Booth *et al.* (2006) and Sherif *et al.* (2019) presented a simple formalism to determine the metric signature of an apparent horizon and demonstrated that for spherically symmetric space–times. We will here discuss it for the purpose to determine some properties of the MOTT of the Kerr–Vaidya geometry. Let χ^μ be the vector field tangential to the MOTT and orthogonal to the MOTSs that foliate the MOTT. Also, assume that the vector field χ^μ generates the flow that preserves the foliation. Then $\mathcal{L}_\chi v = f(v)$, where $\mathcal{L}_\chi$ denotes the Lie derivative along χ and $f(v)$ denotes some function on the foliation v. We similarly denote by $\bar{\chi}^\mu$ the vector field normal to the MOTT, which satisfies $\bar{\chi}_\mu \chi^\mu = 0$. If we assume both χ^μ and $\bar{\chi}^\mu$ are future directed, then one of them is space like and another is time like. Now, in terms of the null vectors l^μ and n^μ given above, we can write

$$\chi^\mu = l^\mu - Cn^\mu, \tag{2}$$

$$\bar{\chi}^\mu = l^\mu + Cn^\mu, \tag{3}$$

for some function C. Along χ^μ, the Lie derivative of an outgoing expansion is zero thereby implying

$$\mathcal{L}_\chi \theta_l = \mathcal{L}_l \theta_l - C\mathcal{L}_n \theta_l = 0 \tag{4}$$

or

$$C = \frac{\mathcal{L}_l \theta_l}{\mathcal{L}_n \theta_l}. \tag{5}$$

Equation (5) can be used to calculate C and as $C \propto \chi^\mu \chi_\mu$, the metric signature of the MOTT is determined by the sign of C. $C > 0$ means that an apparent horizon is space like, $C < 0$ means that an apparent horizon is time like, and $C = 0$ or ∞ means that an apparent horizon is null. The sign of C also determines whether an apparent horizon is expanding or contracting. To see this, let us take an area element of the two surface be $\tilde{q}$ and evaluate its Lie derivative along χ^μ:

$$\mathcal{L}_\chi \tilde{q} = -C\theta_n \tilde{q}. \tag{6}$$

As $\theta_n < 0$ on the trapped surface, $C > 0$ implies that an apparent horizon is expanding, $C < 0$ implies that the horizon is receding and $C = 0$ or ∞ implies that the horizon is isolated. Thus, the space like apparent horizon is expanding and the time like apparent horizon is receding, while isolated apparent horizons are null.

1.3 *Slowly evolving horizon*

We are working in the semi-classical domain for which the quasi-static evaporation law given in Levi & Ori (1995) and Brout *et al.* (2016) is valid. In this domain, an apparent horizon of the Kerr–Vaidya metric can be approximated as a slowly evolving horizon. So, we present some characteristics of the slowly evolving horizon here. An almost-isolated trapping horizon is defined as a slowly evolving horizon and the invariant characterizations of such a horizon are made by Booth & Fairhurst (2004, 2007). We have mentioned above that an apparent horizon is null and isolated if $C = 0$. It thus seems that a slowly evolving horizon can be characterized by the space-like or time-like horizon for which C is small. C, however, can be arbitrarily varied by rescaling $n^\mu \to n^\mu/\alpha$ and $C \to C/\alpha^2$, for some α. The slowly evolving horizon should thus be identified in a scaling-independent manner and for this, we write the area evolution law of Equation (6) as

$$\mathcal{L}_\chi \tilde{q} = -\|\chi\| \left(\sqrt{\frac{C}{2}} \theta_n \tilde{q} \right) \tag{7}$$

such that the term in parentheses is independent of rescaling. This term gives an invariant area evolution and if this term is small, then the horizon is slowly evolving. Thus, one of the conditions proposed by Booth & Fairhurst (2004) for the horizon to be slowly evolving over some MOTS in a given foliation is

$$\sqrt{C}\theta_n < \frac{\epsilon}{R}, \tag{8}$$

for arbitrary small ϵ, where R is the areal radius of the MOTS. Another condition is the choice of scaling of the null vectors, such that, $\|\chi\| \sim \epsilon$. There are other conditions given in Booth & Fairhurst (2004, 2007) requiring that the horizon evolves smoothly over a long period of time. These conditions will be automatically satisfied for the slowly evolving Kerr–Vaidya metric where the horizon offsets from the isolated horizon, at most, by an order of M_v. The reason behind this is every parameter calculated at the horizon, in this case, differs from the stationary Kerr metric by an order of M_v.

2. Trapped region in advanced Kerr–Vaidya space–time

2.1 *Some examples*

We have the line element for Kerr–Vaidya space–time in advanced coordinates $(v, r, \theta, \tilde{\phi})$ given as (Senovilla & Torres 2015)

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2M(v)r}{\rho^2}\right)dv^2 + 2dvdr + \rho^2 d\theta^2 \\ &- \frac{4aM(v)r\sin^2\theta}{\rho^2}d\tilde{\phi}dv - 2a\sin^2\theta d\tilde{\phi}dr \\ &+ \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\tilde{\phi}^2, \end{aligned} \tag{9}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2M(v)r + a^2$. This metric reduces to the familiar Kerr metric for some advanced time v if $M(v)$ is constant (Kerr 1963). The trapped region associated with this metric can be determined by the calculation of the expansion scalars θ_l and θ_n. To find the trapped surface for this metric, we use the future-directed null vectors of the form (Senovilla & Torres 2015)

$$\begin{aligned} l_\mu &= \frac{\rho^2}{2\Omega^2}(-\Delta,\ r^2 + a^2 + \Omega,\ 0,\ 0), \\ n_\mu &= \left(-1,\ \frac{r^2+a^2-\Omega}{\Delta},\ 0,\ 0\right), \end{aligned} \tag{10}$$

where $\Omega = \sqrt{(r^2+a^2)\rho^2 + 2M(v)ra^2\sin^2\theta}$. These null vectors also being normalized satisfy the relations

$$l_\mu l^\mu = n_\mu n^\mu = 0 \quad \text{and} \quad l_\mu n^\mu = -1. \tag{11}$$

Moreover, the calculation of the geodesic equation $l^\mu l_{v;\mu} = \lambda l_v$ for some parameter λ shows that the vector l^μ is tangent to the null geodesic only on $\Delta = 0$ surface. We now use Equation (1) to calculate the expansion scalars using these null vectors. The expansions thus obtained are given as

$$\begin{aligned} \theta_l = &\frac{1}{2\Omega^3}(\Delta(2r^3 + 2a^2r - a^2r\sin^2\theta + a^2M\sin^2\theta) \\ &+ a^2rM_v\sin^2\theta(r^2+a^2+\Omega)), \\ \theta_n = &-\frac{1}{2\rho^2\Delta\Omega}(\Delta(2r^3 + 2a^2r - a^2r\sin^2\theta + a^2M\sin^2\theta) \\ &+ a^2rM_v\sin^2\theta(r^2+a^2-\Omega)), \end{aligned} \tag{12}$$

where $M_v = \partial M/\partial v$. Now, an MOTS is the two surface where outgoing null expansion (θ_l) vanishes and this marks the boundary of the trapped region. Senovilla & Torres (2015) have argued that the solution for $\theta_l = 0$ does not exist in general for the $v = \text{constant}$ foliation considered. This is indeed true and the reason behind this is that the null vectors are geodesic only on the $\Delta = 0$ surface and on this surface, $\theta_l = 0$ only when $M_v = 0$. They have thus considered the surface of intersection between $\Delta = 0$ and $M_v = 0$, where $r_g = M \pm \sqrt{M^2 - a^2}$ are the two real solutions of $\theta_l = 0$. For dynamical space–time considered, $M_v = 0$ is not true in general and might occur only for some $v = v_0$. Thus, the v, constant surface does not foliate an apparent horizon in general.

The solution of $\theta_l = 0$ is simpler and more evident if we calculate the expansion scalars for the null geodesics with the following tangent vectors:

$$\begin{aligned} l_\mu &= \frac{\Delta}{2\rho^2}\left(-1,\ \frac{2\rho^2}{\Delta},\ 0,\ a\sin^2\theta\right), \\ n_\mu &= (-1,\ 0,\ 0,\ a\sin^2\theta). \end{aligned} \tag{13}$$

l_μ satisfies the geodesic equation $l^\mu l_{v;\mu} = \lambda l_v$, for some parameter λ defining the geodesic curve only when $M_v = 0$. The expansions using these pairs of null vectors are as follows:

$$\theta_l = \frac{r\Delta}{\rho^4}, \quad \theta_n = -\frac{2r}{\rho^2}. \tag{14}$$

We thus get $\theta_l = 0$ when $\Delta = 0$. Hence, the MOTS should be the intersection between $\Delta = 0$ and $M_v = 0$ leading us again to the conclusion that v, constant surface does not foliate an apparent horizon in general.

2.2 *MOTSs are not unique*

Here, we will first show that depending on the choice of the null vectors, the MOTS (the surface for which $\theta_l = 0$) can extend all the way from the singularity to the infinity. For this, we take the general null vector of the form:

$$l_\mu = (-\alpha,\ 1,\ \beta,\ \gamma),\ n_\mu = (-1,\ 0,\ 0,\ a\sin^2\theta), \tag{15}$$

where $\alpha = \alpha(v, r, \theta, M(v))$. The outgoing vector l_μ will be future directed for $\alpha > 0$ and its null implies that

$l_\mu l^\mu = 0$. This allows us to express γ in terms of α and β as

$$\gamma = (-a + \alpha a \pm \sqrt{(1-2\alpha)a^2 - (\Delta - 2\alpha(a^2+r^2) + \beta^2)\csc^2\theta})\sin^2\theta. \tag{16}$$

With this value of γ, the vectors l_μ and n_μ satisfy all the relations given in Equation (11). We can now calculate the outgoing expansion θ_l and its value is obtained as follows:

$$\theta_l = \theta_{\text{Kerr}} + \frac{2aM_v\left(r + \rho^2\frac{\partial\alpha}{\partial M} - \beta\frac{\partial\beta}{\partial M}\right)}{\rho^2\sqrt{(1-2\alpha)a^2 - (\Delta - 2\alpha(a^2+r^2) + \beta^2)\csc^2\theta}}, \tag{17}$$

where θ_{Kerr} is an expression containing terms that do not have M_v. This corresponds to the outgoing expansion for the Kerr metric with M, constant. However, for the Kerr geometry, we should have

$$\theta_{\text{Kerr}} = \frac{\Delta g(t,r,\theta)}{\rho^2\sqrt{(1-2\alpha)a^2 - (\Delta - 2\alpha(a^2+r^2) + \beta^2)\csc^2\theta}}, \tag{18}$$

where $g(t,r,\theta)$ is an arbitrary function. However, we should have $g(t,r,\theta) > 0$ to ensure that $\theta_{\text{Kerr}} > 0$ for $\Delta > 0$. We thus obtain

$$\theta_l = \frac{g(\Delta + \frac{2aM_v}{g}(r + \rho^2\frac{\partial\alpha}{\partial M} - \beta\frac{\partial\beta}{\partial M}))}{\rho^2\sqrt{(1-2\alpha)a^2 - (\Delta - 2\alpha(a^2+r^2) + \beta^2)\csc^2\theta}}. \tag{19}$$

Again, to ensure that $\theta_l > 0$ outside the horizon, we should have

$$\Delta + \frac{2aM_v}{g}\left(r + \rho^2\frac{\partial\alpha}{\partial M} - \beta\frac{\partial\beta}{\partial M}\right) > 0, \tag{20}$$

as $g > 0$. We will analyse the two cases of $\Delta > 0$ (for the possibility of horizon lying outside $M + \sqrt{M^2 - a^2}$) and $\Delta < 0$ (for the possibility of horizon lying inside $M + \sqrt{M^2 - a^2}$) individually.

First, let us consider the case of $\Delta > 0$ at the horizon. To achieve this, that is, to get $\theta_l = 0$, we should have from Equation (19)

$$\begin{aligned} r + \rho^2\frac{\partial\alpha}{\partial M} - \beta\frac{\partial\beta}{\partial M} &> 0 \quad \text{for} \quad M_v < 0 \quad \text{and} \quad g > 0, \\ r + \rho^2\frac{\partial\alpha}{\partial M} - \beta\frac{\partial\beta}{\partial M} &< 0 \quad \text{for} \quad M_v > 0 \quad \text{and} \quad g > 0. \end{aligned} \tag{21}$$

Without the risk of losing the generality of the result, we will assume $\beta = 0$ for the rest of the analysis. This gives

$$\begin{aligned} r + \rho^2\frac{\partial\alpha}{\partial M} &> 0 \quad \text{for} \quad M_v > 0, \\ r + \rho^2\frac{\partial\alpha}{\partial M} &< 0 \quad \text{for} \quad M_v < 0. \end{aligned} \tag{22}$$

The lower/upper bound of this inequality gives $\alpha = \Delta/2\rho^2$ with the choice of $(a^2+r^2)/2$ as an integration constant and this is the null vector given in Equation (13) to obtain $\Delta = 0$ as the horizon. In general, the solution for $\theta_l = 0$ from Equation (19) is given as

$$\Delta + \frac{2aM_v}{g}\left(r + \rho^2\frac{\partial\alpha}{\partial M}\right) = 0 \tag{23}$$

or

$$\left(1 + \frac{2aM_v}{g}\frac{\partial\alpha}{\partial M}\right)r^2 - 2\left(M - \frac{aM_v}{g}\right)r + a^2\left(1 + \frac{2aM_v}{g}\frac{\partial\alpha}{\partial M}\cos^2\theta\right) = 0. \tag{24}$$

The solution to this equation will be diverging as $\partial\alpha/\partial M \to -(g/2aM_v)$, which is the reasonable possibility. We will here give an example to support this argument. But, let us first assume $M_{vv} \approx 0$ (only for this example), which is the reasonable approximation in the semi-classical limit. We then take

$$\alpha = \frac{a^2Q^2 + M^2\csc^2\theta}{2a^2Q^2}, \tag{25}$$

where $Q = Q(v)$ is some arbitrary function. With this choice of α, we obtain

$$\gamma = \frac{M^2 - a^2Q^2\sin^2\theta + 2\sqrt{MQ^2(\rho^2M + 2a^2rQ^2\sin^2\theta)}}{2aQ^2}. \tag{26}$$

We thus have the pair of null vectors given as

$$\begin{aligned} l_\mu &= \left(-\frac{a^2Q^2 + M^2\csc^2\theta}{2a^2Q^2},\ 1,\ 0,\ \frac{M^2 - a^2Q^2\sin^2\theta + 2\sqrt{MQ^2(\rho^2M + 2a^2rQ^2\sin^2\theta)}}{2aQ^2}\right), \\ n_\mu &= (-1,\ 0,\ 0,\ a\sin^2\theta). \end{aligned} \tag{27}$$

They satisfy the relations given in Equation (11) and thus can be used for the calculation of expansion scalars. The value of θ_l is obtained as follows:

$$\theta_l = \frac{1}{\rho^2}\left(r - M + \frac{MQ(rM + a^2Q^2\sin^2\theta) + QM_v(\rho^2 M + a^2 rQ^2\sin^2\theta) - \rho^2 M^2 Q_v}{\sqrt{MQ^4(\rho^2 M + 2a^2 rQ^2\sin^2\theta)}}\right), \tag{28}$$

where $Q_v = \partial Q/\partial v$. The solution of $\theta_l = 0$ for an apparent horizon will be convenient when $Q^2 \to M_v^2$, for which case, $Q_v \sim 0$. The solution for r in that regime would be

$$r = \left|\frac{2MM_v}{M_v^2 - Q^2}\right|. \tag{29}$$

We thus have diverging $r = r_g$ as $Q^2 \to M_v^2$.

Now, we consider the case of $\Delta < 0$ at the horizon. Then, to ensure that $\theta_l = 0$, we should have

$$r + \rho^2\frac{\partial\alpha}{\partial M} < 0 \quad \text{for} \quad M_v < 0,$$
$$r + \rho^2\frac{\partial\alpha}{\partial M} > 0 \quad \text{for} \quad M_v > 0. \tag{30}$$

Again, the upper/lower bound of this inequality gives $\alpha = \Delta/2\rho^2$ with the choice of $(a^2 + r^2)/2$ as an integration constant, which is the null vector given in Equation (13). The solution for $\theta_l = 0$ from Equation (19) is given as

$$\left(1 + \frac{2aM_v}{g}\frac{\partial\alpha}{\partial M}\right)r^2 - 2\left(M - \frac{aM_v}{g}\right)r + a^2\left(1 + \frac{2aM_v}{g}\frac{\partial\alpha}{\partial M}\cos^2\theta\right) = 0. \tag{31}$$

Both for $M_v < 0$ and $M_v > 0$, a proper choice of the value of $\partial\alpha/\partial M$ could make the value of $r = r_g$ recede up to the singularity. This can be seen by minimizing r with respect to $\partial\alpha/\partial M$ in this equation, whose solution is $r^2 + a^2\cos^2\theta = 0$. Thus, depending on the choice of null vectors (which in fact determines the choice of the foliation), the value of the MOTS can extend from the singularity to the infinity. As we have seen in the previous two examples, all the MOTSs do not foliate an apparent horizon. However, it is not clear which MOTSs dynamically evolve to foliate the apparent horizon and if the apparent horizon is unique (Ashtekar & Galloway 2005; Hawking & Ellis 2011). Moreover, to find the trapping horizon, we have to check each apparent horizon hypersurfaces to confirm whether space–time inside it is trapped and outside it is normal. So, the problem of locating the trapping horizon extends further beyond the complexity of identifying the apparent horizon.

2.3 *An approximate apparent horizon*

Besides the problem of foliation dependence, there is another more serious problem in general dynamical axially symmetric space–times in the calculation of the trapped surfaces. In general, for time-dependent axial symmetry, it is observed that the future-directed outgoing null vector is not tangent to the geodesic everywhere. This is because, the general future-directed outgoing null vector has three independent parameters and can be written as $l_\mu = (-\alpha, 1, \beta, \gamma)$, where $\alpha \neq 0$. However, there are four constraints altogether to satisfy by this vector: three independent constraints from the geodesic equation $l^\mu l_{v;\mu} = \lambda l_v$, for some parameter λ (one of the equation $l^\mu l_{r;\mu} = \lambda l_r$ is satisfied identically) and one constraint from the null condition $l^\mu l_\mu = 0$. There are thus four equations to satisfy by three variables and this will hold only on some region/hypersurface of the space–time. This is in contrast to the situation in both the time-dependent spherical symmetry and the stationary axial symmetry, where an arbitrary null vector can be made geodesic by the proper choice of a parameter called an affine parameter defining the curve. Thus, an outgoing null vector is a tangent to the geodesic only on some surface in axial symmetry in general, and on that surface, the outgoing expansion θ_l does not vanish usually. A way to address this problem is to calculate an approximate apparent horizon.

To approximate an apparent horizon for the Kerr–Vaidya geometry, let us take the null vectors of Equation (13) for which the solution of $\theta_l = 0$ is clearly given by $\Delta = 0$. However, as concluded above, the intersection of v is constant and $\Delta = 0$ surface does not foliate the apparent horizon in general. The reason behind this is that the vectors of Equation (13) are not tangent to the null geodesics in general. To see this, the geodesic deviation equation $l^\mu l_{v;\mu} - \kappa l_v$ for some parameter κ is given as follows:

$$l^\mu l_{v;\mu} - \kappa l_v = \frac{a^2 r \sin^2\theta M_v}{\rho^4}, \tag{32}$$

$$l^\mu l_{r;\mu} - \kappa l_r = 0, \tag{33}$$

$$l^\mu l_{\theta;\mu} - \kappa l_\theta = 0, \tag{34}$$

$$l^\mu l_{\phi;\mu} - \kappa l_\phi = -\frac{ar(a^2+r^2)\sin^2\theta M_v}{\rho^4}. \tag{35}$$

Thus, the null vectors are offset from being the geodesics by an order of M_v, which is obviously small in the semi-classical limit as pointed out in Section 1.3. So in the semi-classical region, we can approximate $\Delta = 0$ as an apparent horizon, and we will explain below that this satisfies all the properties for being the slowly evolving horizon. However, before that, we will present a technique that is similar to the perturbation expansion to approximate an apparent horizon. Possibly, the validity of this approximation method would be for all $|M_v| < 1$ and not only for very small M_v.

For this, we proceed forward by making a slight modification on the null vectors of Equation (13)

$$l_\mu = \left(-\frac{\Delta + \lambda(r,\theta)M_v}{2\rho^2},\ 1,\ \zeta(r,\theta)M_v,\ \frac{a\Delta\sin^2\theta + v(v,r,\theta)}{2\rho^2}\right),$$
$$n_\mu = (-1,\ 0,\ 0,\ a\sin^2\theta). \tag{36}$$

The null condition $l^\mu l_\mu = 0$ gives

$$v = \sin^2\theta\left(a\lambda M_v - 2a\rho^2 + 2\rho^2\sqrt{a^2 + \csc^2\theta\lambda M_v - \csc^2\theta\zeta^2 M_v^2}\right). \tag{37}$$

We now calculate the outgoing expansion θ_l at $r = r_g$ given by the solution of equation $M = (a^2 + r^2 + f(t,r,\theta)M_v)/2r$, where $f(t,r,\theta)$ is another arbitrary function. Assuming $M_{vv} \sim 0$, we get from the solution of $\theta_l = 0$

$$f = \frac{2\rho^2\left(\left(2\frac{\partial\zeta}{\partial\theta} - \frac{\partial\lambda}{\partial r}\right)\sqrt{\csc^2\theta M_v(\lambda - M_v\zeta^2) + a^2} + 2\zeta\left(\cot\theta\sqrt{\csc^2\theta M_v(\lambda - M_v\zeta^2) + a^2} - aM_v\frac{\partial\zeta}{\partial r}\right) + a\frac{\partial\lambda}{\partial r}\right)}{4r\sqrt{\csc^2\theta M_v(\lambda - M_v\zeta^2) + a^2}} + \lambda. \tag{38}$$

Again, solving the geodesic deviation equation at $M = (a^2 + r^2 + f(t,r,\theta)M_v)/2r$, with the value of f given by Equation (38), we get

$$l^\mu l_{v;\mu} - \kappa l_v = \frac{M_v\left((r^2-a^2)\lambda + 4a^2r^2\sin^2\theta\right)}{4r\rho^4} + \mathcal{O}\left(M_v^2\right), \tag{39}$$

$$l^\mu l_{r;\mu} - \kappa l_r = 0, \tag{40}$$

$$l^\mu l_{\theta;\mu} - \kappa l_\theta = \frac{(a^2 - r^2)M_v\zeta}{2r\rho^2} + \mathcal{O}\left(M_v^2\right), \tag{41}$$

$$l^\mu l_{\phi;\mu} - \kappa l_\phi = \frac{(a^2+r^2)M_v\left((a^2-r^2)\lambda - 4a^2r^2\sin^2\theta\right)}{4a\rho^4 r} + \mathcal{O}\left(M_v^2\right), \tag{42}$$

where $\mathcal{O}\left(M_v^2\right)$ represents the terms of order M_v^2 and higher. Now, requiring the geodesic deviation of the null tangent vectors to be of at least $\mathcal{O}\left(M_v^2\right)$ we should have

$$\lambda = \frac{4a^2r^2\sin^2\theta}{a^2 - r^2} \quad \text{and} \quad \zeta = 0. \tag{43}$$

Substituting this in Equation (38), we get

$$f = \frac{2a^2\sin^2\theta\left(2a^4\cos^2\theta + 2a^2r^2 - (a^4\cos 2\theta + a^4 + 2r^4)\sqrt{\frac{4r^2M_v}{a^2-r^2} + 1}\right)}{(a-r)^2(a+r)^2\sqrt{\frac{4r^2M_v}{a^2-r^2} + 1}}$$
$$= \frac{4a^2r^2\sin^2\theta}{a^2-r^2} + \mathcal{O}(M_v). \tag{44}$$

Thus, the solution of equation $M = (a^2 + r^2 + f(t,r,\theta)M_v)/2r$ with the value of f from Equation (44) gives the value of r. This r corresponds to the apparent horizon of the Kerr–Vaidya space–time up to the first-order correction in M_v. Specifically,

$$r = M + \sqrt{M^2 - a^2} + \frac{a^2\sin^2\theta\left(2M^2 - a^2 + 2M\sqrt{M^2-a^2}\right)}{(M^2-a^2)\left(M + \sqrt{M^2-a^2}\right)}M_v. \tag{45}$$

For the Kerr–Vaidya metric in advanced coordinates, when $M_v < 0$, the apparent horizon is ellipsoid flattened at the equator, provided that a is small. Similarly, when $M_v > 0$, for small a, the apparent horizon is ellipsoid flattened at the pole. It can be shown that this approximate horizon is not the unique approximate apparent horizon. There exist other surfaces where the null vectors satisfy geodesic equations up to the second order in M_v and have the vanishing outgoing expansion θ_l (e.g. the exact procedures for the calculation of an approximate horizon applied to the null vectors of Equation (10) give different surfaces as an approximate horizon). However, these surfaces differ from each other and the stationary black hole horizon, at most, by an order of M_v.

The MOTSs foliating the apparent horizon is unique (Ashtekar & Galloway 2005). So, if an apparent horizon is known *a priori*, then we can infer that this has been foliated by a dynamical evolution of the unique MOTS. This is unlike the case of isolated horizons where foliations are freely deformable. However, we could not say anything about the uniqueness of the apparent horizon itself. (Remark: the assumption of the finite time formation of an apparent horizon constrains its location. However, it is not sufficient constraint to give the unique apparent horizon.)

The approximate apparent horizon calculated here has promising features, which we list below, and this might provide a clue for the appropriate choice of null vectors for the calculation of the trapped surface. The null vectors given in Equation (36) have the vanishing shear tensor in the leading order approximation in M_v. The shear tensor $\sigma_{\mu\nu}$ can be calculated by using the relation

$$\sigma_{\mu\nu} = \tilde{B}_{\mu\nu} - \frac{\theta}{2} h_{\mu\nu}, \tag{46}$$

where $h_{\alpha\beta} = g_{\alpha\beta} + l_\alpha n_\beta + n_\alpha l_\beta$ and $\tilde{B}_{\alpha\beta} = \frac{1}{2}\theta h_{\alpha\beta}$ (see Poisson (2004) for details). Now, at the apparent horizon, $\theta_l = 0$. The direct calculation using Equation (46) gives

$$\sigma^l_{\mu\nu}\sigma^{\mu\nu}_n = \frac{4a^2 r_g^2 \cos^2\theta}{\rho^4(a^2 - r_g^2)} M_v + \mathcal{O}(M_v^2), \tag{47}$$

which is already the first order in M_v. The choice of null vectors with a vanishing shear tensor, at least, in the leading order approximation is the natural choice of null vectors (and hence the natural choice of foliation) for the calculation of the trapped surface. This can be explained as follows.

It is undoubtedly true that for the calculation of trapped surfaces in spherically symmetric space–times, we should choose the foliation that respects the symmetry of the space–time (Faraoni *et al.* 2017). For this spherical foliation of choice, the null geodesic congruence normal to this surface is radial and shear free. Some choices of non-spherical foliations, even in stationary spherically symmetric space–times, do not contain trapped surfaces (Krishnan 2014). Obviously, choosing an arbitrary axially symmetric foliation does not work for axially symmetric space–times. Shear-free geodesic congruence is the axially symmetric analogy of radial geodesic in the spherical symmetric space–time (Chandrasekhar 1985). Thus, the foliation associated with the shear-free geodesic in axially symmetric space–time complements the symmetry respecting foliation of the spherically symmetric space–time.

Furthermore, our calculation of an approximate apparent horizon also follows from an assumption of the validity of the geometric horizon conjecture stated in Section 1.1. The corollary of the Goldberg & Sachs (2009) theorem given in (Chandrasekhar (1985), p. 63) implies that if the congruence formed by the two principal null directions l^μ and n^μ are geodesic and shear free then space–time is algebraically special. As the null geodesics of Equation (36) for the calculation of an approximate horizon is shear free, at least in first-order approximation in M_v, the congruence formed by them is more algebraically special. This implies that the approximate apparent horizon is also an approximate geometric-trapped surface. Moreover, the procedure for the calculation of an approximate apparent horizon is the generalization of the third procedure for the calculation of the geometric horizon, explained above, to the more general case of axial symmetry.

2.4 *Features of the apparent horizon*

We now take the pair of null vectors given in Equation (36) with the value of respective parameters ν given in Equation (37) and λ and μ given in Equation (43). The direct calculation using these null vectors yields

$$\mathcal{L}_l\theta_l = l^\mu(\theta_l)_{;\mu} = -\frac{2r^2 M_v}{\rho^4} + \mathcal{O}(M_v^2),$$

$$\mathcal{L}_n\theta_l = n^\mu(\theta_l)_{;\mu} = \frac{a^2 - r^2}{\rho^4} - \frac{4a^2 r^2 (a^2 + r^2)\sin^2\theta}{\rho^4 (a^2 - r^2)^2} M_v + \mathcal{O}(M_v^2). \tag{48}$$

We substitute this in Equation (5) to get

$$C = \frac{2r^2 M_v}{r^2 - a^2} + \mathcal{O}\big(M_v^2\big). \tag{49}$$

From this equation, $M_v < 0$ gives $C < 0$ thereby implying that the apparent horizon of the Kerr–Vaidya line element in advanced coordinates is time like and receding in $M_v < 0$ domain. These are important features, as the $M_v < 0$ regime of the Kerr–Vaidya line element in advanced coordinate is believed to be the evaporating black hole solution of the Einstein equation. Similarly, for $M_v > 0$, the apparent horizon is space like and advancing.

Again, for the null vectors of Equation (36), we get $\theta_n = -2r/\rho^2$. We thus have the condition for the slowly evolving horizon

$$\sqrt{|C|}\theta_n = \sqrt{\frac{8r^4|M_v|}{\rho^4(r^2 - a^2)}} < \frac{\epsilon}{r^2 + a^2}, \tag{50}$$

that holds only when the space–time is slowly rotating, that is, $a \ll r$ and $\sqrt{|M_v|} \sim \mathcal{O}(\epsilon)$. However, in the semi-classical limit, for a slowly rotating case near the apparent horizon r_g, $|M_v| \approx r_g' \sim 10^{-3}–10^{-4}$ (Brout *et al.* 1995; Levi & Ori 2016). This gives $\epsilon \sim 0.01 \ll 1$. Thus, the apparent horizon of an advanced Kerr–Vaidya line element, in the semi-classical domain, satisfies the condition to be called as the slowly evolving horizon provided $a \ll r$.

Similarly, the calculation using the pair of null vectors of Equation (13) yields $C \sim M_v$ for $a \ll r$. This again leads to the conclusion that the surface $\Delta = 0$ can be called the slowly evolving horizon when $\sqrt{M_v} \sim \epsilon$. Thus, $\Delta = 0$ is the slowly evolving horizon of the Kerr–Vaidya metric in advanced coordinates provided $a \ll r$.

3. Possible characterization of a black hole boundary

To begin with, we first summarize the complications we encounter in identifying trapped surfaces as a black hole boundary. Identification of trapped surfaces as a black hole boundary is straightforward and poses no problem in stationary space–times (where the event horizon can also be undoubtedly located) and in dynamical spherically symmetric space–times (where the location of the event horizon is problematic). However, trapped surfaces turned out to be the ill-defined concept when we try to explore them beyond these two classes of space–times. This is because of the two fundamental reasons:

- The problem of the foliation dependence of trapped surfaces: even in the simplest class of space–times such as Schwarzschild, Vaidya (1951), the trapped surface depends on the choice of foliation on which the family of null congruence is orthogonal.
- An arbitrary outgoing null vector l_μ is not geodesic everywhere in general axisymmetric space–time. In general, the region where l_μ is geodesic is not where the outgoing null expansion θ_l vanishes. Because of this, even an exact location of a geometrical trapped surface, which is a foliation-independent entity is not possible.

In an attempt to address these problems, we present an alternative way of characterization of a black hole boundary. The common procedure for locating a black hole boundary is by looking for some identically non-zero invariant quantity whose value vanishes on the horizon. For example, the expansion scalar of an outgoing null geodesic vanishes at the trapped surface. Similarly, the norm of the Killing vector field or some particular combinations of the scalar polynomial curvature invariants or Cartan invariants vanishes at the geometrical horizon. So, analogously, we are also looking for an invariant quantity that is identically zero at the horizon and non-zero elsewhere. However, for us, that quantity should be foliation/observer-independent and applicable to the more general case of dynamical axial symmetry. Such a quantity would be an arbitrary radial trajectory l^r, which becomes null on the horizon, that is, $l^\mu l_\mu = 0$, given l^θ and l^ϕ are null (which can be constructed by choice).

To demonstrate that all the radial trajectories indeed become null at the horizon, we here give an example of the general spherically symmetric space–time

$$\mathrm{d}\tau^2 = -\mathrm{e}^{2h(t,r)} f(t,r)\mathrm{d}t^2 + \frac{1}{f(t,r)}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2. \tag{51}$$

Now, for this space–time, the equation of motion in the θ-direction is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\mathrm{d}}{\mathrm{d}\tau}(r^2\dot{\theta}) = -\frac{\partial \mathcal{L}}{\partial \theta} = r^2 \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2, \tag{52}$$

where $\dot{\theta} = \mathrm{d}\theta/\mathrm{d}\tau$. So, if we initially choose $\theta = \pi/2$ and $\dot{\theta} = 0$, then $\ddot{\theta} = 0$. This implies that the geodesic motion can be described in an invariant plane and we choose that plane to be the equator $\theta = \pi/2$. Thus, the radial equation of motion for the space–time of Equation (51) can be written as

$$\dot{r}^2 = -\delta f(t,r) + (e^{h(t,r)} f(t,r)\dot{t})^2 - \frac{L^2}{r^2} f(t,r), \quad (53)$$

where $\delta = 0$ for null geodesic, $\delta = 1$ for time like geodesic and $L = r^2 \mathrm{d}\phi/\mathrm{d}\tau$ is a constant. As $\dot{t}$ does not depend on the geodesic being null or time like and as L is arbitrary, the only surface where the radial trajectory $\dot{r}$ is always null is $f(t,r) = 0$. So, in dynamical spherically symmetric space–times, all the radial trajectories become null on some hypersurface, and this three surface is uniquely characterized as the black hole boundary by our prescription. Also, looking at the geodesic equations for the Kerr space–time given in Chandrasekhar (1985), we can see that all of the time like radial geodesics approach null geodesic on the $r^2 + a^2 - 2rM = 0$ surface. This surface is, in fact, both the event and the apparent horizon of the Kerr space–time.

Hence, it might be reasonable to characterize a black hole boundary as the hypersurface where every time like radial geodesics approaches null geodesics. This is the asymptotic three surface, and the motivation for this is the classical picture of the black hole as an asymptotic state of the gravitational collapse. For most of the space–times, the black hole boundary in this characterization is given by the solution of $g^{rr} = 0$. Some space–times, such as Vaidya, has $g^{rr} = 0$ identically and non-zero g^{tr}. Coordinate transformation is possible for such space–times to make $g^{tr} = 0$, and then the solution of $g^{rr} = 0$ gives the black hole boundary (otherwise, the full geodesic equation should be solved).

We now take an example of the general axisymmetric space–time with two parameters

$$\mathrm{d}\tau^2 = e^{2h}\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{4e^h aMr\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\phi - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 - \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\mathrm{d}\phi^2, \quad (54)$$

where $h = h(t,r,\theta)$ and $M = M(t,r,\theta)$. To calculate the boundary of the black hole for this space–time, we assume the trajectory of the form

$$l^\mu = \beta(1,\ \alpha,\ \gamma,\ \delta), \quad (55)$$

where β, α, γ, and δ are the functions of t, r, and θ, respectively. Now, assuming l^μ to be a time like trajectory, it should satisfy $l^\mu l_\mu = -1$ and this gives

$$\alpha_t = \sqrt{\frac{\Delta}{\rho^2}\left(-\frac{1}{\beta^2} - \frac{2rM(e^h - a\sin^2\theta\delta)^2}{\rho^2} + e^{2h} - \rho^2\gamma^2 - (a^2+r^2)\sin^2\theta\delta^2\right)}. \quad (56)$$

Now, this time like radial trajectory coincides identically with the null trajectory given by

$$\alpha_n = \sqrt{\frac{\Delta}{\rho^2}\left(-\frac{2rM(e^h - a\sin^2\theta\delta)^2}{\rho^2} + e^{2h} - \rho^2\gamma^2 - (a^2+r^2)\sin^2\theta\delta^2\right)}, \quad (57)$$

on the surface $\Delta = 0$ irrespective of the form of the parameters e^h, γ and δ assuming that the normalization factor $\beta \neq 0$ there (this is expected otherwise, the trajectory is identically zero at $\Delta = 0$. Thus, for the general form of the axisymmetric metric given by Equation (54), adopting the definition presented here gives $g^{rr} = 0$ as the black hole boundary.

4. Discussions and conclusion

It seems that the quasi-local measures for determining a black hole boundary are not suitable for the general axisymmetric space–times. In a spherical symmetry, there exists a preferred choice of foliation, which obeys the symmetry of the space–time and gives a unique MOTS. However, such a preferred choice of foliations cannot be made in axisymmetric space–times. We have shown that for axisymmetric space–times, MOTS is not unique and lies anywhere from the singularity to the infinity depending on the choice of the null vectors. Although all these MOTSs do not foliate apparent horizons, many of them could. However, there is no preferred choice of MOTS foliating an apparent horizon that acts as a true black hole boundary.

Another problem in the calculation of an apparent horizon as explained in Section 2.2 is that the outgoing null vectors are not geodesic everywhere, in general, in axisymmetric space–times. At least, for Kerr–Vaidya line element, it is found that the surface where $\theta_l = 0$ does not coincide with the hypersurface where the null vectors are geodesic. We expect the same situation to occur in most of the general axisymmetric space–times. If this is the case, then the calculation of the trapped surface as a black hole boundary does not even make a sense for axisymmetric space–times. One way to get out of this problem was the calculation of an approximate apparent horizon, explained in Section 2.3. The approximate horizon exists only for small M_v, and when it exists, Section 2.4 shows that it has some promising features. The problem, however, is that the approximate apparent horizon calculated in the way given in Section 2.3 is not unique.

We have obtained an approximate apparent horizon that has some promising features. This encouraging result has been obtained by using the null vectors of Equation (36) that has the vanishing shear tensor $\sigma_{\mu\nu}$ in the leading order. The null congruence used for the calculation of an approximate horizon being more algebraically special makes it eligible to be called the geometric trapped surface. The null vectors with vanishing shear tensor $\sigma_{\mu\nu}$, at least in the leading order, could thus be a natural choice for calculation of the trapped surface. The choice of null vectors determines the choice of foliation.

We have also proposed a new way to characterize a black hole boundary. Although our procedure for locating the black hole boundary requires solving the full radial geodesic equation of the space–time in general, for some forms of space–time it is just the solution of $g^{rr} = 0$. Equation (54) is an example of such types of space–times. The validity of our approach for locating the black hole boundary lies in the fact that such a surface exists in the black hole solution of the Einstein equation (e.g. the Schwarzschild and the Kerr solution). The universality of our approach relies on whether such surface where all the time like radial geodesics approaching the null geodesics exists in all black hole solutions. From our study on the black hole boundary, we have noticed that the definition of the black hole boundary plays a crucial role in our understanding of black holes. The new definition is therefore exciting and is believed to meet this expectation.

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