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Review

On the BTZ black hole and the spinning cosmic string

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Abstract. We review the Bañados–Teitelboim–Zanelli (BTZ) black hole solution in connection with the spinning string solution. We find a new exact solution, which can be related to the (2 + 1)-dimensional spinning point particle solution. There is no need for a cosmological constant, and so the solution can be uplifted to (3 + 1) dimensions. The exact solution in a conformally invariant gravity model, where the space–time is written as $g_{\mu\nu} = \omega^2 \tilde{g}_{\mu\nu}$, is horizon free and has an ergo-circle, while $\tilde{g}_{\mu\nu}$ is the BTZ solution. The dilaton ω determines the scale of the model. It is conjectured that the conformally invariant non-vacuum BTZ solution will solve the boundary and causality problems which one encounters in spinning cosmic string solutions.

Keywords. Spinning cosmic string—BTZ black hole—conformal invariance.

1. Introduction

Besides the well-studied Schwarzschild and Kerr solution in general relativity theory (GRT), there is another black hole solution in $(2+1)$-dimensional space–times, i.e., the Bañados–Teitelboim–Zanelli (BTZ) black hole (Bañados *et al.* 1993). The BTZ geometry solves Einstein's equations with a negative cosmological constant in $(2+1)$ dimensions. The solution asymptotes for large r to a global anti-de Sitter (AdS_3) space–time. In general, $(2+1)$ dimensional gravity has been widely recognized as an interesting laboratory, for studying not only GRT, but also quantum-gravity models (Compère 2019). It is conjectured that this genuine solution will be of importance when one considers thermodynamic properties close to the horizon, i.e., Hawking radiation ('t Hooft 2017). The BTZ solution is comparable with the spinning point particle solution (or "cosmon" Deser *et al.* (1984, 1992); Deser & Jackiw (1989)) of the dimensional reduced spinning cosmic string or Kerr solution. The $(2+1)$-dimensional gravity without matter, implies that the Ricci and Riemann tensor vanish, and so matter-free regions are flat pieces of space–time. When locally a mass at rest is present, it cuts out a wedge from the two-dimensional space surrounding it and makes the space conical. The angle deficit is then proportional to the mass (Garfinkle 1985). The important fact is that the spinning point particle has a physically acceptable counterpart in $(3+1)$ dimensions, i.e., the spinning cosmic string. The z-coordinate is suppressed, because there is no structure in that direction altogether. It is the unconventional range and jump properties of the coordinates that remind us that there are sources somewhere. One can prove that the source of the $(3+1)$-dimensional cosmic string cannot be infinite thin (Geroch & Traschen 1987). This implies matching problems at the boundary (Janca 2007; Krisch 2003). This problem could be overcome in a conformally invariant setting (Slagter & Duston 2019). The BTZ solution, however, when up-lifted, needs a zero cosmological constant, and so a different solution emerges. It is not a surprise that these models are used in constructing quantum gravity models. In these models one uses locally Minkowski space–time, and so planar gravity fits in very well. It is conjectured ('t Hooft 1996) that $(2+1)$-dimensional gravity with matter could be quantized in an unambiguous way. In this context, one can also consider conformally invariant (CI) gravity models, specially after the recognition that the

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asymptotically AdS_3 is related to a two-dimensional conformal field theory (CFT) (Strominger 1997). Conformal invariance was originally introduced by Weyl (1918). The idea was to introduce a new kind of geometry, in relation to a unified theory of gravitation and electromagnetism. This approach was later abandoned by the birth of modern gauge field theories. Quite recently the anti-de Sitter/conformal field-theory (AdS/CFT) correspondence renewed the interest in conformal gravity. AdS/CFT is a conjectured relationship between two kinds of physical theories. AdS spaces are used in theories of quantum gravity while CFT includes theories similar to the Yang–Mills theories that describe elementary particles (Maldacena 1999). It is now believed that CI can help us to move a little further along the road to quantum gravity. CI in GRT considered as exact at the level of the Lagrangian, but spontaneously broken, just as in the case of the Brout–Englert–Higgs (BEH) mechanism in standard model of particle physics, is an approved alternative for disclosing the small-distance structure when one tries to describe quantum-gravity problems ('t Hooft 2011). It can also be used to model scale-invariance in the cosmic microwave background radiation (CMBR) (Bars *et al.* 2014). Another interesting application can be found in the work of Mannheim on conformal cosmology (Mannheim 2005). This model could serve as an alternative approach to explain the rotational curves of galaxies, without recourse to dark matter and dark energy (or cosmological constant). The key problem is the handling of asymptotic flatness of isolated systems in GRT, specially when they radiate and the generation of the metric $g_{\mu\nu}$ from at least Ricci-flat space–times. In the non-vacuum case one can construct a Lagrangian where space–time and the fields defined on it, are topologically regular and physically acceptable. This can be done by considering the scale factor (or warp factor in higher-dimensional models) as a dilaton field besides, for example, a conformally coupled scalar field or other fields. Conformally invariant gravity distinguishes itself by the notion that the space–time is written as $g_{\mu\nu} = \omega^2 \tilde{g}_{\mu\nu}$, with ω a dilaton field which contains all the scale dependencies and $\tilde{g}_{\mu\nu}$ the "un-physical" space–time, related to the $(2+1)$-dimensional Kerr and BTZ black hole solution.

In this paper, we present a new solution of the BTZ-type and compare the solution with the conformally invariant counterpart solution. We will not consider here, for the time being, the quantum mechanical implications of the model.

2. The BTZ solution revised

If one solves the Einstein equations $G_{\mu\nu} = \Lambda g_{\mu\nu}$ for the space–time

$$ds^2 = -N(r)^2 dt^2 + \frac{1}{N(r)^2} dr^2 + r^2 (d\varphi + N^\varphi(r) dt)^2, \tag{1}$$

one obtains

$$N(r)^2 \equiv \alpha^2 - \Lambda r^2 + \frac{16G^2J^2}{r^2},$$
$$N^\varphi(r) \equiv -\frac{4GJ}{r^2} + S, \tag{2}$$

where S, J and α are integration constants (Bañados *et al.* 1993; Compère 2019). The parameters α and J represent the standard ADM mass ($\alpha^2 = \pm 8\ GM$) and angular momentum and determine the asymptotic behaviour of the solution. The Λ represents the cosmological constant. There is an inner and outer horizon and an ergo-circle just as in the case of the Kerr space–time. However, if one lifts-up this space–time to $(3+1)$-dimensions, one must take $\Lambda = 0$, which can easily be verified by the Einstein equations. So we consider here the case $\Lambda = 0$, and we write the space–time as

$$ds^2 = -[8G(JS - M) - S^2 r^2] dt^2 + \frac{r^2 r_H^2}{16G^2J^2(r_H^2 - r^2)} dr^2 + r^2 d\varphi^2 + 2r^2 \left(S - \frac{4GJ}{r^2}\right) dt d\varphi, \tag{3}$$

where r_H represents the horizon $r_H = \sqrt{(2G/M)}J$. In the case of $S = 0$, which is also done in the original BTZ solution, one can transform the space–time to

$$ds^2 = -\left(\alpha dt + \frac{4GJ}{\alpha} d\varphi\right)^2 + dr'^2 + \alpha^2 r'^2 d\varphi^2 \tag{4}$$

by $r'^2 = (16G^2J^2 + \alpha^2 r^2)/\alpha^4$. This is just the spinning particle space–time (Deser & Jackiw 1989) without horizons. There are now evidently CTC's for $r' < (4GJ/\alpha^2)$. By re-defining $\varphi \to \alpha\varphi$, in order to obtain asymptotically the Minkowski space–time, we can identify the mass parameter α with an angle deficit (ignoring for the moment the time re-definition in order to get rid of J). As already mentioned, we then run into problems (apart from the zero cosmological constant) when lifting-up to $(3+1)$-space–time. The mass of the general relativistic cosmic string is determined by the parameters of scalar-gauge fields, i.e., by taking the integral of the energy density over a $t =$ constant, $z =$ constant two-surface. So there is an obscurity by defining the mass parameter M of the bounded $(2+1)$-dimensional BTZ black hole by using the method of the surface charges associated with the 2 Killing vectors (Compère 2019). One must keep in mind that the original asymptotic solution of the general relativistic cosmic string is given by Garfinkle (1985)

$$ds^2 = -e^{a_0}(dt^2 - dz^2) + dr^2 + e^{-2a_0}(kr + a_1)^2 d\varphi^2, \tag{5}$$

where a_0 and a_1 integration constants. k follows directly from the field equations. The transformations $r \to r + (a_1/k), \varphi \to ke^{-a_0}\varphi, z \to e^{-a_0/2}z, t \to e^{-a_0/2}t$ will now deliver Minkowski without a wedge, i.e., a conical space–time

$$ds^2 = -dt'^2 + dz'^2 + dr'^2 + (1 - 4G\mu)^2 r'^2 d\varphi^2, \tag{6}$$

where $1 - 4G\mu = ke^{-a_0}$. The angle deficits are then given by the parameters k and a_0 and are in general determined by the string variables, i.e., the scalar and gauge field (and by the metric variables due to the fact that we are dealing with a coupled set of differential equations). No *a priori* connection with the mass of a black hole is necessary: the mass density of the cosmic string is directly related to the angle deficit when lifting-up to 4D.

2.1 *The new solution*

For $S = 2M/J$ we can write the space–time of Equation (3) for the transformation

$$r^* = -\frac{r}{8GM} + \frac{\sqrt{2}J}{8M\sqrt{GM}} \text{arctanh}\left(\frac{r}{r_H}\right), \tag{7}$$

in the form

$$ds^2 = 16G^2J^2\left(\frac{1}{r^2} - \frac{1}{r_H^2}\right)(-dt^2 + dr^{*2}) + r^2\left[d\varphi + 4GJ\left(\frac{1}{r_H^2} - \frac{1}{r^2}\right)dt\right]^2. \tag{8}$$

The angular velocity of the null generator, $\Omega_A \equiv (d\varphi/dt) = (g_{t\varphi}/g_{\varphi\varphi})$ becomes

$$\Omega_A = 4GJ\left(\frac{1}{r_H^2} - \frac{1}{r^2}\right). \tag{9}$$

The $dtd\varphi$ term will then be zero at the horizon, which means that locally non-rotating observers has no coordinate angular velocity, i.e., there is no dragging of inertial frames at the horizon. We can plot the Penrose diagram by defining the appropriate null coordinates $U = r^* - t, V = r^* + t$ and switch to compactified coordinates $U = \tan((p+q)/2)$, $V = \tan((p-q)/2)$. So $r^* = \sin(p)/\cos(p) + \cos(q)$ as in the original BTZ case. The metric then becomes

$$ds^2 = \frac{16G^2J^2\left((1/r^2) - (1/r_H^2)\right)}{(\cos(p) + \cos(q))^2}[dp^2 - dq^2] + r^2\left[d\varphi + 4GJ\left(\frac{1}{r_H^2} - \frac{1}{r^2}\right)dt\right]^2. \tag{10}$$

In figure 1 we plotted r^* for the original BTZ solution in 3D and our solution of Equation (7). The only difference is the behaviour of r^* at $r = r_H$: there is, in our solution, no jump in r^* when crossing the horizon. Further, for $r \to \pm\infty$, we have $r^* \to \mp\infty$.

In figure 2 we plotted the Penrose diagram of our solution. The only difference with the original BTZ Penrose diagram is the location of the lines $r = \infty$. The maximal extension is then achieved by gluing together the regions II, III and IV.

3. The conformally invariant solution

We consider now the 4D space–time (compare with Equation (1))

$$ds^2 = \omega(r)^2\left[-N(r)^2dt^2 + \frac{1}{N(r)^2}dr^2 + dz^2 + r^2(d\varphi + N^\varphi(r)dt)^2\right], \tag{11}$$

where ω represents the dilaton field. So we can write the metric as $g_{\mu\nu} = \omega^2\tilde{g}_{\mu\nu}$, where $\tilde{g}_{\mu\nu}$ is now called an "un-physical" space–time. The conformal invariant Lagrangian (Wald 1984; 't Hooft 2015) is

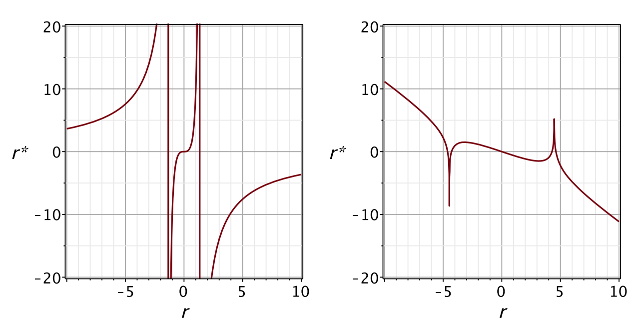

Figure 1. Graphs of r^* for the original BTZ solution (left) and our solution of Equation (7) (right).

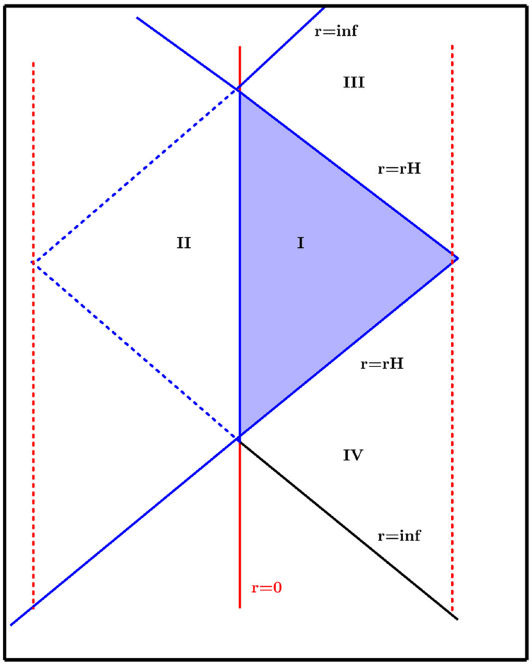

Figure 2. Penrose diagram of the new solution in (p, q)-coordinates.

$$\mathcal{S} = \int d^4x\sqrt{-\tilde{g}}\{\omega^2\tilde{R} + 6\partial_\alpha\omega\partial^\alpha\omega + \kappa^2\Lambda\omega^4\}. \quad (12)$$

One can easily prove that this Lagrangian is locally conformally invariant under

$$\tilde{g}_{\mu\nu}(\mathbf{x}) \to \Omega(\mathbf{x})^2\tilde{g}_{\mu\nu}(\mathbf{x}), \qquad \omega(\mathbf{x}) \to \frac{1}{\Omega(\mathbf{x})}\omega(\mathbf{x}). \quad (13)$$

The field equations become

$$\tilde{G}_{\mu\nu} = \frac{1}{\omega^2}\left(\tilde{T}^{(\omega)}_{\mu\nu} + \frac{1}{6}\tilde{g}_{\mu\nu}\Lambda\kappa^2\omega^4\right),$$
$$\tilde{\nabla}^\alpha\partial_\alpha\omega - \frac{1}{6}\tilde{R}\omega - \frac{1}{9}\Lambda\kappa^2\omega^3 = 0, \quad (14)$$

with

$$\tilde{T}^{(\omega)}_{\mu\nu} = \left(\tilde{\nabla}_\mu\partial_\nu\omega^2 - \tilde{g}_{\mu\nu}\tilde{\nabla}_\alpha\partial^\alpha\omega^2\right) - 6\left(\partial_\mu\omega\partial_\nu\omega - \frac{1}{2}\tilde{g}_{\mu\nu}\partial_\alpha\omega\partial^\alpha\omega\right). \quad (15)$$

The cosmological constant Λ could be ignored from the point of view of naturalness in order to avoid the inconceivable fine-tuning. Putting Λ as zero increases the symmetry of the model. One formulates the cosmological constant problem as a mismatch between particle physics and cosmology. However, in conformally invariant models, where the exact symmetry is spontaneously broken by the vacuum (as in the Brout–Englert–Higgs mechanism), a cosmological constant could be induced by fermionic matter-fields. The zero-point fluctuations could then cancel the induced cosmological constant (Mannheim 2008). In the conformally invariant model considered here, a slightly different formulation is used. If one calculate the beta-functions of the model ('t Hooft 2010), they all should be zero, because conformal invariance has to be kept. The number of beta-functions is equal to the number of freely adjustable parameters, such as the cosmological constant (it appears in the Lagrangian by $\Lambda\omega^4$). So they will be fixed in Planck units. It is conjectured that this will lead to an almost zero cosmological constant, if one incorporates all the allowed matter fields in the Lagrangian, such as fermion fields and non-Abelian gauge fields. This approach could solve the long-standing hierarchy problem. One should also investigate the stability of the BTZ solution against a small cosmological constant in the non-vacuum situation. In an earlier study (Slagter 2020), it was found numerically in the non-vacuum case (i.e., a scalar-gauge field), that a (tiny) cosmological constant has no effect at all on the essential behaviour of the dilaton and scalar field.

In the field equations, Equation (14), the covariant derivatives are taken with respect to $\tilde{g}_{\mu\nu}$ and $\tilde{R}$ is associated with $\tilde{g}_{\mu\nu}$.

The set of differential equations become

$$N'' = \frac{1}{3r^2N}[r^2(N')^2 - 3rNN' + 2N^2], \quad (16)$$

$$N^{\varphi''} = \frac{N^{\varphi'}}{3rN}(4rN' - 7N), \quad (17)$$

$$\omega'' = \frac{2\omega}{9r^2N^2}(2rN' + N)^2, \quad (18)$$

where we eliminated ω' from the equations by the remaining constraint equation

$$\omega' = -\frac{\omega}{3Nr}(2rN' + N). \quad (19)$$

An exact solution can be found,

$$N = \frac{(c_1r^{4/3} + c_2)^{3/2}}{r},$$

$$N^\varphi = c_3 + c_4\left(\frac{c_2^2}{4r^{8/3}} + \frac{c_1c_2}{r^{4/3}} - \frac{2}{3}c_1^2\log(r)\right),$$

$$\omega = (c_1r^{4/3} + c_2)^2[c_5H_+(r)r^{1/2+1/6\sqrt{17}} + c_6H_-(r)r^{1/2-1/6\sqrt{17}}], \quad (20)$$

with

$$H_{\pm}(r) = \text{Hypergeom}\left(\left[\frac{7}{8} \pm \frac{1}{8}\sqrt{17}, \frac{25}{8} \pm \frac{1}{8}\sqrt{17}\right], \left[1 \pm \frac{1}{4}\sqrt{17}\right], \frac{c_4 r^{4/3}}{c_3}\right) \tag{21}$$

a hyper-geometrical function. The constants c_1 and c_2 are related to the angular momentum and mass. By considering BTZ space–time as the unphysical metric $\tilde{g}_{\mu\nu}$ in the conformally invariant setting, we can then consider the dilaton as the scale factor. $\tilde{g}_{\mu\nu}$ has evidently the horizon at $r = \pm(c_2/c_1)^{3/4}$ and an ergo-circle, while $g_{\mu\nu}$ has no horizon, although it has the ergo-circle at $r = \pm(c_2/c_1)^{3/4}$. So one could say that the boundary of the object lies at $r = \pm(c_2/c_1)^{3/4}$. In figure 3 we plotted a typical solution for some values of c_i. It is clear that the solution for $g_{\mu\nu}$ is then regular and does not suffer from the problems encountered in the BTZ solution, such as the up-lifting problem to $(3+1)$-dimensional models, the formation of closed time-like curves and naked singularities. However, it must be remarked that this is a vacuum solution, and so not resembling reality: one needs matter terms in the field equations. The constraint equation will then be used in a numerical approach.

3.1 *The 3D counterpart solution*

If we disregard the dz^2 term, we obtain the set of equations

$$\omega'' = 2\frac{\omega'^2}{\omega}, \qquad N^{\varphi''} = -N^{\varphi'}\left(2\frac{\omega'}{\omega} + \frac{3}{r}\right),$$
$$N'' = -12N\frac{\omega'^2}{\omega^2} - 10\frac{\omega' N'}{\omega} - \frac{N'^2}{N} - 6N\frac{\omega'}{r\omega} - 3\frac{N'}{r} \tag{22}$$

with solution

$$\omega = \frac{1}{c_1 r + c_2},$$
$$N^{\varphi} = c_3 + c_4\left[c_1^2 \log(r) - \frac{c_2}{2r^2}(4c_1 r + c_2)\right],$$
$$N = \frac{(c_1 r + c_2)^4}{\sqrt{10r}}. \tag{23}$$

In figure 4 we plotted a typical solution. If we compare this solution with the 4D counterpart solution of Equation (20), we observe that only the behaviour of ω differs significantly. This can be explained as follows. Locally, on small scales, an observer experiences ω as given by Equation (23). At larger scales, the 4D counterpart model describes a different ω given by Equation (20). The metrics $\tilde{g}_{\mu\nu}$ do not differ significantly. The dilaton, locally unobservable, will be fixed by the choice of our coordinate system. Another argument in favour of our model concerning the physical acceptability, is the issue of the mass of the object. This will be treated in the next section. One must keep in mind that the dilaton field must be treated as a quantum field, when approaching smaller scales ('t Hooft 2015). If one incorporates matter fields in the model, then the local conformal invariance will be broken. These issues will not be further pursued here.

4. The spinning cosmic string connection

In an earlier study (Slagter & Duston 2019) we found an exact Ricci-flat conformally invariant solution of the exterior of a spinning cosmic string of the metric

$$ds^2 = \omega(r)^2[-(dt - J(r)d\varphi)^2 + b(r)^2 d\varphi^2 + e^{2\mu(r)}(dr^2 + dz^2)], \tag{24}$$

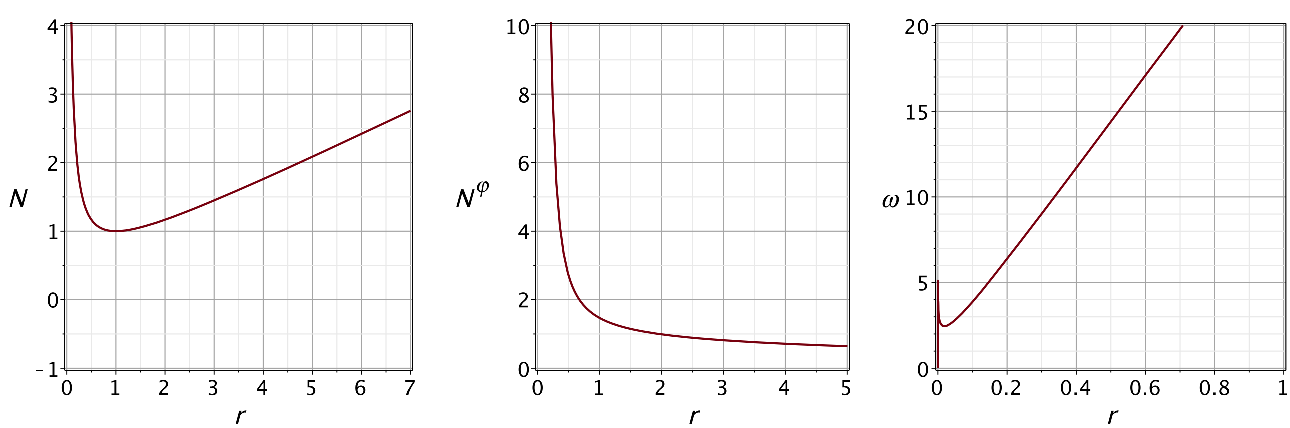

Figure 3. Example of a conformally invariant solution of the BTZ 4D space–time.

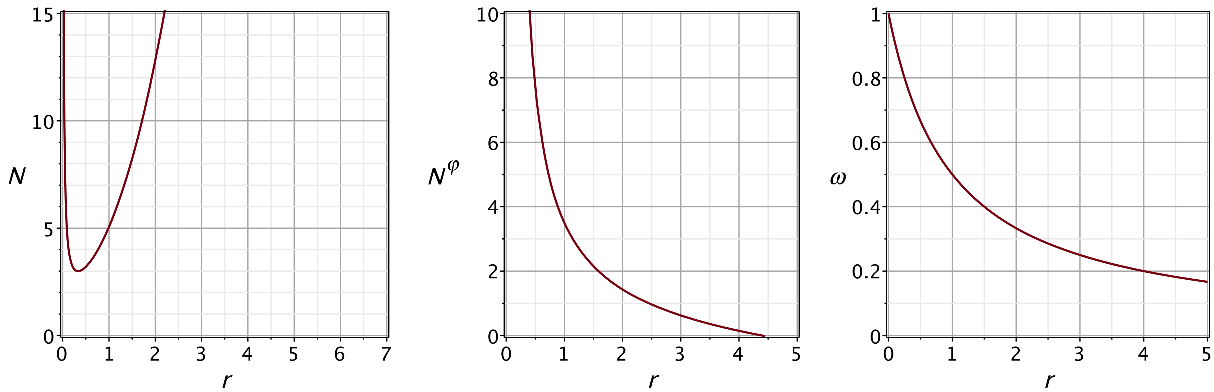

Figure 4. Example of a conformally invariant solution of the BTZ in 3D space–time.

which could be matched on the interior of the $U(1)$ scalar-gauge field solution. The most important results of the solution were the correct asymptotic behaviour of $J(r)$ and the absence of horizons. For the exterior it was found that

$$J(r) = \text{const.} \int \frac{b(r)}{\omega(r)^2} dr. \tag{25}$$

A comparable relation can be found here in the 4D as well as in the 3D case, i.e.,

$$N^{\varphi}(r) = \text{const.} \int \frac{1}{r^3 \omega(r)^2} dr. \tag{26}$$

In the original BTZ solution, localized matter has no influence on the local geometry of the source free regions and effects only the global space–time. The asymptotic symmetry group (considered as gauge transformations) is then applied and boundary conditions are adopted at spatial infinity. After the coordinate transformation (see Equation 4), the "jump" in the coordinate time was related to the angular momentum and the angle deficit with the mass. Only at $r' = 0$ there is an obstruction. In our model, after up-lifting to 4D, we have a boundary determined by the matter fields of the spinning cosmic string. One can easily check that in the case of global strings, with a scalar field Φ present, the relation between the angular momentum and the dilaton field of Equation (27) changes into (Slagter & Duston 2019)

$$J(r) = \text{const.} \int \frac{b(r)}{\omega(r)^2 + \eta^2 \Phi(r)^2} dr, \tag{27}$$

where η is the vacuum expectation value. Our solution of Section 3 suggest that it is the dilaton field that determines the global behaviour of the space–time and not an infinite thin line mass. At very small scales when $\omega \to 0$, $J(r)$ remains finite. Moreover, the angular momentum has already the correct asymptotic behaviour. In the up-lifted situation, ω is truly a scale factor. It was also recognized (Deser *et al.* 1992; Janca 2007), that the spinning cosmic string solution in $(2+1)$ dimensions ("cosmon") encounters inconsistencies if one uplifts the solution to our real $(3+1)$-dimensional space–time. The viewpoint of a particle is no longer tenable: an infinite thin cosmic string (an infinite line mass) is physically not acceptable (Geroch & Traschen 1987). Moreover, the weak energy condition is not fulfilled. In the conformally invariant model, these problems are not present. It is the dilaton field ω which plays the role of a massless (quantized) scalar field.

5. Conclusion

A new solution is found for the BTZ space–time, without a cosmological constant. The solution shows some different features with respect to the standard BTZ solution. A local non-rotating observer has no coordinate angular velocity, i.e., no frame dragging and there is no jump in the radial coordinate (in a suitable coordinate system) when crossing the horizon. The conformal invariant counterpart model shows no horizon and can be related to the spinning point particle solution of the dimensional reduced spinning cosmic string. The next step will be the investigation of the dynamical BTZ solution in the vacuum as well as in the non-vacuum case. This is currently under investigation by the author.

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