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Modified Newtonian Dynamics (MOND) as a Modification of Newtonian Inertia

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Abstract. We present a modified inertia formulation of Modified Newtonian dynamics (MOND) without retaining Galilean invariance. Assuming that the existence of a universal upper bound, predicted by MOND, to the acceleration produced by a dark halo is equivalent to a violation of the hypothesis of locality (which states that an accelerated observer is pointwise inertial), we demonstrate that Milgrom's law is invariant under a new space–time coordinate transformation. In light of the new coordinate symmetry, we address the deficiency of MOND in resolving the mass discrepancy problem in clusters of galaxies.

Keywords. Dark matter—modified dynamics—Lorentz invariance

1. Introduction

The modified Newtonian dynamics (MOND) paradigm posits that the observations attributed to the presence of dark matter can be explained and empirically unified as a modification of Newtonian dynamics when the gravitational acceleration falls below a constant value of $a_0 \simeq 10^{-10}\,\mathrm{m\,s^{-2}}$. Milgrom (1983) noticed that the rotation curves of disk galaxies can be specified, given only the distribution of visible (baryonic) matter, using the formula

$$g\mu(g/a_0) = g_N \tag{1}$$

which relates the observed gravitational acceleration g to the Newtonian gravitational acceleration g_N as calculated from the baryonic mass distribution. The interpolating function $\mu(g/a_0)$ satisfies $\mu(g/a_0) = g/a_0$ when $g \ll a_0$, and $\mu(g/a_0) = 1$ when $g \gg a_0$. The direct observational evidence for Milgrom's formula is the fact that the mass discrepancy in galaxies of all sizes always appears below the acceleration scale a_0 (Famaey & McGaugh 2012). It follows from the appearance of an acceleration scale where dark matter halos are needed, that there is a universal upper bound to the acceleration that a dark halo can produce (Brada & Milgrom 1999). The difference between the MOND acceleration g and the Newtonian acceleration g_N can be explained by the presence of a fictitious dark halo and the upper bound is inferred by writing the excess (halo) acceleration as a function of the MOND acceleration,

$$g_D(g) = g - g_N = g - g\mu(g/a_0). \tag{2}$$

It seems from the behavior of the interpolating function as dictated by Milgrom's formula (Brada & Milgrom 1999) that the acceleration equation (2) is universally bound from the above by a value of the order a_0,

$$a_\dagger = \eta a_0, \tag{3}$$

where η is a dimensionless constant which is of order unity. This prediction was confirmed from the rotation curves for a sample of disk galaxies (Milgrom and Sanders 2005).

On the other hand, in Einstein's special theory of relativity, when Lorentz invariance is extended to accelerated observers, it is assumed that the behavior of measuring rods and clocks is independent of acceleration (Mashhoon 1990). This in fact, is a statement of the hypothesis of locality which asserts that an accelerated observer makes the same measurements as a hypothetical momentarily co-moving inertial observer. For instance, the rate of an accelerated clock is assumed to be independent of its acceleration and identical to that of the instantaneously co-moving inertial clock 'the clock hypothesis' (Moller 1972; Rindler 1982).

If, however, we assume that the characteristic maximum acceleration that appears in the behavior of dark halos predicted by MOND equation (3) is invariant under transformations from inertial to accelerated reference frames, then our assumption apparently contradicts the kinematic rule that the acceleration $a_\dagger$ as measured in an inertial frame S is given by French (1971): $a_\dagger = a'_\dagger + A$, where $a'_\dagger$ is the acceleration as measured in an accelerated frame S' and A is the acceleration of the frame S' with respect to the inertial frame S. Thus, if we assume that $a_\dagger = a'_\dagger$, then the accelerated measuring rods and clocks must behave in such a way that the relative acceleration between the two reference frames S' and S becomes undetectable and therefore it becomes unreasonable to assume that the hypothesis of locality is still valid upon making such assumptions about the maximum halo acceleration. So, probably the most suitable way to derive the maximum halo acceleration equation (3) from physical assumptions (not from the near coincidence of a_0 with cosmological parameters (Milgrom 1983) or introducing any new assumptions about the nature of dark matter) is to assume that the hypothesis of locality is false in the low acceleration limit $g \ll a_0$.

Our derivation of the maximum halo acceleration is based on interpreting Milgrom's formula as a modification of Newton's second law of motion (Milgrom 1994),

$$F = mg\mu(g/a_0), \tag{4}$$

where F is the total force exerted on the particle, and m is the inertial mass (response of the particle to all forces). However, this law is not enough by itself to represent a consistent modification of Newtonian inertia, because, if we consider an isolated system consisting of two bodies interacting gravitationally with small masses m_1 and m_2 such that equation (4) applies, and differentiate the total momentum $p = p_1 + p_2$ using equation (4) we obtain (Felten 1984)

$$\dot{p} = \sqrt{a_0 F}(\sqrt{m_1} - \sqrt{m_2}). \tag{5}$$

The total momentum of this isolated system is not conserved, $\dot{p} \neq 0$ unless $m_1 = m_2$. This problem can be avoided if there is a nonstandard kinetic action from which the equation of motion equation (4) is derived. Milgrom (1994) constructed such modified kinetic actions and showed that they must be time-nonlocal to be Galilean invariant, but it is unclear how to construct a relativistic generalization of such a scheme. It should be mentioned here that there are other possible approaches to MOND inertia, for example, inertia could be the product of the interaction of an accelerating particle with the vacuum (Milgrom 1999; McCulloch 2012).

The problems of modified inertia formulations of MOND can be alleviated by interpreting Milgrom's formula as a modification of Newtonian gravity. Bekenstein and Milgrom (1984) proposed a non-relativistic theory of MOND as a modification of Newtonian gravity (called AQUAL); the theory contains a modified gravitational action while the kinetic action takes its standard form, and thus conservation laws are preserved. The attempts to formulate a covariant generalization of AQUAL culminated with the emergence of the Tensor-Vector-Scalar theory (TeVeS) (Bekenstein 2004), the first consistent relativistic gravitational field theory for MOND. But even in this theory there is still a need for a predefined interpolating function that interpolates between the Newtonian and MONDian regime.

Instead of focusing our attention on constructing modified actions for MOND, let us reconsider the non-conservation of momentum problem from a mathematical point of view; the non-conservation of momentum exhibited in equation (5) can be attributed to the fact that each body in the isolated system is subject to a non-Newtonian force of magnitude ma_0, thus

$$\dot{p} = \sqrt{F}(\sqrt{m_1 a_0} - \sqrt{m_2 a_0}), \tag{6}$$

which means that it is not possible to isolate the two interacting bodies in the low acceleration limit from the influence of all sorts of external forces. So, why should we expect Newton's first law to be valid in the MOND regime? Perhaps, the isolated body in the MOND regime behaves differently from the isolated body in the Newtonian regime.

Newton's first law states that for an isolated body, far removed from all other matter, the vector sum of all forces vanishes $\vec{F} = 0$, hence, the isolated body moves with uniform velocity. Let us instead assume that in the case of an isolated body in the MOND regime; the sum of the magnitudes of all forces is constant and is proportional to ma_0, thus, $F = \eta m a_0$ where η is the proportionality factor, a_0 is the MOND acceleration constant, and m is the inertial mass. The isolated body then moves with uniform acceleration $g\mu(g/a_0) = \eta a_0$, we will refer to this assertion as the modified Newton's first law. Note that this is an assertion that cannot be confirmed experimentally, like Newton's first law.

Suppose an observer is placed in a reference frame S in which the modified Newton's first law holds, then the observer in this frame will measure a force $F = mg\mu(g/a_0) = m\eta a_0$. If there is another observer in a frame S' which is moving with respect to S with acceleration, then the second observer will also measure the same force $F' = mg'\mu(g'/a_0) = m\eta a_0$ (assuming that the mass and the acceleration constant ηa_0 are the same in S as in S'). The relative acceleration between the two frames S and S' is not dynamically detectable due to the invariance of ηa_0.

Therefore, the modified Newton's first law, coupled with the invariance of ηa_0, defines an infinite class of equivalent reference frames in accelerated motion relative to one another, and suppresses the appearance of inertial forces. Hence, the uniformly accelerated frames S and S' are equivalent and Milgrom's law is the same in both frames $F = mg\mu(g/a_0) = mg'\mu(g'/a_0) = F'$. When a MOND theory is fully compatible with this coordinate symmetry (i.e. the impossibility of detecting a coordinate change) it must satisfy conservation laws such as the conservation of momentum, because the coordinate symmetry implies the homogeneity of space. The modified Newton's first law is a key feature of our derivation of the upper bound equation (3) from physical assumptions.

2. The maximum halo acceleration and its consequences

Any physical process that involves the dynamics of particles and fields plays out on a background of space and time. Consequently, the physical laws must be adapted to any changes that might occur in the background (such as replacing the Galilean transformation by the Lorentz transformation); this scientific way of thinking about space and time, initiated by Einstein, led to modifications of the existing physical laws that are not Lorentz invariant (Rindler 1982). Therefore, we can establish an elegant physical basis for MOND, if Milgrom's law equation (4) is invariant under a new space–time coordinate transformation.

Let us consider a test particle of mass m freely falling in a uniform gravitational field of a dark matter distribution, where the density distribution of dark matter is derived from the rotation curves of disk galaxies; the discrepancy between the rotation curve expected from the distribution of baryonic matter, $v_B^2 = GM_B/r$, and the rotation curve measured by utilizing the Doppler effect, $v^2 = GM/r$, yields the distribution of dark matter, $v_D^2 = v^2 - v_B^2 = GM_D/r$. Then, the force acting on the test particle is given by Newton's second law

$$m\frac{d^2x}{dt^2} = mg_D = \frac{GmM_D}{r^2}. \tag{7}$$

As a consequence of the equality of inertial and gravitational mass, a freely falling reference frame constitutes an inertial reference frame; the uniform gravitational field cannot be detected in the freely falling frame. We can demonstrate that the inertial mass of the test particle governed by the equation of motion equation (7) is equivalent to its gravitational mass by performing the space–time coordinate transformations,

$$x' = x - \frac{1}{2}g_D t^2, \tag{8}$$

$$t' = t, \tag{9}$$

where the spatial origins of the two coordinate systems S and S' coincide at $t' = t = 0$, the unprimed system is the freely falling frame and the primed system is an inertial frame. Since the gravitational field g_D is uniform (it does not depend on t or x), the equation of motion becomes

$$m\frac{d^2x'}{dt'^2} = 0, \tag{10}$$

the gravitational force mg_D is canceled by an inertial force. Hence, at any space–time point in a uniform gravitational field we can specify a locally inertial reference frame in accordance with the principle of equivalence. But, motivated by the existence of the acceleration scale equation (3), let us assume that at some space–time points in the uniform gravitational field g_D, we cannot specify a locally inertial frame, in particular, let us postulate that there exists a universal constant of the order of the MOND acceleration constant $a_\dagger = \eta a_0$, which is invariant under transformations from inertial to accelerated frames. Thus, by performing space–time coordinate transformations analogous to equations (8) and (9) when the magnitude of the gravitational field g_D is equal to $a_\dagger$, the equation of motion equation (7) must become

$$m\frac{d^2x'}{dt'^2} = ma_\dagger, \tag{11}$$

this result is what we referred to earlier as the modified Newton's first law, an observer placed in the reference frame S' is assumed to be isolated from the influence of any other matter; and yet the observer experiences a force of magnitude $ma_\dagger$. This is due to the fact that $a_\dagger$ is an invariant of coordinate transformations,

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2} = a_\dagger. \tag{12}$$

Then, according to our postulate, the space–time coordinate transformations equations (8) and (9) must be accommodated to the condition equation (12). Although this postulate has not been confirmed by any experiment and cannot be demonstrated from first principles, we will demonstrate that this postulate is the main reason for the emergence of the upper limit equation (3) that has been confirmed from observations.

Consider a uniformly accelerated reference frame S' moving with an acceleration g_D relative to an inertial reference frame S, if the origins of both reference frames coincide at $t' = t = 0$ the origin of the reference frame S' which has the coordinate $x' = 0$ will be at a distance $x = \frac{1}{2}g_D t^2$ from the reference frame S. Thus, it is reasonable to assume that x' is proportional to the same $(x - \frac{1}{2}g_D t^2)$ factor as in the familiar coordinate transformation equation (8):

$$x' = \alpha\left(x - \frac{1}{2}g_D t^2\right), \tag{13}$$

where α is the proportionality factor. The same argument applies if we take the coordinate system S' to be the inertial frame, in this case, the origin of the reference frame S has the coordinate $x = 0$ and moves with acceleration $-g_D$ relative to the reference frame S', so that $x' = -\frac{1}{2}g_D t'^2$. Hence, the space transformation now takes the form

$$x = \alpha'\left(x' + \frac{1}{2}g_D t'^2\right), \tag{14}$$

where α' is the proportionality factor. In order to determine the relation between the two factors α and α', let us consider two observers placed in the uniformly accelerated frames S and S'. Since the observers in both frames experience the inertial force caused by the acceleration g_D (only the sign of g_D is different in the two frames), the uniformly accelerated frames S and S' must be equivalent for the description of physical events, for example, the length of the same measuring rod moving in these frames at the same acceleration g_D must be the same.

Suppose a rod of length $l = x'_2 - x'_1$ is at rest in the reference frame S' which is moving with an acceleration g_D relative to the reference frame S. The observer in S, who wants to measure the length of this rod, must measure the coordinates of the ends of the rod at the same time t. Using the coordinate transformation equation equation (13), we have

$$x'_1 = \alpha\left(x_1 - \frac{1}{2}g_D t^2\right), \quad x'_2 = \alpha\left(x_2 - \frac{1}{2}g_D t^2\right). \tag{15}$$

Therefore, the length of the rod as measured in the reference frame S is

$$\frac{l}{\alpha} = \frac{x'_2 - x'_1}{\alpha} = x_2 - x_1. \tag{16}$$

Let us now interchange S and S'. Suppose the same rod is at rest in S, where length $l = x_2 - x_1$, the observer in S', who wants to measure the length of this rod, must measure the coordinates of the ends of the rod at the same time t'. Using the coordinate transformation equation (14), we have

$$x_1 = \alpha'\left(x'_1 + \frac{1}{2}g_D t'^2\right), \quad x_2 = \alpha'\left(x'_2 + \frac{1}{2}g_D t'^2\right). \tag{17}$$

Then, the length of the rod as measured in the reference frame S' is

$$\frac{l}{\alpha'} = \frac{x_2 - x_1}{\alpha'} = x'_2 - x'_1, \tag{18}$$

if both frames S and S' are equivalent and the length of the same rod moving in these frames at the same acceleration g_D must be the same, we must have $l/\alpha = l/\alpha'$. Consequently,

$$\alpha = \alpha'. \tag{19}$$

In accordance with the postulate equation (12), if there is an object moving at acceleration $a_\dagger$ in an accelerating reference frame S', then the trajectory of this object as measured by an observer in an inertial reference frame S is

$$x = \frac{1}{2}a_\dagger t^2, \tag{20}$$

while the trajectory of the same object as measured by an observer in the accelerating frame S' is

$$x' = \frac{1}{2}a_\dagger t'^2. \tag{21}$$

Substituting these trajectories into equations (13) and (14), we obtain

$$a_\dagger t'^2 = \alpha t^2(a_\dagger - g_D), \tag{22}$$

$$a_\dagger t^2 = \alpha t'^2(a_\dagger + g_D), \tag{23}$$

from which we obtain a Lorentz-type factor

$$\alpha^2 = \frac{a_\dagger^2}{(a_\dagger + g_D)(a_\dagger - g_D)}, \tag{24}$$

$$\alpha = \frac{1}{\sqrt{1 - g_D^2/a_\dagger^2}}. \tag{25}$$

The appearance of the Lorentz-type factor concludes our derivation of the maximum halo acceleration from physical assumptions, since it implies that it is a physical impossibility for a spherical dark matter halo to attain a surface density greater than $a_\dagger G^{-1}$.

To get the time transformation, we can substitute equation (13) into equation (14) to obtain

$$x = \alpha\left(\alpha\left(x - \frac{1}{2}g_D t^2\right) + \frac{1}{2}g_D t'^2\right) \tag{26}$$

and

$$\frac{1}{2}g_D t'^2 = \frac{x}{\alpha} - \alpha\left(x - \frac{1}{2}g_D t^2\right) = \alpha\frac{1}{2}g_D t^2 + \left(\frac{1}{\alpha} - \alpha\right)x \quad (27)$$

whereas, from equation (24), we have

$$\frac{1-\alpha^2}{\alpha} = \sqrt{1 - \frac{g_D^2}{a_\dagger^2}}\frac{-\left(\frac{g_D^2}{a_\dagger^2}\right)}{\left(1 - \frac{g_D^2}{a_\dagger^2}\right)} = -\alpha\frac{g_D^2}{a_\dagger^2}. \quad (28)$$

which leads to

$$\frac{1}{2}g_D t'^2 = \alpha\frac{1}{2}g_D t^2 - \alpha\frac{g_D^2}{a_\dagger^2}x. \quad (29)$$

Thus, the space–time coordinate transformations equations (8) and (9) are the low acceleration limit $g_D \ll a_\dagger$ of the Lorentz-type transformations

$$x' = \alpha\left(x - \frac{1}{2}g_D t^2\right), \quad (30)$$

$$t'^2 = \alpha\left(t^2 - \frac{2g_D}{a_\dagger^2}x\right). \quad (31)$$

Unlike the Galilean and Lorentz transformations, the transformation equations (8) and (9), and the Lorentz-type transformation equations (30) and (31) are nonlinear in the time coordinate. Hence, if we represent the nonlinear transformation for the space and time coordinates by a 4 × 4 matrix

$$x'^\mu = \sum_{\nu=0}^{3}\Lambda^\mu_\nu x^\nu, \quad (32)$$

then the elements of the transformation matrix Λ must be coordinate dependent $\Lambda(x)$. However, the new transformation equations (30) and (31) have an advantage over the classical equations (8) and (9): it has a free parameter with the dimensions of acceleration $a_\dagger$. Since, $a_\dagger$ is the same in all coordinate systems, it can be recognized (in analogy to the speed of light) as a conversion factor that converts time measurements in seconds to meters $t\,(\mathrm{m}) = \frac{1}{2}a_\dagger \times t^2(\mathrm{s}^2)$. Thus, we can define x^0 with the factor $a_\dagger$ so that x^0 has the dimensions of length,

$$x^0 = \frac{1}{2}a_\dagger t^2 \quad (33)$$

and if we number the x, y, z coordinates, so that

$$x^1 = x, \quad x^2 = y, \quad x^3 = z \quad (34)$$

then we can rewrite the Lorentz-type transformation equations (30) and (31) in the four-vector notation:

$$x'^1 = \alpha\left(x^1 - \frac{g_D}{a_\dagger}x^0\right), \quad x'^2 = x^2, \quad x'^3 = x^3, \quad (35)$$

$$x'^0 = \alpha\left(x^0 - \frac{g_D}{a_\dagger}x^1\right) \quad (36)$$

which we can write in the matrix form as equation (32), where the components of the transformation matrix are:

$$\Lambda^\mu_\nu = \begin{pmatrix} \alpha & -\alpha g_D/a_\dagger & 0 & 0 \\ -\alpha g_D/a_\dagger & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (37)$$

Note that the components of the transformation matrix do not depend on the coordinates, as long as the gravitational field g_D is uniform.

A four-vector can now be defined as any set of four components that transform under the Lorentz-type transformation equation (32), the same way (x^0, x^1, x^2, x^3) does; for example, the time difference between any two events and their spatial separation can be represented by the displacement four-vector

$$dx^\mu = \begin{pmatrix} a_\dagger t\,dt \\ dx \\ dy \\ dz \end{pmatrix}. \quad (38)$$

the components of this vector as specified relative to the coordinate system S are related to the components of the same vector dx'^μ as specified relative to the coordinate system S' by the Lorentz-type transformation equation (32)

$$dx'^\mu = \sum_{\nu=0}^{3}\Lambda^\mu_\nu dx^\nu. \quad (39)$$

Let the inverse transformation to equation (39) read as follows:

$$dx^\nu = \sum_{\nu=0}^{3}\Lambda^\nu_\beta dx'^\beta, \quad (40)$$

where the matrices Λ^μ_ν and Λ^ν_β are inverse to each other;

$$\Lambda^\nu_\beta = \begin{pmatrix} \alpha & \alpha g_D/a_\dagger & 0 & 0 \\ \alpha g_D/a_\dagger & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (41)$$

By substituting equation (40) into equation (39), we can verify that the transformation matrix of the Lorentz-type transformation equation (32) satisfies the orthogonality condition

$$\sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} \Lambda^{\nu}_{\beta} = \delta^{\mu}_{\beta}, \tag{42}$$

where δ^{μ}_{β} is the Kronecker delta.

Therefore, the scalar product of dx^{μ} with itself is an invariant quantity

$$dx^{\mu} dx_{\mu} = (a_{\dagger} t dt)^2 - (dx)^2 - (dy)^2 - (dz)^2, \tag{43}$$

$$dx^{\mu} dx_{\mu} = (a_{\dagger} t' dt')^2 - (dx')^2 - (dy')^2 - (dz')^2. \tag{44}$$

If an observer is at rest in the frame S', then the spatial components of the displacement vector in this frame is zero

$$\begin{aligned} dx^{\mu} dx_{\mu} &= (a_{\dagger} t dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \\ &= (a_{\dagger} t' dt')^2. \end{aligned} \tag{45}$$

Hence, the scalar product $dx^{\mu} dx_{\mu}$ is proportional to the time interval measured by an observer in its rest frame. We can employ this fact to define a transformation-invariant coordinate time (the proper time τ) which will allow us to obtain a four-vector when differentiating a four-vector,

$$dx^{\mu} dx_{\mu} = (a_{\dagger} t dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 = (a_{\dagger} \tau d\tau)^2. \tag{46}$$

Thus

$$\begin{aligned} \tau d\tau &= t dt \left(1 - \frac{1}{a_{\dagger}^2} \frac{dx^2 + dy^2 + dz^2}{t^2 dt^2} \right)^{1/2} \\ &= t dt \left(1 - g_D^2 / a_{\dagger}^2 \right)^{1/2}, \end{aligned} \tag{47}$$

the invariant time unit can be obtained by integrating both sides

$$\tau = t \left(1 - g_D^2 / a_{\dagger}^2 \right)^{1/4}. \tag{48}$$

The trajectory of the observer in the frame S can be parameterized by the proper time τ;

$$x^{\mu}(\tau) = \begin{pmatrix} \frac{1}{2} a_{\dagger} t^2(\tau) \\ \mathbf{x}(\tau) \end{pmatrix}, \tag{49}$$

where $\mathbf{x}$ is the three-dimensional position, therefore the excess acceleration four-vector can be obtained by differentiating the position four-vector equation (49) with respect to the proper time equation (48)

$$\mathcal{A}^{\mu} = \frac{d^2 x^{\mu}}{d\tau^2} = \alpha \begin{pmatrix} a_{\dagger} \\ d^2\mathbf{x}/dt^2 \end{pmatrix} = \alpha \begin{pmatrix} a_{\dagger} \\ g_D \end{pmatrix}. \tag{50}$$

Hence, we can generalize Newton's second law equation (7) to the covariant form

$$\mathcal{F}^{\mu} = m \mathcal{A}^{\mu} = m\alpha \begin{pmatrix} a_{\dagger} \\ g_D \end{pmatrix} = m\alpha \begin{pmatrix} a_{\dagger} \\ GM_D/r^2 \end{pmatrix}, \tag{51}$$

where $\mathcal{F}^{\mu}$ is a four-vector force (the force due to the excess acceleration four-vector). We can find an appropriate interpretation of the time component of the force four-vector by Taylor, expanding the α factor,

$$\mathcal{F}^0 = \frac{m a_{\dagger}}{\sqrt{1 - g_D^2/a_{\dagger}^2}} = m a_{\dagger} + \frac{1}{2} m \frac{g_D^2}{a_{\dagger}} + \cdots. \tag{52}$$

Let us now return to Milgrom's formula equation (1) and choose the following form of the interpolating function as chosen by (Bekenstein 2004)

$$\mu(x) = \frac{\sqrt{1+4x} - 1}{\sqrt{1+4x} + 1}, \tag{53}$$

where $x = g/a_0$, hence, in the low acceleration limit the total acceleration due to Newtonian gravity can be expressed as follows

$$g = g_N + \sqrt{a_0 g_N}. \tag{54}$$

Thus, the excess acceleration $g_D = g - g_N$ can take the following form

$$g_D = \sqrt{a_0 g_N}. \tag{55}$$

Since, in equation equation (52) we can neglect the terms divided by $a_{\dagger}^2$ and higher in the limit of small accelerations $g_D \ll a_{\dagger}$, we can assume that $\mathcal{F}^0$ is made up of two parts: the first part gives identical results to Milgrom's law equation (4) in the low acceleration limit (when the numerical factor takes the value $\eta = \frac{1}{2}$) and in the presence of Newtonian gravitational forces

$$\mathcal{F}_1^0 = \frac{m a_{\dagger}}{\sqrt{1 - g_D^2/a_{\dagger}^2}} - m a_{\dagger} = m g_N \tag{56}$$

and the second part is a constant force

$$\mathcal{F}_2^0 = m a_{\dagger} \tag{57}$$

which is the force experienced by an observer at rest, and it can be interpreted as a location independent weight. In contrast, the weight of an object in Newtonian physics is defined as the product of the object's mass and the magnitude of the gravitational acceleration which depends on the location. We deduce from this that $\mathcal{F}^0 = \mathcal{F}_1^0 + \mathcal{F}_2^0 = F$ is the total force acting on the particle mass m.

In the Newtonian limit $g \gg a_0$ or equivalently $a_{\dagger} \to \infty$, i.e. $\alpha = 1$, the spatial components of the force four-vector equation (51) reduce to Newton's second law equation (7). Note that there is no need for a predefined interpolation function, but instead, it is the Lorentz-type factor α that allows the transition between the Newtonian and MONDian regime.

In order to write the equation of motion equation (51) using the Lagrangian formalism, the classical Lagrangian $L = \frac{1}{2}m\mathbf{u}^2 - V(\mathbf{x})$ must be invariant under the Lorentz-type transformations. The Lagrangian must be a function of the coordinates equation (49) and their derivatives with respect to the invariant parameter – the proper time τ. Suppose the force equation (51) is a conservative force derivable from a potential,

$$\mathcal{F}^\mu = -\frac{\partial V(x^\mu)}{\partial x^\mu}. \tag{58}$$

Using the velocity four-vector

$$u^\mu = \frac{dx^\mu}{d\tau} = \alpha^{\frac{1}{2}}\begin{pmatrix} a_\dagger t \\ \mathbf{u} \end{pmatrix}, \tag{59}$$

we can suggest the covariant Lagrangian

$$L = \frac{1}{2}mu^\mu u_\mu - V(x^\mu). \tag{60}$$

It follows from Hamilton's variational principle

$$\delta S = \delta \int_a^b L\, d\tau = 0 \tag{61}$$

that the Lagrangian equation (60) must satisfy the Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0. \tag{62}$$

Since the MOND effects can be attributed to the presence of a fictitious dark halo, the time component of Lagrange's equations determines the distribution of the dark halo from the distribution of baryonic mass

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^0}\right) = \frac{ma_\dagger}{\sqrt{1 - g_D^2/a_\dagger^2}}, \tag{63}$$

$$\frac{\partial L}{\partial x^0} = -\frac{\partial V}{\partial x^0} = mg_N + ma_\dagger, \tag{64}$$

where $\mathcal{F}^0 = mg_N + ma_\dagger$ is the total force exerted on the particle. While the spatial components of Lagrange's equations determine the motion of test particles in the gravitational field of the dark halo

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^i}\right) = m\alpha\frac{d\mathbf{u}}{dt}, \tag{65}$$

$$\frac{\partial L}{\partial x^i} = -\frac{\partial V}{\partial x^i} = m\alpha g_D. \tag{66}$$

The formulation of MOND, illustrated above, not only reproduces the predictions of Milgrom's law but it also leads to a number of physical consequences that arise from replacing the space–time coordinate transformation equations (8) and (9) by the Lorentz-type transformation equations (30) and (31).

Let us first write the transformations equations (30) and (31) in the differential form

$$dx' = \alpha\,(dx - g_D t dt)\,, \tag{67}$$

$$t'dt' = \alpha\left(tdt - \frac{g_D}{a_\dagger^2}dx\right). \tag{68}$$

Consider a rod of length dx' placed at rest in a frame of reference S' which is moving relative to a frame of reference S with an acceleration of g_D. To measure the rod's length in the frame S, the end points of the rod must be observed at the same time t. Since the observer in the frame S must measure the distance between the two end points simultaneously $dt = 0$, we have from equations (67),

$$dx = \left(1 - g_D^2/a_\dagger^2\right)^{1/2} dx'. \tag{69}$$

If, for example, D is the distance between two stars in a binary pair whose positions are observed simultaneously, then

$$D = \left(1 - g_D^2/a_\dagger^2\right)^{1/2} D_0, \tag{70}$$

where D_0 is the distance between the two stars as measured in a frame of reference in which g_D is equal to zero. Therefore, the distance between two uniformly accelerating stars with an acceleration of g_D is reduced by a factor $(1 - g_D^2/a_\dagger^2)^{1/2}$.

Consider a clock placed at rest in a frame of reference S' and it measures a time interval dt', suppose that S' is moving relative to a frame of reference S with an acceleration of g_D. Since the clock is stationary (there is no spatial displacement $dx' = 0$) in the frame S', we have from equation (67),

$$dx' = \alpha(dx - g_D t dt) = 0 \tag{71}$$

Substituting equation (71) into equation (68) we obtain

$$t = \frac{t'}{\left(1 - g_D^2/a_\dagger^2\right)^{1/4}}, \tag{72}$$

we conclude from the above formula that a uniformly accelerating clock with an acceleration of g_D runs slow by a factor $(1 - g_D^2/a_\dagger^2)^{1/4}$ relative to the clocks in the frame S.

Radiation emitted from a source while moving directly toward a receiver will be shifted in frequency by the Doppler effect. Suppose a light source is at rest in a frame of reference S' which is moving towards a receiver in a frame of reference S with speed u, thus if the light source sends a signal at a time interval dt as measured by a co-moving observer at the source, then during that time the signal is sent from position

$x = udt$, so this signal arrives at the receiver time

$$dT = dt - dtu/c \tag{73}$$

apart. But the effect of time dilation equation (72) modifies equation (73) to

$$dT = \frac{dt' - dt'u/c}{\left(1 - g_D^2/a_\dagger^2\right)^{1/4}}, \tag{74}$$

where dt' and dT are inversely proportional to the frequency ν_0 of the source in S' and the frequency ν of the source as seen by the observer in S, respectively,

$$\frac{\nu_0}{\nu} = \frac{1 - u/c}{\left(1 - g_D^2/a_\dagger^2\right)^{1/4}}. \tag{75}$$

Therefore, if we consider a spectroscopic binary star system placed in a frame of reference in which g_D is not equal to zero, then equation (75) predicts that there should be a detectable Doppler shift at any given time even when the inclination of the star's orbit relative to the line of sight is zero, i.e. the radial component of the star's velocity is zero $u = 0$. In contrast, the classical Doppler relation for non-relativistic speeds $\nu_0 = \nu(1 - u/c)$ predicts that there will be no detectable Doppler shift at instants of time when $u = 0$.

3. MOND and clusters of galaxies

We shall illustrate how the weak equivalence principle or the universality of free fall which allows us to equate the Newtonian gravitational field with an accelerated reference frame can be incorporated into our formulation of MOND. In Newtonian mechanics, an object freely falling in a uniform gravitational field is considered to be weightless, however, it seems from the modified Newton's first law $F = m\eta a_0$ that the state of weightlessness cannot actually be achieved even if the object is in a state of free fall in a uniform gravitational field. So combining the universality of free fall with the modified Newton's first law yields the following empirical formula that replaces Milgrom's formula,

$$g\mu(g/a_0) = g_N + \eta a_0, \tag{76}$$

where the interpolating function can be chosen to resemble the μ-function of Milgrom's formula, such that it satisfies $\mu(x) = 1$ when $x \gg \eta$, and $\mu(x) = x$ when $x \ll \eta$. The dimensionless constant η is equal to $1/2$ according to the argument above equation (56).

Then, in high acceleration systems $g \gg \frac{1}{2}a_0$ the term $\frac{1}{2}a_0$ appears as an anomalous acceleration

$$g = g_N + \frac{1}{2}a_0, \tag{77}$$

while in the low acceleration limit $g \ll \frac{1}{2}a_0$, we obtain

$$g = \sqrt{a_0 g_N + \frac{1}{2}a_0^2} \tag{78}$$

which might becomes relevant within large clusters of galaxies, particularly within their central regions, since MOND fails to completely resolve the mass discrepancy problem in these systems (Sanders 2003; Aguirre *et al.* 2001). We can, in principle, demonstrate this by considering a cluster in hydrostatic equilibrium, using equation (78) the dynamical mass can be determined from the density and temperature distribution of the X-ray emitting gas

$$\sqrt{a_0 GM + \frac{1}{2}a_0^2 r^2} = -\frac{kT}{\mu m_p}\left(\frac{d\ln\rho}{d\ln r} + \frac{d\ln T}{d\ln r}\right). \tag{79}$$

This relation is apparently more convenient for clusters than the mass-temperature relation $M \propto T^2$ predicted by MOND (Aguirre *et al.* 2001), because clusters are mostly isothermal, and isothermality (in the case of the mass-temperature relation) corresponds to a point mass not to an extended object.

Even though, the formula equation (76) seems to be helpful in removing the remaining mass discrepancy in MOND, it is not obvious how this formula is consistent with the fact that rotation curves are asymptotically flat. The resolution of this apparent contradiction lies in the principle of equivalence; since the ratio of inertial to gravitational mass is the same for all bodies $m_i = m_g$ then it should be possible, in the case of uniform gravitational field, to transform to space–time coordinates such that the effect of a gravitational force will not appear.

$$m_i g'\mu(g'/a_0) = (m_g - m_i)g_N + m_i\eta a_0 = m_i\eta a_0. \tag{80}$$

Hence, the analogue of Einstein's principle of equivalence in MOND states that: it is possible, in a sufficiently small regions of space–time such that the Newtonian gravitational field changes very little throughout it, to specify a coordinate system in which matter satisfies the law of motion equation (51), and hence it is possible in these regions to observe the asymptotic flatness of rotation curves. Therefore, any consistent generalization of the afore mentioned Lorentz-type invariance to non-uniformly accelerated coordinate systems must reproduce the results of the formula equation (76).

4. Conclusion

In this paper, we have presented a relativistic formulation of the MOND hypothesis based on the assumption that accelerated measuring rods and clocks are affected by acceleration in the low acceleration regime. The proposed relativistic formulation produces a prediction that does not result from either the MOND hypothesis or the dark matter hypothesis; it predicts that spectroscopic binary star systems in the low acceleration $g \ll a_0$ and low velocity $u \ll c$ regime should exhibit a non-classical Doppler shift, as expressed by the formula equation (75), due to a time dilation effect. We also showed that the mass discrepancy in clusters of galaxies can be accounted for by a consistent generalization of the Lorentz-type symmetry to non-uniformly accelerated coordinate systems.

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