



Classical Orbital Paramagnetism in Non-equilibrium Steady State

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Abstract. We report the results of our numerical simulation of classical-dissipative dynamics of a charged particle subjected to a non-Markovian stochastic forcing. We find that the system develops a steady-state orbital magnetic moment in the presence of a static magnetic field. Very significantly, the sign of the orbital magnetic moment turns out to be *paramagnetic* for our choice of parameters, varied over a wide range. This is shown specifically for the case of classical dynamics driven by a Kubo–Anderson type non-Markovian noise. Natural spatial boundary condition was imposed through (1) a soft (harmonic) confining potential, and (2) a hard potential, approximating a reflecting wall. There was no noticeable qualitative difference. What appears to be crucial to the orbital magnetic effect noticed here is the non-Markovian property of the driving noise chosen. Experimental realization of this effect on the laboratory scale, and its possible implications are briefly discussed. We would like to emphasize that the above steady-state classical orbital paramagnetic moment complements, rather than contradicts the Bohr–van Leeuwen (BvL) theorem on the absence of classical orbital diamagnetism in thermodynamic equilibrium.

Keywords. Fluctuation phenomena—random processes—noise—Brownian motion—diamagnetism—paramagnetism—superparamagnetism.

1. Introduction

The Bohr–van Leeuwen (BvL) theorem on the absence of classical orbital diamagnetism in thermodynamic equilibrium has been a surprise of theoretical physics (Bohr 1911; van Leeuwen 1921; van Vleck 1932; Peierls 1979; Ma 1985). This BvL null result is statistical-mechanically exact. It is, however, counter-intuitive inasmuch as a charged particle orbiting classically under the Lorentz force exerted by an externally applied magnetic field, is equivalent to an amperian current loop in the interior of the sample, and hence to a non-zero orbital magnetic moment. Moreover, the orbital moment is expected to be diamagnetic following the Lenz’s law. A physically appealing resolution was advanced by Bohr (1911) suggesting an exact cancellation, on the average, of the diamagnetic orbital magnetic moment in the interior of the sample by the paramagnetic moment subtended by the particle skipping the boundary in the opposite (paramagnetic) sense – the edge current (Ma 1985; van Vleck 1932).

The BvL theorem on the absence of classical orbital (dia-)magnetism was re-examined by Kumar and Kumar

(2009), where they considered dynamics of the charged particle in a finite but unbounded space, namely the surface of a sphere (recall that, strictly speaking, a boundary has no boundary). They had found, through numerical simulations, that an equilibrium non-zero orbital diamagnetism could possibly exist for the case of a charged-particle motion on the surface of a sphere, but not a plane.

Soon thereafter, in Pradhan and Seifert (2010), the role of the boundary was treated explicitly through a solution of the Fokker–Planck equation associated with the classical Langevin dynamics of the charged particle on the surface of a sphere. Again, the orbital magnetic moment turned out to be zero.

More recently, however, Deshpande *et al.* (2012) negated the result of Kumar and Kumar (2009) through similar numerical simulation, but now with finer time steps, for the motion of the charged particle on the surface of a sphere – in fact, in the long-time limit (i.e. in thermal equilibrium) the orbital moment indeed again turned out to vanish with decreasing time step, just as per the BvL theorem! This trend revealed the possible role of finite temporal correlation scale of random

forcing in inducing non-zero steady-state orbital magnetic moment.

Our analyses strongly suggest that the vanishing of the classical orbital diamagnetism is a direct consequence of detailed-balance (the microscopic reversibility), namely that there are *no cycles* in thermodynamic equilibrium. Now, the detailed balance is, of course, conditioned mathematically by the second fluctuation-dissipation (II-FD) theorem of Kubo (Kubo 1954, 1966; Anderson 1954). In terms of the classical Langevin equation governing the stochastic dissipative dynamics, that underlies equilibrium statistical mechanics, the F-D relation requires the driving noise to be Markovian. Indeed, it has been shown (Kumar 2012) that a Markovian steady state can always be transformed to an equivalent thermodynamic state in equilibrium. It is our conjecture, therefore, that a stochastic dissipative system driven by a non-Markovian noise may have, in general, a non-equilibrium steady state with finite orbital magnetic moment – without conflicting with the BvL theorem for thermodynamic equilibrium. In what follows, we have addressed this issue through a detailed numerical simulation of the relevant stochastic dissipative dynamics involved. The results of our simulation support our conjecture, namely that the orbital magnetic moment is indeed finite for the non-Markovian case. There is, however, a new surprise now, namely that the sign of the magnetic moment turns out to be paramagnetic! Moreover, for a classical gas of such charged particles with paramagnetic orbital moment, the inherently positive feedback may lead to an enhanced magnetic susceptibility – possibly even to a spontaneous ordering of the classical orbital magnetic moments.

2. Stochastic dissipative dynamics in a magnetic field

Consider the stochastic-dissipative classical dynamics of a charged particle (charge = $-e$, $e > 0$) and mass m in the xy -plane in the presence of a uniform magnetic field of magnitude B directed along the positive z -axis. Let the particle be confined harmonically in the xy -plane. (In this simple model, the motion along the z -axis factors out.) Harder confinement (reflecting wall) will be introduced later. The equation of motion can now be written down straightforwardly in the polar co-ordinate system as

$$m\ddot{r} = mr\dot{\theta}^2 - kr - \Gamma\dot{r} - \frac{eB}{c}r\dot{\theta} + F_r(t, \Delta t), \quad (1a)$$

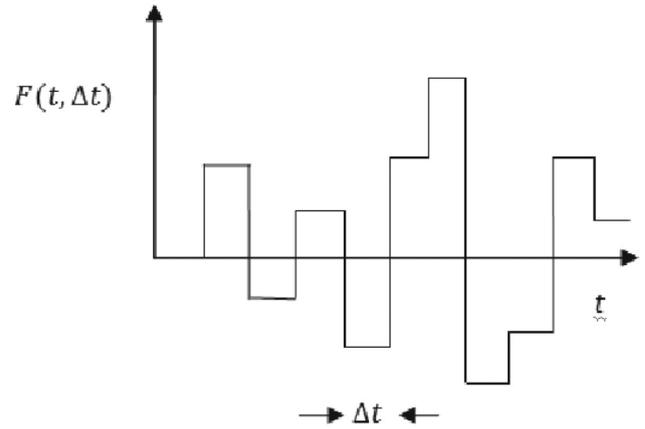


Figure 1. Shows schematically one possible realization of such a non-Markovian noise comprising equi-spaced rectangular pulses of random Gaussian height modeled on a Kubo–Anderson process.

$$m\ddot{\theta} = -2m\dot{r}\dot{\theta} - \Gamma r\dot{\theta} + \frac{eB}{c}\dot{r} + F_{\theta}(t, \Delta t), \quad (1b)$$

where $F_r(t, \Delta t)$ and $F_{\theta}(t, \Delta t)$ are the Kubo–Anderson-type noise terms. The K–A noise, as depicted schematically in Fig. 1, is best described as a sequence of random rectangular pulses of equal pulse-width Δt . The pulse heights are, however, identically independently distributed Gaussian random variates, of mean zero and finite variance.

It is to be emphasized that Δt here is not necessarily a small quantity in any sense – it is to be regarded as a physical input parameter that makes the noise tunably non-Markovian. The numerical simulation, however, must cover these not necessarily small time intervals Δt with much finer sub-divisions $\delta t \ll \Delta t$ so as to ensure high numerical accuracy in treating the systematic parts of the driving forces present in equations (1a) and (1b). More specifically, in our numerical simulation the number of finer time-step subdivisions has been taken to be >500 (Figures 1–3). Moreover, the time of integration has been kept long (5×10^8 sub-steps) in all these cases.

It is convenient to define here the parameters $\Omega_0 \equiv$ confining harmonic frequency = $\sqrt{k/m}$, $\Omega_c =$ the cyclotron frequency eB/mc , and $\gamma = \Gamma/m$, the Stokes friction. Also, we write that $F_r(t, \Delta t) = m\gamma^2\sigma f_r(t, \Delta t)$, and similarly for $F_{\theta}(t, \Delta t)$. Note that σ has the dimension of length. Further, we introduce the dimensionless quantities $\omega_0 = \Omega_0/\gamma$, $\omega_c = \Omega_c/\gamma$, $\tau = \gamma t$ and $R = r/\sigma$ the dimensionless radial coordinate. With these re-definitions, equations (1a) and (1b) can now be re-written in the fully dimensionless form as

$$\ddot{R} = R\dot{\theta}^2 - \omega_0^2 R - \dot{R} - \omega_c R\dot{\theta} + f_r(\tau, \Delta\tau), \quad (2a)$$

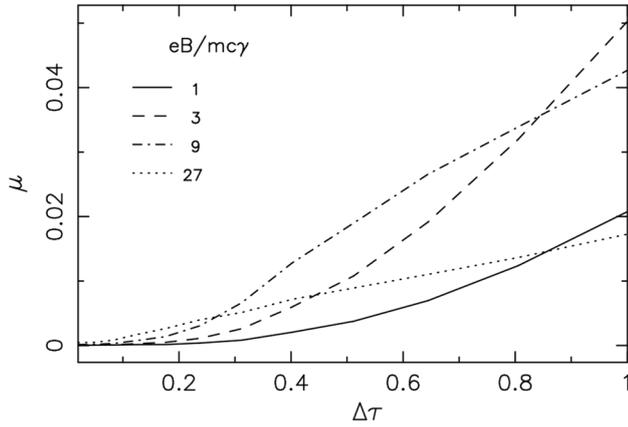


Figure 2. Plot of dimensionless steady-state orbital magnetic moment μ against the pulse-width $\Delta\tau$ for some chosen values of the magnetic field (measured in terms of ω_c). This is for the case of harmonic (soft) confinement. For details, see the text.

$$R\ddot{\theta} = -2\dot{R}\dot{\theta} - R\ddot{\theta} + \omega_c\dot{R} + f_\theta(\tau, \Delta\tau), \quad (2b)$$

where $F_r(t, \Delta t) \equiv f_r(\tau, \Delta\tau)$ and $F_\theta(t, \Delta t) \equiv f_\theta(\tau, \Delta\tau)$. Here the overhead dot denotes derivative with respect to the dimensionless time τ . Also, $f_r(\tau, \Delta\tau)$ and $f_\theta(\tau, \Delta\tau)$ are uncorrelated Gaussian variates with mean zero and variance unity.

The steady-state orbital magnetic moment M can now be written as

$$M = \langle\langle -\frac{e}{2c}(\mathbf{r} \times \dot{\mathbf{r}}) \rangle\rangle = -\left(\frac{e\gamma\sigma^2}{2c}\right) \langle\langle R^2\dot{\theta} \rangle\rangle \quad (3)$$

giving the dimensionless orbital magnetic moment

$$\mu = \frac{M}{\left(\frac{e\gamma\sigma^2}{2c}\right)} = -\langle\langle R^2\dot{\theta} \rangle\rangle. \quad (4)$$

Here the double angle bracket denotes averaging over an ensemble of realizations of the random noise as well as over time in the long-time limit. We identify μ with the steady-state value of the dimensionless orbital magnetic moment.

In Fig. 2, we have plotted the dimensionless magnetic moment μ against the pulse-width $\Delta\tau$, for some chosen values of the magnetic field, measured in terms of the dimensionless cyclotron frequency ω_c . The parameters ω_0 and σ have been set equal to unity.

In Fig. 3, we have repeated, for the sake of comparison, the plot in Fig. 2, but now for the case of a hard confinement. The latter is realized by modifying the soft, harmonic restoring force term $-kr$ in the radial equation (1a) to $-kr(r/a)^n$, and choosing a large integer exponent n . The length parameter a here denotes the radial position of the effectively reflecting wall.

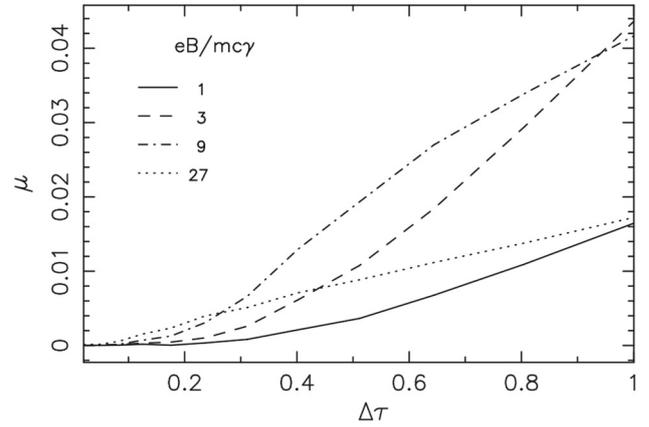


Figure 3. Plot of dimensionless steady-state orbital magnetic moment μ against the pulse-width $\Delta\tau$ for some chosen values of the magnetic field (measured in terms of $\omega_c = eB/mc\gamma$). This is for the case of hard confinement of nearly reflecting boundary as described in the text.

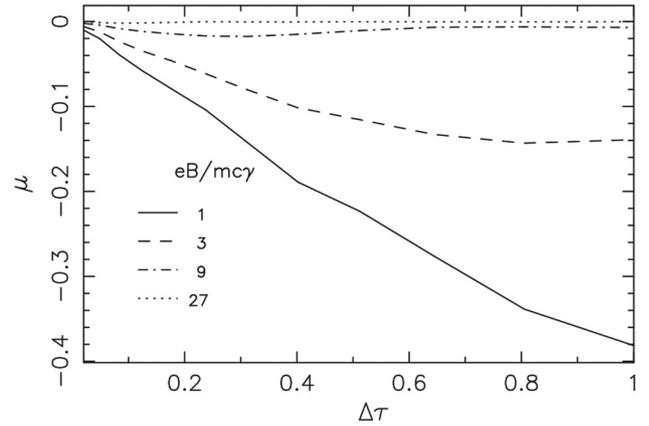


Figure 4. Plot of dimensionless steady-state orbital magnetic moment μ against the pulse-width $\Delta\tau$ for the same chosen values of parameters as in Figures 2 and 3, except now for $\omega_0 = 0$ (no confinement).

With this, the term $-\omega_0^2 R$ in equation (2a) becomes $-\omega_0^2 R(R/A)^n$, with $A = a/\sigma$. In Fig. 3, we have chosen $n = 20$ and $A = 1$. (Note that the limit $n = 0$ corresponds to the soft (harmonic) confining potential.)

Finally, for completeness, in Fig. 4 we have repeated the plot (of μ against $\Delta\tau$) for the case of no confining potential, i.e., for $\omega_0 = 0$, with the other parameters remaining the same.

3. Discussion

Through our numerical simulation of a classical model for the stochastic-dissipative dynamics of a charged particle, moving in a magnetic field and driven by a non-Markovian noise, we have demonstrated the appearance

of an induced orbital magnetic moment in the steady state. Most significantly, the orbital moment turns out to be *paramagnetic*! The appearance of a non-zero classical orbital magnetic moment in the non-equilibrium steady state implies deviation of the dynamical system from the second fluctuation-dissipation (II-FD) relation. The latter would have otherwise enforced a detailed-balance (meaning no cycles), and hence no induced orbital magnetic moment. The moment, of course, vanishes in the limit of delta-correlated white noise (i.e., $\Delta\tau \rightarrow 0, \sigma^2 \rightarrow \infty$, with the product $\Delta\tau\sigma^2$ finite) that renders the system II-FD theorem compliant. The essential point to be emphasized here is the non-Markovian nature of the driving noise used in the stochastic model. This point is consistent with the proven result of [Prost et al. \(2009\)](#) that for a Markovian dynamics, the non-equilibrium steady state can always be transformed into an effectively equilibrium state. It may be noted that our results hold for a soft (harmonic) potential confinement as well as for a hard (reflecting wall type) potential confinement. Also, the magnitude of the orbital paramagnetic moment is non-monotonic in the externally applied magnetic-field strength – initially increasing with increasing magnetic field strength and then decreasing in the high-field regime. Further, the magnetic moment scales as the variance of the random pulse-height for a given pulse width in the case of harmonic confinement (see equation (3)). In the case of the hard confinement, however, there is no simple scaling, but the qualitative behaviour remains the same.

For the sake of completeness, we also carried out a similar simulation for the case of no potential-confinement, i.e., with $\omega_0 = 0$. Interestingly, but not surprisingly though, we obtained as in Fig. 4 a steady-state orbital magnetic moment which has the opposite sign – it is diamagnetic! This is, however, consistent with the physical picture of Bohr ([Bohr 1911](#); [van Leeuwen 1921](#); [van Vleck 1932](#); [Peierls 1979](#); [Ma 1985](#)): without the confinement, there are naturally no orbits skipping the boundary, i.e., no edge current, which would have sub-tended a paramagnetic moment leading to the cancellation. We are thus left only with the Maxwell cycles well within the interior that constitute the amperian current loops giving the diamagnetic moment (the Lenz’s law). This avoided cancellation was seen also in the analytical solution for the simple case of a Markovian (delta-correlated) noise ([Jayannavar and Kumar 1981](#)). This now turns out to be true even for the non-Markovian case as shown in our simulation. The overall picture is one in which the particle, initially at the origin, say, diffuses outwards on the average, but the orbital diamagnetic moment reaches its steady-state

value on a relatively short time scale which is determined by the parameters appearing in equations (1a) and (1b). The outward diffusion simply spreads out the total orbital magnetic moment ($\propto \mathbf{r} \times \dot{\mathbf{r}}$) over the ever increasing area covered, but without changing its time-averaged value – the steady-state diamagnetic moment.

4. Concluding remarks

The paramagnetic sign of the induced orbital magnetic moment can have interesting consequences of considerable physical significance. After all, the paramagnetic sign of the moment inherently signifies a positive feedback effect when we consider not just one but a system of many charged particles. Here, the mean self-field can, in principle, lead to a spontaneous macroscopic orbital magnetic moment.

As for a possible experimental realization of such a confined system, we begin by noting that what is really essential for obtaining the classical orbital paramagnetism in a non-equilibrium steady state is the non-Markovian nature of the stochastic forcing. Thus, a micron-sized sample of a semi-metal (such as bismuth) trapped in an optical tweezer, and irradiated with random laser pulses in the presence of an external magnetic field would constitute a possible candidate system. The laser impulses should impart high enough kinetic energy (high nominal temperature) so as to create a non-degenerate (classical) gas of charged particles (electrons and holes). Then, the high temperature washes away the quantum signature, i.e., the discrete quantum level-spacings in the micron-sized sample, leaving behind a classical charged particle system.

Recall that the orbital magnetic moment does not depend on the sign of the charge on the particle, e.g., be it an electron or a hole. It is then reasonable to expect the total induced orbital magnetic moment to scale up with the number of charged particles in the confined system.

We would like to summarize now the main findings of our work as reported above. In the process, we hope to clarify some of the key points made therein. The classical (and indeed the classic) Bohr–van Leeuwen (BvL) theorem implies a complete absence of the equilibrium orbital magnetic moment inasmuch as the partition function then turns out to be independent of the magnetic *vector potential*. This global result has received an exact microscopic treatment based on the classical Langevin equation for the stochastic thermal motion of a charged (spinless) particle, the electron, in a static

magnetic field, in the presence of friction (dissipation) and the random forcing (fluctuation) that satisfies the celebrated fluctuation-dissipation (F-D) theorem in thermal equilibrium. It turned out, however, that a deviation from the equilibrium F-D theorem did induce a *non-zero orbital magnetic moment in the system* (Kumar 2012). Moreover, it could be shown that the sign of the orbital magnetic moment (paramagnetic or diamagnetic) depends on the relative strengths of fluctuation and the dissipation effects that could be readily parametrized rather simply. Finally, these stochastic treatments based on the underlying Markovian noise is replaced here by the non-Markovian Kubo–Anderson process, providing finite correlation time-scale $\Delta\tau$ for noise driving without any memory from earlier states, and solved for now numerically.

The present results, owing to the details of system and conditions considered here being different, do not render ready comparison with the earlier findings across the different parameters probed in Kumar (2012). A limited comparison, however, is possible in the dependence of the steady-state orbital magnetic moment on the magnitude of the magnetic field strength. More specifically, as can be viewed from the set of curves in Figures 2 and 3, the orbital magnetic moment at any given $\Delta\tau$ first increases with magnetic field, and then reduces with further increase in the magnetic field. This behaviour is qualitatively consistent with the trend apparent in Fig. 1 of Kumar (2012) for the paramagnetic signature (i.e., when $\eta = 0.5$). The Kubo–Anderson process (where the transition rates are independent of original state the transition came from) is admittedly more realistic (physical) than the usual Gaussian processes.

It is our conjecture that this non-zero classical orbital paramagnetism of the non-equilibrium steady-state, driven by the Kubo–Anderson stochastic noise, should hold true in general across a wider range of situations. If indeed it turns out to be the case, it would be a surprise of theoretical physics akin to the Bohr–van Leeuwen (BvL) theorem for the equilibrium quantum mechanics! This mechanism may even generate spontaneously

a macroscopic magnetic mean field — such as the *seed* field of interest in the astrophysical context.

One of the issues central to the compact stars (related astrophysics) is the origin of magnetic effects (magnetic fields) associated with the stellar objects¹, in particular, their electron-gas effects due to the *motion* of the charged particles (electrons). This *orbital* (not the spin) effect is precisely what has been addressed in our work! The idea underlying our work is that a deviation from the F-D (fluctuation-dissipation (theorem)) naturally leads to a magnetic effect (susceptibility) that can indeed be paramagnetic in a *confined* system of charged particles (electrons), without involving their spin. It ultimately depends on which of the two — the fluctuations or the dissipation — dominates! In our model, this is controlled by the stochastic parameters, such as the finite correlation time scale of the random forcing (in comparison with the cyclotron period).

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¹Magnetic fields have indeed threaded Srinivasa's works on compact stars, particularly that elucidating the fine details of evolution of both and the remarkable interplay.