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Irreversible thermodynamics of the universe in $f(R)$ gravity

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Abstract. In this study, the irreversible thermodynamics of the universe in the framework of $f(R)$ gravity has been studied following standard Eckart theory of non-equilibrium thermodynamics. For a spatially flat universe, validity of GSLT and thermodynamic equilibrium have been examined both for apparent and event horizon as bounding horizons.The general result has been verified for exponential models of $f(R)$ gravity and it is shown that GSLT holds on both apparent and event horizon, but thermodynamic equilibrium does not hold on event horizon.

Keywords. Irreversible thermodynamics—$f(R)$gravity—GSLT—Thermodynamic equilibrium.

1. Introduction

In 1970s Stephen Hawking (Hawking 1975) made a revolutionary discovery that black hole behaves as a black body and emits thermal radiation (Hawking radiation). It was also discovered (Bardeen *et al.* 1973) that the four laws of thermodynamics are actually analogous to the four laws of black hole mechanics, with temperature and entropy, respectively, playing the role of surface gravity and area of the surface of bounding horizon. Further Jacobson (1995) derived Einstein's field equations from first law of thermodynamics and Padmanavan (2002) derived first law of thermodynamics from Einstein's field equations for a general static spherically symmetric space-time. These two deductions strongly support the idea that black hole thermodynamics can be extended to more general space-time. More specifically, it can be said that not only a black hole but the whole universe can be considered to be a thermodynamical object. Now, universal thermodynamics should be irreversible in nature, neither reversible nor quasi-reversible. It was first realized by Eling *et al.* (2006), when following the work of Jacobson (1995), they tried to reproduce Einstein's field equations from first law of thermodynamics in $f(R)$ gravity and found out that a non-equilibrium thermodynamic treatment is necessary for this successful derivation. Also, it will be worth to mention that processes like decoupling of neutrinos from cosmic plasma, decoupling of photons from matter during recombination era, nucleosynthesis, etc., need non-equilibrium thermodynamics for explanation.

Now, current observations like supernovae Ia (SNeIa) (Riess *et al.* 1998), cosmic microwave background (CMB) (Bennet *et al.* 2003), large-scale structure (LSS) (Hawkins *et al.* 2003), etc., reveal that our universe is going through an accelerating phase. There are two approaches to explain such acceleration of universe. One approach is to assume the existence of an exotic matter called dark energy which has negative pressure. Another approach is to consider the theory of modified gravity. $f(R)$ gravity is one of the popular models among modified gravity models where negative and positive powers of Ricci curvature scalar R naturally combine the inflation at early times and the cosmic acceleration at late times. In Sobouti (2007), it was also argued that the $f(R)$ gravity can serve as dark matter (DM). The above discussions motivated us to investigate thermodynamic properties of universe in irreversible thermodynamic context in $f(R)$ gravity model. We therefore structure this paper as follows: In Section 2 the basic features of $f(R)$ gravity have been described. In Section 3, a general prescription of irreversible thermodynamics in the frame of $f(R)$ gravity model has been given. In Section 4, for a general $f(R)$ gravity model, we investigated the validity of the generalized second law of thermodynamics (GSLT) and

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thermodynamic equilibrium (TE). In Section 5 we discuss these results for a particular viable model of $f(R)$ gravity, namely, exponential model both for apparent and event horizon as bounding horizon. Finally, in Section 6 we discuss the summary and conclusion of this work.

2. $f(R)$ gravity framework

The modified Einstein–Hilbert action in the Jordan frame in $f(R)$ gravity is given by (Nojiri & Odintsov 2011)

$$S_J = \int \sqrt{-g} d^4x \left[\frac{f(R)}{16\pi G} + L_{matter}\right] \tag{1}$$

where G, g, R and L_{matter} are the gravitational constant, the determinant of the metric $g_{\mu\nu}$, the Ricci scalar and the Lagrangian density of the matter inside the universe respectively. $f(R)$ is an arbitrary function of the Ricci scalar. One can derive the following equation on varying the action (1) with respect to $g_{\mu\nu}$:

$$FG_{\mu\nu} = 8\pi G T^m_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(RF - f) + \nabla_\mu \nabla_\nu F - g_{\mu\nu}\Box F \tag{2}$$

Here $F = \frac{df}{dR}$, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ and $T^m_{\mu\nu}$ is the energy-momentum tensor of the matter.

The gravitational field equation in its standard form (Starobinsky 1980; Motohashi *et al.* 2010) is written as

$$G_{\mu\nu} = 8\pi G \left(T^m_{\mu\nu} + T^D_{\mu\nu}\right) \tag{3}$$

where

$$8\pi G T^D_{\mu\nu} = (1 - F)G_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(RF - f) + \nabla_\mu \nabla_\nu F - g_{\mu\nu}\Box F \tag{4}$$

If one takes $T^{\mu(m)}_\nu = diag(-\rho, p, p, p)$ in the perfect fluid form, then for a spatially flat universe, the FRW equations read as

$$3H^2 = 8\pi G(\rho + \rho_D) \tag{5}$$

$$2\dot{H} = -8\pi G(\rho + \rho_D + p + p_D) \tag{6}$$

where

$$8\pi G\rho_D = \frac{1}{2}(RF - f) - 3H\dot{F} + 3H^2(1 - \dot{F}) \tag{7}$$

$$8\pi G p_D = \left[-\frac{1}{2}(RF - f) + \ddot{F} + 2H\dot{F} - (1 - F)(2\dot{H} + 3H^2)\right] \tag{8}$$

and

$$R = 6(\dot{H} + 2H^2) \tag{9}$$

Here $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Also ρ_D and p_D are the curvature contribution to the energy density and pressure, respectively.

3. General prescription of irreversible thermodynamics in $f(R)$ gravity

In this work we have followed the theory of irreversible thermodynamics previously discussed by Gong *et al.* (2009) and Chakraborty & Biswas (2013a, b). The summary of the theory is as follows:

In non-equilibrium thermodynamics. the Clausius relation $dS = \frac{\delta Q}{T}$ is replaced by the entropy balance equation:

$$dS_T = \frac{\delta Q}{T} + d_i S = d_e S + d_i S \tag{10}$$

where the extra term $d_i S$ arises due to internal production process. In particular

$$d_i S = \begin{cases} > 0, & \text{for irreversible process;} \\ 0, & \text{for reversible process.} \end{cases}$$

Classically δQ represents the heat transfer between the system and its surrounding, while $d_i S$ is the change of entropy associated with the uncompensated heat when the system undergoes an irreversible process.

Let us denote σ and $\overrightarrow{J_s}$ to be the internal entropy production density and the entropy flow density respectively. Assuming local equilibrium, one can write:

$$\frac{d_e S}{dt} = -\int_{\sum} \overrightarrow{J_s}.d\sum \tag{11}$$

$$\frac{d_i S}{dt} = \int_V \sigma dV \tag{12}$$

Here V = volume inside the bounding horizon of the system, $\sum$ being the surface of the horizon. If heat conduction is considered to be responsible for causing the entropy flow, then it can be written that

$$\frac{d_e S}{dt} = \frac{A\overrightarrow{J_q}}{T} \tag{13}$$

$$\frac{d_i S}{dt} = \sigma.V \tag{14}$$

where $\overrightarrow{J_q}$ = heat current, T = temperature of the system and A = surface area of the horizon. $\overrightarrow{J_q}$ and σ can be calculated from the following relations:

$$\overrightarrow{J_s} = \frac{\overrightarrow{J_q}}{T}$$

$$\sigma = \overrightarrow{J_q}.\nabla\left(\frac{1}{T}\right)$$

Let us consider the universe be a thermodynamical system with $R_\sum$ the radius of the bounding horizon. Now if we assume the universe to be a Bekenstein system bounded by apparent/event horizon, the entropy of the horizon in $f(R)$ gravity is then given by $S = \frac{AF}{4}$.

Therefore, one can obtain

$$\frac{d_e S}{dt} = \frac{\dot{A}F + A\dot{F}}{4} \tag{15}$$

Now, comparing the above equation with Equation (13) and putting $A = 4\pi R_\sum^2$(area of the surface of the horizon), we can derive the following equation:

$$|\overrightarrow{J_q}| = \frac{TF}{2}\left(\frac{\dot{F}}{2F} + \frac{\dot{R_\sum}}{R_A}\right) \tag{16}$$

where for a flat universe the Hawking temperature $T = \frac{1}{2\pi R_\sum}\left(1 - \frac{\dot{R_\sum}}{2}\right)$.

Now, according to Eckart–Fourier law:

$$\overrightarrow{J_q} = -\lambda\overrightarrow{\nabla}T$$

which states that there will be an energy flux if there is a temperature gradient ($\lambda > 0$) is the thermal conductivity). Now, putting the value of $|\overrightarrow{J_q}|$ in the equation for σ and using the Fourier law, one can obtain:

$$\begin{aligned}
\sigma &= \overrightarrow{J_q}.\nabla\left(\frac{1}{T}\right) \\
&= \overrightarrow{J_q}.\left(-\frac{1}{T^2}\right)\nabla T \\
&= \left(-\frac{1}{T^2}\right)\overrightarrow{J_q}.\frac{\overrightarrow{J_q}}{-\lambda} \\
&= \frac{|\overrightarrow{J_q}|^2}{\lambda T^2} \\
&= \frac{T^2F^2}{4\lambda T^2}\left(\frac{\dot{F}}{2F} + \frac{\dot{R_\sum}}{R_\sum}\right)^2 \\
&= \frac{F^2}{4\lambda}\left(\frac{\dot{F}}{2F} + \frac{\dot{R_\sum}}{R_\sum}\right)^2
\end{aligned} \tag{17}$$

Using Equations (14), (15) and (17), the change of total entropy can be obtained as

$$\frac{dS_T}{dt} = 2\pi R_\sum X\left(1 + \frac{X}{6\lambda}\right) \tag{18}$$

where $X = \dot{R_\sum}F + \frac{\dot{F}R_\sum}{2}$.

The above expression shows that total entropy change of the system depends on the non-equilibrium factor λ.

In case of reversible system, the non-equilibrium factor $\lambda \to \infty$. Infinitely large value of thermal conductivity λ means the heat transfer rate is infinitely large and therefore the temperature will eventually be same everywhere and therefore the system will be in equilibrium. We see that when $\lambda \to \infty$, the contributing part corresponding to irreversibility in Equation (18) will become zero and the equation then perfectly represents the reversible system.

4. Generalized second law of thermodynamics and thermodynamic equilibrium

In the context of isolated macroscopic systems, the entropy should never decrease, because such a system always evolves towards thermodynamic equilibrium, a state having maximum entropy. Thus, for a matter filled universe, bounded by a horizon, the generalized second law of thermodynamics (GSLT) and thermodynamical equilibrium (TE) hold if the following conditions hold:

I $\frac{dS_T}{dt} \geq 0$ (for GSLT)
II $\frac{d^2S_T}{dt^2} < 0$ (for Thermodynamic Equilibrium)

From Equation (18), it can be easily conjectured that GSLT is valid if either of the following conditions is hold:

- $X > 0$,
- $X < -6\lambda$.

Now, from Equation (18), one can calculate $\frac{d^2S_T}{dt^2}$ as

$$\frac{d^2S_T}{dt^2} = \left[\frac{\dot{R_\sum}}{R_\sum} + \frac{2\dot{X}}{X}\right]\frac{dS_T}{dt} - 2\pi R_\sum\dot{X} \tag{19}$$

Now, if we assume GSLT holds, then following cases may arise in order to make decision on thermodynamic equilibrium (TE):

5. Validity of GSLT and TE in exponential model of $f(R)$ gravity

$f(R)$ theory of modified gravity as an alternative of dark energy has been widely discussed in recent past (Sobouti 2007; Capozziello & Laurentis 2011; Clifton *et al.* 2011; Felice & Tsujikawa 2010; Nojiri & Odintsov 2011) and we do not have enough scope in the present article to discuss all the features of every viable model of $f(R)$ gravity analysed before. In this work our aim was to investigate the thermodynamic properties of the universe on the basis of the theory discussed in the previous section in one viable model of $f(R)$ gravity, namely, exponential model of $f(R)$ gravity. Some recent analysis on exponential gravity model suggests that observational constraints can be well satisfied from the cosmological point of view, in such a way that $f(R)$ gravity and ΛCDM model turn out to be nearly indistinguishable, as investigated in a previous analysis (Bamba *et al.* 2010a, b; Yang *et al.* 2010; Chen *et al.* 2015). In addition, the exponential gravity model can be extended to cover the inflationary stage as well.

The exponential model is given by

$$f(R) = R - \beta R_S\left(1 - e^{-\frac{R}{R_S}}\right) \tag{20}$$

where β and R_S are two constant parameters of this model. Following Bamba *et al.* (2010a, b), we put $\beta = 1.8$ and $R_S = \frac{18\Omega_{m0}H_0^2}{\beta}$.

From Equation (20), we calculate F, $\dot{F}$ and $\ddot{F}$ as follows:

$$F = 1 - \beta e^{-\frac{R}{R_S}} \tag{21}$$

$$\dot{F} = \frac{6H^3\beta(j-q-2)e^{-\frac{R}{R_S}}}{R_S} \tag{22}$$

$$\ddot{F} = \beta e^{-\frac{R}{R_S}}\left[\frac{6H^4\left(s+q^2+8q+6\right)}{R_S} - \frac{36H^6\,(j-q-2)^2}{R_S^2}\right] \tag{23}$$

where q, j, s are respectively the deceleration parameter, jerk parameter and state finder parameter defined by $q = -1 - \frac{\dot{H}}{H^2}$, $j = \frac{\ddot{H}}{H^3} - 3q - 2$, $s = \frac{H^{(3)}}{H^4} + 4j + 3q(q+4) + 6$.

Now, we examine GSLT and TE for the exponential model of $f(R)$ gravity on apparent and event horizon as bounding horizon of the universe in two separate cases.

Apparent Horizon: In the case of apparent horizon, Equations (18) and (19) take the following forms:

$$\frac{dS_T}{dt} = 2\pi H^{-1}X\left(1 + \frac{X}{6\lambda}\right) \tag{24}$$

$$\frac{d^2S_T}{dt^2} = \left[H(q+1) + \frac{2\dot{X}}{X}\right]\frac{dS_T}{dt} - 2\pi H^{-1}\dot{X} \tag{25}$$

Here $X = (q+1)F + \frac{\dot{F}H^{-1}}{2}$ and $\dot{X} = H(2q^2 + q - j)F + \frac{\ddot{F}}{2H} + \frac{3}{2}(q+1)\dot{F}$

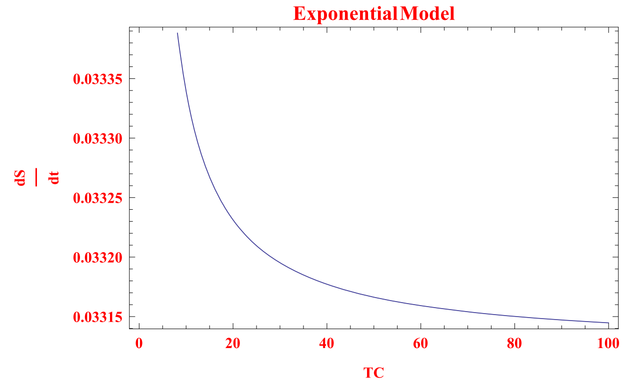

Figure 1. Graph for $\frac{dS_T}{dt}$.

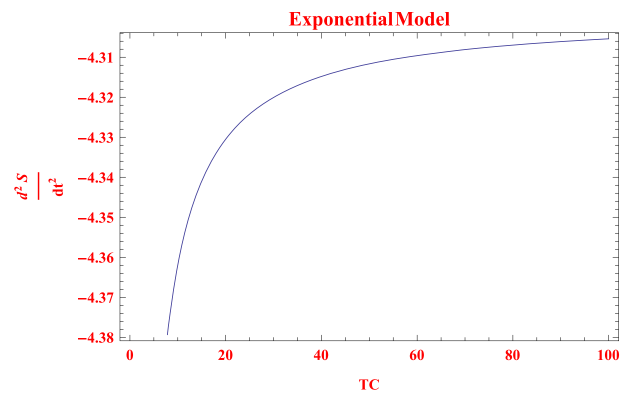

Figure 2. Graph for $\frac{d^2S_T}{dt^2}$ and $\frac{d^2S_T}{dt^2}$ with $\Omega_{m0} = 0.23$, $H_0 = 74.3$, $q_0 = -0.588$, $j_0 = 1$, $s_0 = -0.238$.

We plot graphs for $\frac{dS_T}{dt}$ and $\frac{d^2S_T}{dt^2}$ against the non-equilibrium factor λ (thermal conductivity) with the use of current observed value of matter density parameter and the cosmographic parameters obtained for standard ΛCDM model (Xia *et al.* 2012) as follows:

$$\Omega_{m0} = 0.23,\ H_0 = 74.3,$$
$$q_0 = -0.588,\ j_0 = 1,\ s_0 = -0.238.$$

From Figures 1 and 2 it is evident that at present epoch both GSLT and TE holds on apparent horizon.

From analytic expression of $\frac{dS_T}{dt}$ and $\frac{d^2S_T}{dt^2}$ in equation (24) and (25) respectively it is not possible to understand which particular condition depicted in Table 1 the model follows for the validity of GSLT and TE. Therefore, we consider an arbitrary value of $\lambda = 0.1$ (as λ is a small positive constant) and then calculate the terms X, $\dot{X}$, $\frac{dS_T}{dt}$, $H(q+1) + 2\frac{\dot{X}}{X}$ and obtain the following results:

$$X = 0.391688,\ \dot{X} = -62.8283,$$
$$\frac{dS_T}{dt} = 0.0547463,\ H(q+1) + 2\frac{\dot{X}}{X} = -290.196$$

Table 1. Condition for TE when $\frac{dS}{dt} > 0$.

	Cases	Conclusion
I	$\dot{X} > 0, \frac{\dot{R_\Sigma}}{R_\Sigma} + \frac{2\dot{X}}{X} > 0$	TE hods if $\frac{dS_T}{dt} < \frac{2\pi R_\Sigma \dot{X}}{\frac{\dot{R_\Sigma}}{R_\Sigma} + \frac{2\dot{X}}{X}}$
II	$\dot{X} < 0, \frac{\dot{R_\Sigma}}{R_\Sigma} + \frac{2\dot{X}}{X} > 0$	TE does not hold
III	$\dot{X} > 0, \frac{\dot{R_\Sigma}}{R_\Sigma} + \frac{2\dot{X}}{X} < 0$	TE holds
IV	$\dot{X} < 0, \frac{\dot{R_\Sigma}}{R_\Sigma} + \frac{2\dot{X}}{X} < 0$	TE holds if $\frac{dS_T}{dt} > \frac{2\pi R_\Sigma \lvert \dot{X} \rvert}{\left\lvert \frac{\dot{R_\Sigma}}{R_\Sigma} + \frac{2\dot{X}}{X} \right\rvert}$

Based on the above result we can naively comment that this model of $f(R)$ gravity GSLT follows the condition $X > 0$ in order to satisfy GSLT and for holding TE it follows the last condition of Table 1, i.e.

$$\dot{X} < 0,\ H(q+1) + \frac{2\dot{X}}{X} < 0,\ \frac{dS_T}{dt} > \frac{2\pi R_A \left|\dot{X}\right|}{\left|H(q+1) + \frac{2\dot{X}}{X}\right|}.$$

Event Horizon In case of event horizon, Equation (18) and Equation (19) take the following forms:

$$\frac{dS_T}{dt} = 2\pi R_E X \left(1 + \frac{X}{6\lambda}\right) \quad (26)$$

$$\frac{d^2 S_T}{dt^2} = \left[H - R_E^{-1} + \frac{2\dot{X}}{X}\right] \frac{dS_T}{dt} - 2\pi R_E \dot{X} \quad (27)$$

where R_E is the radius of event horizon and $X = (HR_E - 1) F + \frac{R_E \dot{F}}{2}$, $\dot{X} = -HF(1 + qHR_E) + 3(HR_E - 1) + \frac{R_E \ddot{F}}{2}$

Using the current observed value for matter density parameter and cosmography parameters as before $\Omega_{m0} = 0.23$, $H_0 = 74.3$, $q_0 = -0.588$, $j_0 = 1$, $s_0 = -0.238$, we obtain:

$$X = 71.5419 R_E - 0.971414$$
$$\dot{X} = 3209.71 R_E - 70.2737$$

Therefore, $X \lesseqgtr 0$ accordingly $R_E \lesseqgtr 0.0135783$ and $\dot{X} \lesseqgtr 0$ accordingly $R_E \lesseqgtr 0.0218941$.

Since it is well known that $R_E > R_A = \frac{1}{H} = 0.0134589502$, we can conclude that both X and $\dot{X}$ are positive. Since $X > 0$, GSLT is satisfied on event horizon.

In order to check whether TE holds on event horizon, we plot $\frac{d^2 S_T}{dt^2}$ against R_E and thermal conductivity λ.

From Figure 3 it is evident that thermodynamic equilibrium does not hold on event horizon.

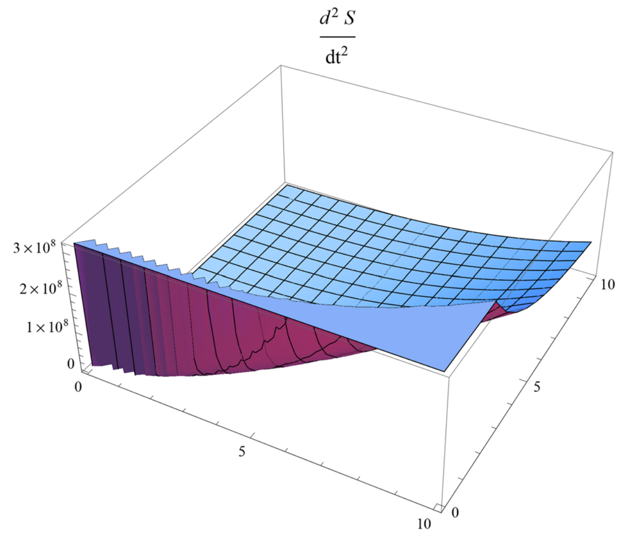

Figure 3. Graph for $\frac{d^2 S_T}{dt^2}$.

6. Conclusion

In this paper we have studied thermodynamic properties of universe in $f(R)$ gravity in the context of irreversible thermodynamics. Following standard Eckart theory of irreversible thermodynamics, we first derived condition for validity of GSLT and thermodynamic equilibrium both on apparent and event horizon for a general $f(R)$ gravity model. Later we considered the exponential model of $f(R)$ gravity as a particular example. Using current observed values of matter density parameter and cosmography parameters, we have shown that though GSLT is satisfied both on apparent horizon and event horizon, but thermodynamic equilibrium does not hold on event horizon. Since there is a constant transfer of energy (in this case, heat) between the system and surroundings, thermal equilibrium is not possible, i.e. maximum entropy can not be attained. Consequently, we can say that when a system undergoes irreversible thermodynamic process, thermodynamic equilibrium should not hold. In this context, we can conclude that event horizon is more preferable than apparent horizon.

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