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Out-of-Plane Equilibrium Points in the Photogravitational CR3BP with Oblateness and P-R Drag

Jagadish Singh & Tajudeen Oluwafemi Amuda*
[1]*Department of Mathematics, Faculty of Science, Ahmadu Bello University, P.M.B. 2222 Sokoto Road, Samaru, Zaria-Nigeria.*
**e-mail: teejaymath@gmail.com*

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Abstract. This paper investigates the motion of a test particle around the out-of-plane equilibrium points in the circular photogravitational restricted three-body problem when the effect of radiation pressure from the smaller primary and its Poynting-Robertson (P-R) drag are taken into account, and the bigger primary is modeled as an oblate spheroid. These points lie in the xz-plane almost directly above and below the center of the oblate primary. The equilibrium points are sought, and we observe that, there are two coordinate points $L_{6,7}$ which depend on the oblateness of the bigger primary, and the radiation pressure force and P-R drag of the smaller primary. The positions and linear stability of the problem are investigated both analytically and numerically for the binary system Cen X-4. The out-of-plane equilibrium points are found to be unstable in the sense of Lyapunov due to the presence of a positive real root.

Key words. R3BP—P-R drag—oblateness—out-of-plane equilibrium points.

1. Introduction

The three-body problem (3BP) is one of the most important problems in celestial mechanics. It has many applications in scientific researches, especially in the field of astrophysics and astrodynamics. The three-body problem is concerned with the motion of three particles attracting each other according to the Newtonian law of gravitation and that are free to move in space. Researchers like Bruno (1994), Gutzwiller (1998), Valtonen & Karttunen (2006) and Chenciner (2007) investigated the complete solution of the general three-body problem. There are also various forms of three-body problems in general relativity (Fokker 1921; Nordvedt 1968; Renzetti 2012; Iorio 2014). The restricted three-body problem describes the motion of a particle of negligible mass in the vicinity of two massive bodies, called primaries, which move in circular orbits around their common center of mass, on account of their mutual gravitational attraction. Interesting and significant results have been produced by renown mathematicians and scientists attempting to understand and predict the motion of natural bodies.

In the classical restricted three-body problem, only gravitational forces come into play; it is the case here when at least one of the interacting bodies is an intense emitter of radiation. It is altogether inadequate to consider solely the gravitational force in certain solar or stellar dynamics problems. For example, when a star acts upon a particle in a cloud of gas and dust, the dominant factor is not gravity, but the repulsive force of the radiation pressure (Radzievsky 1950). Several studies have considered one or both primaries as a source of radiation, to mention a few, Hamilton & Burns (1992), Singh (2009), Pinto *et al.* (2011, 2014), Pitjeva & Pitjev (2012), Singh & Leke (2014), and Fienga *et al.* (2014).

In estimating the light radiation force, some studies of the photogravitational R3BP took into account, just one of the three components of the light pressure field which is due to the central force, namely, the gravitational and radiation pressure forces. The other two component forces arise from the Doppler shift and the absorption and subsequent re-emission of the incident radiation. These last two components constitute the so-called Poynting-Robertson (P-R) drag. The P-R drag causes small particles of the solar system to spiral into the Sun at a cosmically rapid rate. Poynting (1903) first pointed out this problem, where he considered the effect of absorption and subsequent re-emission of sunlight by small isolated particles in the solar system. His work was modified by Robertson (1937), in formulating the expression of the total radiation force using precise relativistic treatments of the first order in the ratio of the velocity of the particle to that of light. The solar radiation pressure force Fp is exactly opposite to the gravitational attraction force Fg and change with the distance by the same law, it is possible to consider that the result of the action of this force will lead to reducing the effective mass of the sun or particle. He also suggested that infinitesimal body in solar orbit suffers a gradual loss of angular momentum and ultimately spirals into the sun. It is acceptable to speak about a reduced mass of the particle, as the effect of reducing its mass depends on the properties of the particle itself.

The P-R effect is important in the study of stability of the zodiacal cloud, orbital evolution of cometary meteor streams, asteroidal particles and dust rings around planets. Given the importance of the problem, several authors (Chernikov 1970; Schuerman 1980; Murray 1994; Ragos & Zafiropoulos 1995; Kushvah *et al.* 2007a, c; Kushvah 2008; Kushvah 2009) have done their studies taking P-R effect into account. Recently, Singh & Amuda (2014) examined the Poynting-Robertson drag and oblateness effects on the motion of a test particle around triangular equilibrium points, when the smaller primary is a radiation source. They noted that the motion around triangular equilibrium points is particularly unstable.

A modified study of the restricted three-body problem is the case when one or both primaries are considered as an oblate spheroid. Studies of this type, among many, are Sharma & SubbaRao (1967), Sharma (1987), AbdulRaheem & Singh (2006), Singh (2009), Singh & Mohammed (2012), Singh & Leke (2014) and Narayan *et al*. (2015). Regarding the oblateness in the astrophysical objects, taking a range of situations and scales, we may cite Anand (1965), Hartle & James (1967), Arutyunyan *et al.* (1971), Soffel *et al*. (1988), Damiani *et al*. (2011), Iorio (2007a, b, 2011, 2013a, b), Boshkayev *et al*. (2013), Rozelot & Fazel (2013) and Renzetti (2013, 2014).

The study of the three-dimensional case of the R3BP has been the focus recently. Assuming the bigger primary is an oblate spheroid whose equatorial plane coincides with the plane of motion, Sharma & SubbaRao (1967) investigated the triangular equilibrium points and showed that the oblateness of the primaries resulted in an increase in both the Coriolis and the centrifugal forces. They further established that the range of linear stability of the triangular points decreases, thereby concluding that the Coriolis force is not always a stabilizing force. Using a similar form of their equations of motion, Oberti & Vienne (2003), Douskos & Markellos (2006) and Shankaran *et al*. (2011) examined the out-of-plane points and their stability under different assumptions. Das *et al*. (2009) studied the out-of-plane equilibrium points L_i $(i = 6, 7, 8, 9)$ of passive micron size particle and their stability in the field of radiating-binary stellar systems, taking into account the P-R drag. They proved that the points $L_{6,7}$ are stable in the absence of P-R drag and binary systems like (Kruger 60 and RW-Monocerotis), while they are all unstable in the presence of P-R drag. Singh (2012) examined the out-of-plane equilibrium points $L_{6,7}$ by considering the effect of a small change in Coriolis and centrifugal forces, when the primaries are both radiating and oblate spheroids. Singh & Umar (2013) followed up immediately by studying these equilibrium points in the elliptic restricted three-body problem with radiating and oblate primaries, and apply their study to Gamma Leporis and Altair. Finally, Ershkov (2012) investigated the Yarkovsky effect in a modified photogravitational three-bodies and proved in addition to the well-known nine out-of-plane points the existence of 256 out-of-plane equilibrium points, while Xuetang & Lizhong (1993) examined the nine known out-of-plane points.

In considering the bigger primary as an oblate spheroid and the smaller one as a source of radiation, a different version of the problem came into the picture (Singh & Umar 2012). In this context, they have considered close binary stars in which the bigger primary is a stellar remnant (a neutron star, a white dwarf, a black dwarf) with the smaller one a low-mass stellar companion. A white dwarf, also called a degenerate dwarf, is a highly evolved stellar remnant of a giant star that has blown away its outer layers as planetary nebulae. Such types of binaries are not scarce, as stars of different masses have different life spans (Langer *et al*. 2002). Examples of these are low-mass X-ray binaries (LMXBs), pre-LMXBs, and soft X-ray transient. These are the best natural counterparts in the system under study, which provides test beds for this astrophysical problem (Jonker *et al*. 2007; Shahbaz *et al*. 2014). This constitutes an excellent model for the research under investigation. We study the stability of the system, Cen X-4 in the present problem.

In this paper, our aim is to study the existence and stability of the out-of-plane equilibrium points when the bigger primary is an oblate spheroid and the smaller one, an intense emitter of radiation, together with its P-R drag. The paper is organized as follows. The equations of motion are presented in Section 2. The positions of out-of-plane equilibrium points are discussed in Section 3, while Section 4 examines its linear stability, and Section 5 shows the numerical application. Finally, Section 6 gives the discussion and conclusion on the findings obtained.

2. Equations of motion

We consider the barycentric rotating co-ordinate system $0xyz$ relative to an inertial system with angular velocity n and common z-axis. We take a line joining the primaries as the x-axis. Let m_1, m_2 be the masses of the bigger and the smaller primaries (Fig. 1), respectively. Here r_1 and r_2 are the distances of the third body from the primaries. Let (x, y, z) be the coordinates of an infinitesimal mass m. We take

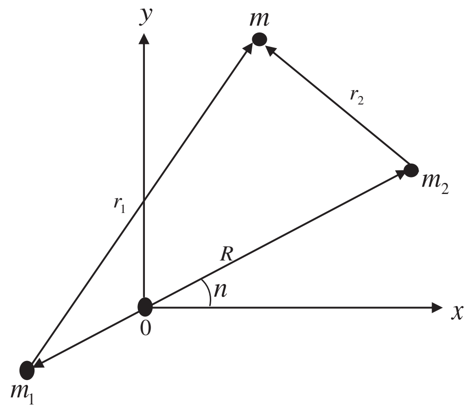

Figure 1. Bigger primary m_1 and smaller primary m_2 moving in circular orbits about their common center of mass unperturbed by the infinitesimal mass m in the CR3BP.

units such that the sum of the masses and the distance between primaries is unity, the unit of time, i.e., time period of m_2 about m_1 consists of 2π units such that the Gaussian constant of gravitational $\gamma = 1$. Then, perturbed mean motion n of the primaries is given by $n^2 = 1 + (3A/2)$, where $A = (r_e^2 - r_\rho^2)/5R^2$ is the oblateness coefficient of m_1 and is such that $0 < A << 1$. Let r_e, r_p be the equatorial and polar radii of m_1, respectively. Let $\mu = m_2/(m_1 + m_2)$ be the mass parameter, then $1 - \mu = m_1/(m_1 + m_2)$ with $m_1 > m_2$. We have considered the dimensionless velocity of light as c_d, which depends on the physical masses of the two primaries and the distance between them. W_2 is the parameter representing the P-R drag. The dots denote differentiation with respect to time t.

The equations of motion of a passively gravitating test body of infinitesimal mass such as meteoroids or small asteroids (about 10 cm to 10 km in diameter; Ershkov 2012), in the CR3BP under the effects of oblateness of the bigger primary, radiation and P-R drag of the smaller primary, have the following form:

$$\vec{a} + 2\vec{w} \times \vec{v} + \vec{w} \times (\vec{w} \times \vec{r}) = -\frac{(1-\mu)\vec{r}_1}{r_1^3} - \frac{3(1-\mu)A\vec{r}_1}{2r_1^5} + \frac{15\,(1-\mu)A\,z^2\vec{r}_1}{2r_1^7} - \frac{\mu\,\vec{r}_2}{r_2^3} + \frac{\mu(1-q)}{r_2^2}\left[\frac{\vec{r}_2}{r_2} - \left\{\frac{(\dot{\vec{r}}_2 + w \times \vec{r}_2).\,\vec{r}_2}{c_d r_2}\right\}\frac{\vec{r}_2}{r_2} - \frac{(\dot{\vec{r}}_2 + w \times \vec{r}_2)}{c_d}\right], \quad (1)$$

where q is the factor characterizing the radiation effects, such that $0 < 1 - q << 1$ (Radzievsky 1950).

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \quad \vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}, \quad \vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}, \quad \vec{w} = n\vec{k},$$
$$\vec{r}_1 = (x+\mu)\vec{i} + y\vec{j} + z\vec{k}, \quad \vec{r}_2 = (x+\mu-1)\vec{i} + y\vec{j} + z\vec{k},$$
$$r_1^2 = (x+\mu)^2 + y^2 + z^2, \quad r_2^2 = (x+\mu-1)^2 + y^2 + z^2,$$
$$\vec{w} \times \vec{v} = -n(\dot{y}\vec{i} - \dot{x}\vec{j}), \quad \vec{w} \times (\vec{w} \times \vec{r}) = -n^2(x\vec{i} + y\vec{j}),$$
$$\dot{\vec{r}}_2 + \vec{w} \times \vec{r}_2 = (\dot{x} - ny)\vec{i} + [\dot{y} + n(x+\mu-1)]\vec{j} + \dot{z}\vec{k},$$
$$(\dot{\vec{r}}_2 + \vec{w} \times \vec{r}_2).\vec{r}_2 = [(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}].$$

Simplifying equation (1) and equating the coefficients of $\vec{i}$, $\vec{j}$ and $\vec{k}$ on both sides, we obtain the system of equations:

$$\begin{aligned}\ddot{x} - 2n\dot{y} &= \Omega_x,\\ \ddot{y} + 2n\dot{x} &= \Omega_y,\\ \ddot{z} &= \Omega_z,\end{aligned} \tag{2}$$

where

$$\Omega_x = n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} + \frac{15(1-\mu)A(x+\mu)z^2}{2r_1^7} - \frac{\mu q(x+\mu-1)}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{(x+\mu-1)}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{x} - ny\right], \tag{3}$$

$$\Omega_y = n^2 y - \frac{(1-\mu)y}{r_1^3} - \frac{3(1-\mu)Ay}{2r_1^5} + \frac{15(1-\mu)Ayz^2}{2r_1^7} - \frac{\mu q y}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{y}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{y} + n(x+\mu-1)\right], \tag{4}$$

$$4\Omega_z = -\frac{(1-\mu)z}{r_1^3} - \frac{9(1-\mu)Az}{2r_1^5} + \frac{15(1-\mu)Az^3}{2r_1^7} - \frac{\mu q z}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{z}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{z}\right], \tag{5}$$

$$r_1^2 = (x+\mu)^2 + y^2 + z^2, \quad r_2^2 = (x+\mu-1)^2 + y^2 + z^2,$$

$$W_2 = \frac{\mu(1-q)}{c_d}, \tag{6}$$

$$n^2 = 1 + \frac{3A}{2}\,. \tag{7}$$

The equations of motion are affected by the shape of the bigger primary; radiation pressure and P-R drag of the smaller primary. Next, we shall discuss the positions of out-of-plane equilibrium points of the test body under these conditions.

3. Position of out-of-plane equilibrium points

The position of the out-of-plane equilibrium points are the solutions of the equations (3), (4) and (5) with $\dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = 0$, that is, they are the solutions of the equations

$$x\left[n^2 - \frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)A}{2r_1^5} + \frac{15(1-\mu)Az^2}{2r_1^7} - \frac{\mu q}{r_2^3}\right] + \frac{\mu q}{r_2^3} + \frac{W_2 n y}{r_2^2} + \mu\left[-\frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)A}{2r_1^5} + \frac{15(1-\mu)Az^2}{2r_1^7} - \frac{\mu q}{r_2^3}\right] = 0\,, \tag{8}$$

$$y\left[n^2-\frac{(1-\mu)}{r_1^3}-\frac{3(1-\mu)A}{2r_1^5}+\frac{15(1-\mu)Az^2}{2r_1^7}-\frac{\mu q}{r_2^3}\right]-\frac{W_2n(x+\mu-1)}{r_2^2}=0\,, \tag{9}$$

$$z\left[-\frac{(1-\mu)}{r_1^3}-\frac{3(1-\mu)A}{r_1^5}-\frac{3(1-\mu)A}{2r_1^5}+\frac{15(1-\mu)Az^2}{2r_1^7}-\frac{\mu q}{r_2^3}\right]=0\,. \tag{10}$$

If $z \neq 0$, equations (8), (9) and (10) are expressed, respectively, as

$$n^2x+\frac{3(1-\mu)Ax}{r_1^5}+\frac{3\mu(1-\mu)A}{r_1^5}+\frac{\mu q}{r_2^3}+\frac{W_2ny}{r_2^2}=0\,, \tag{11}$$

$$n^2y+\frac{3(1-\mu)Ay}{r_1^5}-\frac{W_2n(x+\mu-1)}{r_2^2}=0\,, \tag{12}$$

$$\frac{1-\mu}{r_1^3}+\frac{9(1-\mu)A}{2r_1^5}-\frac{15(1-\mu)Az^2}{2r_1^7}+\frac{\mu q}{r_2^3}=0\,. \tag{13}$$

From equation (12), on simplifying and neglecting the product of A and W_2, we get

$$y=\frac{W_2(x+\mu-1)}{r_2^2}. \tag{14}$$

Now, we substitute equation (14) in (11), to get

$$x_0=-\frac{\mu q}{r_{2(0)}^3}-\frac{3\mu(1-\mu)A}{r_{1(0)}^5}+\frac{3\mu qA}{2r_{2(0)}^3}+\frac{3\mu q(1-\mu)A}{r_{1(0)}^5r_{2(0)}^3}. \tag{15}$$

Substituting (15) in (14), we at once have

$$y_0=\frac{W_2(x_0+\mu-1)}{r_{2(0)}^2}. \tag{16}$$

Also, the use of equations (15) and (16) in (13), gives

$$z_0^2=\frac{3r_{1(0)}^2}{5}+\frac{2r_{1(0)}^4}{15A}+\frac{2\mu qr_{1(0)}^7}{15(1-\mu)r_{2(0)}^3A}. \tag{17}$$

Since the out-of-plane points lies in the $xz-$plane, therefore we proceed by using $x_0=1-\mu,\quad y_0=0,\ 0.1,\ 0.2, z_0=\sqrt{3A}$, as initial approximations for the solutions. The variables $r_{1(0)}, r_{2(0)}$ represent the distance of the equilibrium points from the primaries. Using the software package 'Mathematica', we can determine the position of points $(x_0, y_0, \pm z_0)$ denoted by L_6 and L_7, which we express in power series form to second order terms in A from equations (15), (16) and (17) in the following cases arising from the values of y_0.

Case 1.

$$x_0=-\frac{q\mu}{3\sqrt{3}A^{\frac{3}{2}}}\,,\ z_0\ =\infty,\quad \text{when } y_0=0; \tag{18}$$

Case 2.

$$x_0 = \mu[-1000q + A\{-2.92629 + q(454426 - 2926.29\mu) + 2.92629\mu\} \\ A^2\{21.7299\,(1-\mu) + q(-1.70764\times 10^8 + 1.33856\times 10^6\mu)\}]\ ,$$

$$z_0 = 1.014 + 60691.1q\mu(\mu-1) + \frac{0.00134667 + q\,(138.059 - 138.059\mu)\mu}{A}, \quad (19)$$

when $y_0 = 0.1$;

Case 3.

$$x_0 = \mu[-125q - A\{2.71981\,(1-\mu) - q(14590 - 339.976\mu)\} \\ A^2\{19.614\,(1-\mu) + q(-1.38015\times 10^6 + 40699\mu)\}]\ ,$$

$$z_0 = 1.056 + q\mu(-1957.86 + 1957.86\mu) + \frac{0.00554667 + 19.119\,q(1-\mu)\mu}{A}, \quad (20)$$

when $y_0 = 0.2$.

It is obvious from equations (19) and (20) that these points lie on the xz-plane for fixed values of y_0 and are denoted by $L_{6,7}$. Their positions are symmetrical with respect to the orbital plane and belong to the family of out-of-plane equilibrium points. These points in a simple manner depend on the mass ratio, oblateness of the bigger primary; radiation pressure force and P-R drag of the smaller primary. We observe that when $z = 0$, $q = 1$, $A = 0$ and $W_2 = 0$ in equations (8), (9) and (10), one recovers the classical case which admits the existence of three collinear equilibrium points and two triangular points.

The positions are shown numerically in Tables 1–3 for various assumed values of oblateness of the bigger primary for the binary system Cen X-4. Figures 2, 3 and 4 show the positions of out-of-plane equilibrium points.

Evidently, Tables 1, 2 and 3 represent cases when we set $y_0 = 0.1$, 0.2 and 0.3 of varying oblateness of the bigger primary for the binary system Cen X-4. The numerical computations of the coordinates when $y_0 = 0$ do not suffice, because the solutions turn out to be infinite on the z-axis. This perhaps occurs because, for y_0 to yield zero, either $W_2 = 0$ or $z \to \infty$. In this case, we get $x = -0.003313798$ which reduces to a collinear equilibrium point, since it lies on the line joining the primaries. We note that when $y_0 < 0.1$, these points do not appear realistic. For instance, when

Table 1. Effect of oblateness on the positions of out-of-plane equilibrium points for $y_0 = 0.1$.

A	x_0	$\pm z_0$
0.0022	−31.6200	8.80229
0.0020	−29.6107	17.5585
0.0019	−28.8019	21.0223
0.0018	−28.1237	24.2985
0.0017	−27.5761	27.5022
0.0016	−27.1591	30.7093
0.0015	−26.8726	33.9802
0.0014	−26.7168	37.3693
0.0012	−26.7968	44.7325
0.0010	−27.3991	53.3605
0.005	−114.593	0.+34.8586i
0.010	−517.389	0.+41.5082i

Table 2. Effect of oblateness on the positions of out-of-plane equilibrium points for $y_0 = 0.2$.

A	x_0	$\pm z_0$
0.0022	−3.80958	15.8477
0.0020	−3.87677	16.8333
0.0019	−3.91195	17.3783
0.0018	−3.94818	17.9646
0.0017	−3.98547	18.5979
0.0016	−4.02381	19.2856
0.0015	−4.06320	20.0364
0.0014	−4.10365	20.8615
0.0012	−4.18772	22.7938
0.0010	−4.27600	25.2519
0.005	−3.31177	8.41311
0.010	−4.47906	$0.+0.281765i$

Table 3. Effect of oblateness on the positions of out-of-plane equilibrium points for $y_0 = 0.3$.

A	x_0	$\pm z_0$
0.0022	−1.264660	10.4289
0.0020	−1.277160	10.9780
0.0019	−1.283510	11.2837
0.0018	−1.289930	11.6139
0.0017	−1.296410	11.9722
0.0016	−1.302950	12.3629
0.0015	−1.309560	12.7912
0.0014	−1.316240	13.2639
0.0012	−1.329780	14.3776
0.0010	−1.343580	15.8056
0.005	−1.116830	6.55250
0.010	−0.979358	4.13221

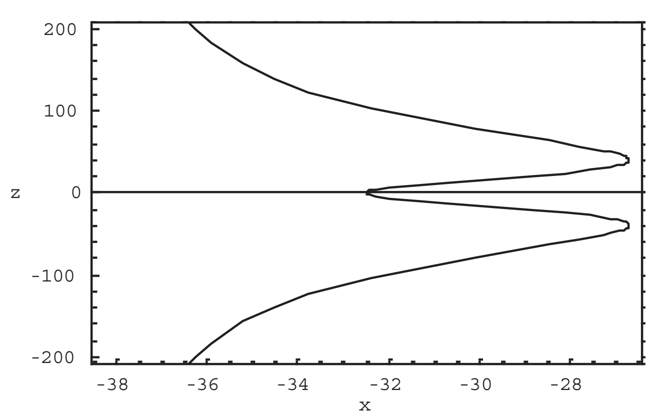

Figure 2. Positions of $L_{6,7}$ for Cen X-4 as a function of A, for $q = 0.993$ and $y_0 = 0.1$.

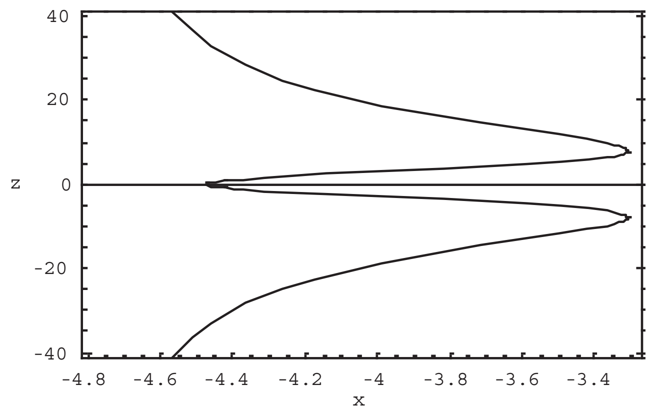

Figure 3. Positions of $L_{6,7}$ for Cen X-4 as a function of A, for $q = 0.993$ and $y_0 = 0.2$.

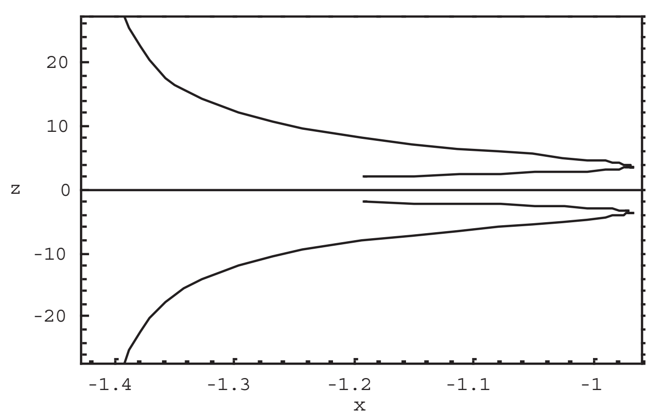

Figure 4. Positions of $L_{6,7}$ for Cen X-4 as a function of A, for $q = 0.993$ and $y_0 = 0.3$.

$y_0 = 0.01$, we get $x = -1.13547 \times 10^7$ and z is imaginary. We have observed that for higher values of oblateness, when $y_0 = 0.1$; 0.2, the out-of-plane equilibrium points cease to exist.

4. Stability of the out-of-plane equilibrium points

We now examine the stability of these equilibrium configurations. By this, we mean, its ability to restrain the body motion in its vicinity. To do this, we displace the third body a little from the equilibrium point by applying a small displacement. If its motion is a rapid departure from the vicinity of the point, we call such a position an unstable equilibrium point. However, if the body merely oscillates about the point, then, it is stable.

Now, let the location of any equilibrium point be denoted by $(x_0, y_0, z_0)$and suppose (ξ, η, ζ) is a small displacement. Then, the body will be displaced to the points

$x = x_0 + \xi$, $y = y_0 + \eta$ and $z = z_0 + \zeta$. Substituting these in equations (2), we obtain the variational equations:

$$\begin{aligned}
\ddot{\xi} - 2n\dot{\eta} &= (\Omega_{xx})^0\xi + (\Omega_{xy})^0\eta + (\Omega_{xz})^0\zeta + (\Omega_{x\dot{x}})^0\dot{\xi} + (\Omega_{x\dot{y}})^0\dot{\eta} + (\Omega_{x\dot{z}})^0\dot{\zeta},\\
\ddot{\eta} + 2n\dot{\xi} &= (\Omega_{yx})^0\xi + (\Omega_{yy})^0\eta + (\Omega_{yz})^0\zeta + (\Omega_{y\dot{x}})^0\dot{\xi} + (\Omega_{y\dot{y}})^0\dot{\eta} + (\Omega_{y\dot{z}})^0\dot{\zeta},\\
\ddot{\zeta} &= (\Omega_{zx})^0\xi + (\Omega_{zy})^0\eta + (\Omega_{zz})^0\zeta + (\Omega_{z\dot{x}})^0\dot{\xi} + (\Omega_{z\dot{y}})^0\dot{\eta} + (\Omega_{z\dot{z}})^0\dot{\zeta}.
\end{aligned} \tag{21}$$

Here, only linear terms in ξ, η and ζ were taken. The second partial derivatives of $\Omega = 0.45$ were denoted by subscripts. The superscript 0, indicates that the derivatives have been computed at the equilibrium points.

The determinant equation corresponding to equations (21) is

$$\begin{vmatrix}
\lambda^2 - \lambda\Omega^0_{x\dot{x}} - \Omega^0_{xx} & -2n\lambda - \lambda\Omega^0_{x\dot{y}} - \Omega^0_{xy} & -\lambda\Omega^0_{x\dot{z}} - \Omega^0_{xz}\\
2n\lambda - \lambda\Omega^0_{y\dot{x}} - \Omega^0_{yx} & \lambda^2 - \lambda\Omega^0_{y\dot{y}} - \Omega^0_{yy} & -\lambda\Omega^0_{y\dot{z}} - \Omega^0_{yz}\\
-\lambda\Omega^0_{z\dot{x}} - \Omega^0_{zx} & -\lambda\Omega^0_{z\dot{y}} - \Omega^0_{zy} & \lambda^2 - \lambda\Omega^0_{z\dot{z}} - \Omega^0_{zz}
\end{vmatrix} = 0.$$

The partial derivatives after ignoring products and higher order terms of very small parameters are

$$\begin{aligned}
\Omega^0_{xx} &= n^2 - \frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)(x+\mu)^2}{r_1^5} - \frac{3(1-\mu)A}{2r_1^5}\\
&\quad + \frac{15(1-\mu)(x+\mu)^2A}{2r_1^7} + \frac{15(1-\mu)Az^2}{2r_1^7}\\
&\quad - \frac{105(1-\mu)(x+\mu)^2Az}{2r_1^9} - \frac{\mu q}{r_2^3} + \frac{3\mu q(x+\mu-1)^2}{r_2^5},\\
\Omega^0_{xy} &= \frac{3(1-\mu)(x+\mu)y}{r_1^5} + \frac{3\mu q(x+\mu-1)y}{r_2^5} + \frac{n}{r_2^2}W_2,\\
\Omega^0_{xz} &= \frac{3(1-\mu)(x+\mu)z}{r_1^5} + \frac{45(1-\mu)(x+\mu)zA}{2r_1^7}\\
&\quad - \frac{105(1-\mu)(x+\mu)z^3A}{2r_1^9} + \frac{3\mu q(x+\mu-1)z}{r_2^5},\\
\Omega^0_{yx} &= \frac{3(1-\mu)(x+\mu)y}{r_1^5} + \frac{3\mu q(x+\mu-1)y}{r_2^5} - \frac{n}{r_2^2}\left[1 + \frac{2(x+\mu-1)^2}{r_2^2}\right]W_2,\\
\Omega^0_{yy} &= n^2 - \frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)A}{2r_1^5} + \frac{15(1-\mu)Az^2}{2r_1^7} - \frac{\mu q}{r_2^3},\\
\Omega^0_{yz} &= \frac{3(1-\mu)yz}{r_1^5} + \frac{3\mu qyz}{r_2^5} + \frac{2n(x+\mu-1)z}{r_2^4}W_2,
\end{aligned} \tag{22}$$

$$\Omega^0_{zx} = \frac{3(1-\mu)(x+\mu)z}{r_1^5} + \frac{45(1-\mu)(x+\mu)zA}{2r_1^7} - \frac{105(1-\mu)(x+\mu)z^3A}{2r_1^9} + \frac{3\mu q(x+\mu-1)z}{r_2^5},$$

$$\Omega^0_{zy} = \frac{3(1-\mu)yz}{r_1^5} + \frac{3\mu q yz}{r_2^5},$$

$$\Omega^0_{zz} = -\frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)z^2}{r_1^5} - \frac{9(1-\mu)A}{2r_1^5} + \frac{45(1-\mu)z^2A}{r_1^7} - \frac{105(1-\mu)Az^4}{2r_1^7} - \frac{\mu q}{r_2^3} + \frac{3\mu q z^2}{r_2^5},$$

$$\Omega^0_{x\dot{x}} = -\frac{1}{r_2^2}\left[1+\frac{(x+\mu-1)^2}{r_2^2}\right]W_2, \quad \Omega^0_{x\dot{y}} = 0, \quad \Omega^0_{x\dot{z}} = -\frac{(x+\mu-1)z}{r_2^4}W_2,$$

$$\Omega^0_{y\dot{x}} = 0, \quad \Omega^0_{y\dot{y}} = -\frac{1}{r_2^2}W_2, \quad \Omega^0_{y\dot{z}} = 0,$$

$$\Omega^0_{z\dot{x}} = -\frac{(x+\mu-1)z}{r_2^4}W_2, \quad \Omega^0_{z\dot{y}} = 0, \quad \Omega^0_{z\dot{z}} = -\frac{1}{r_2^2}\left[1+\frac{z^2}{r_2^2}\right]W_2.$$

Substituting equations (22) in the determinant equation, and simplifying, we get the characteristic equation:

$$\lambda^6 + p\lambda^5 + b\lambda^4 + Q\lambda^3 + d\lambda^2 + e\lambda + f = 0, \tag{23}$$

where p, b, Q, d, e and f are given numerically.

5. Numerical application

The locations and stability of out-of-plane points for the binary system Cen X-4, when the bigger primary is considered to be an oblate spheroid and the smaller one is a source of radiation having its P-R drag are investigated numerically. In table 4, we compute the dimensionless velocity of light for the binary system Cen X-4, using $c_d = c/\sqrt{\frac{\gamma(m_1+m_2)}{a}}$ (Ragos *et al.* 2001), where a is the binary separation (Shahbaz *et al.* 2014), the masses of the primaries and radiation pressure (Singh & Umar 2012) are obtained from literature.

Using the software package *Mathematica*, the coefficients and the six characteristic roots of equation (23) have been computed and presented in Tables 5 and 6 for the binary Cen X-4. We observe that, for the chosen values of the free parameters, there exists negative real part and positive real part of the complex roots. Therefore, the case where all the six roots are purely imaginary quantities or complex numbers with negative real parts do not arise and therefore, the out-of-plane equilibrium points are unstable in the sense of Lyapunov. Our result agrees with previous assertions of

Douskos & Markellos (2006), Shankaran *et al*. (2011), Singh & Umar (2012), and Singh (2012).

6. Discussion and conclusion

The existence and stability of the out-of-plane equilibrium points under the joint effect of oblateness of the bigger primary, radiation pressure of the smaller primary and its P-R drag have been studied. In our model, the three-dimensional equations of motion are affected by the radiation and P-R drag of the smaller primary and oblateness of the bigger primary.

When $z \neq 0$, equations (15), (16) and (17) determine the positions of the out-of-plane equilibrium points of the infinitesimal body. Equations (18), (19) and (20),

Table 4. Numerical data for Cen X-4 binary system.

Binary	Mass (M_0) M_1	Mass (M_0) M_2	Radiation pressure q	Binary separation (R_0) a	Dimensionless velocity of light c_d
Cen X-4	1.9996	0.0801	0.993	4.31	988.323

Table 5. The coefficient of the characteristic equation (23), for varying oblateness of the bigger primary of the binary system Cen X-4.

A	p	b	Q	d	e	f
0.0022	1.72927×10^{-11}	2.0066	2.60993×10^{-11}	8.86593×10^{8}	8.67331×10^{-12}	-1.19910×10^{-8}
0.0020	1.35858×10^{-11}	2.0060	2.04940×10^{-11}	1.30562×10^{9}	6.81229×10^{-12}	-8.38987×10^{-8}
0.0019	1.19603×10^{-11}	2.0057	1.80372×10^{-11}	1.60026×10^{9}	5.99640×10^{-12}	-6.94652×10^{-8}
0.0018	1.04742×10^{-11}	2.0054	1.57918×10^{-11}	1.97658×10^{9}	5.25061×10^{-12}	-5.70636×10^{-8}
0.0017	9.11887×10^{-12}	2.0051	1.37448×10^{-11}	2.46266×10^{9}	4.57059×10^{-12}	-4.64636×10^{-8}
0.0016	7.88638×10^{-12}	2.0048	1.18839×10^{-11}	3.09856×10^{9}	3.95230×10^{-12}	-3.74573×10^{-8}
0.0015	6.76920×10^{-12}	2.0045	1.01976×10^{-11}	3.94245×10^{9}	3.39195×10^{-12}	-2.98566×10^{-8}
0.0014	5.76033×10^{-12}	2.0042	8.67542×10^{-12}	5.08084×10^{9}	2.88602×10^{-12}	-2.34919×10^{-8}
0.0012	4.04181×10^{-12}	2.0036	6.08385×10^{-12}	8.84371×10^{9}	2.02444×10^{-12}	-1.38715×10^{-8}
0.0010	2.68338×10^{-12}	2.0030	4.03683×10^{-12}	1.67162×10^{10}	1.34366×10^{-12}	-7.53852×10^{-9}

Table 6. Roots of the characteristic equation (23), for varying oblateness of the bigger primary of the binary system Cen X-4.

A	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
0.0022	$-122.014+122.018i$	$\pm3.67761\times10^{-9}$	$122.014+122.018i$
0.0020	$-134.410+134.414i$	$\pm8.01621\times10^{-9}$	$134.410+134.414i$
0.0019	$-141.425+141.429i$	$\pm6.58853\times10^{-9}$	$141.425+141.429i$
0.0018	$-149.093+149.097i$	$\pm5.37307\times10^{-9}$	$149.093+149.097i$
0.0017	$-157.519+157.522i$	$\pm4.34364\times10^{-9}$	$157.519+157.522i$
0.0016	$-166.829+166.832i$	$\pm3.47687\times10^{-9}$	$166.829+166.832i$
0.0015	$-177.183+177.186i$	$\pm2.75193\times10^{-9}$	$177.183+177.186i$
0.0014	$-188.784+188.787i$	$\pm2.15026\times10^{-9}$	$188.784+188.787i$
0.0012	$-216.841+216.843i$	$\pm1.25240\times10^{-9}$	$216.841+216.843i$
0.0010	$-254.254+254.256i$	$\pm6.71544\times10^{-10}$	$254.254+254.256i$

respectively, locate the two points L_6 and L_7 forming directly above and below the oblate primary. It is remarkable that the radiating primary produces this shift towards itself, while the oblate one is toward others.

Different researches have considered the out-of-plane points from different perspectives.

In the elliptic problem:

Case 1: Singh & Umar (2012) investigated $L_{6,7}$ with oblateness of the bigger and radiation of the smaller primaries.

Case 2: Singh & Umar (2013) considered the problem when both primaries are sources of radiation and oblate spheroids.

In the circular problem:

Case 3: Douskos & Markellos (2006) considered first oblateness of the primaries and, then, with radiation of the bigger primary.

Case 4: Das *et al*. (2009) studied the out-of-plane equilibrium points L_i $(i = 6, 7, 8, 9)$ of passive micron size particle and their stability in the field of radiating-binary stellar systems taking into account the P-R drag.

Case 5: Shankaran *et al*. (2011) included the effects of an inverse square distance radiation pressure force on the infinitesimal mass due to the primaries, which are both radiating.

Case 6: Singh (2012) considered both primaries as luminous and oblate as well, together with effects of small perturbations in Coriolis and centrifugal forces.

In the absence of P-R drag in the present paper and with circular orbits, the results are in conformity with those of Singh & Umar (2012), and Singh (2012) when the small perturbations in Coriolis and centrifugal forces; radiation pressure of the primary and the oblateness of the secondary is ignored in the latter study. Cases 3, 5 and 6 have neglected P-R drag and Case 4 does not include oblateness. We have examined the out-of-plane points with the Poynting-Robertson (P-R) drag, occurring due to the radiation of the smaller primary, and contained in the expression of y_0 coordinate with oblateness of the bigger primary. In all these communications, however, the stability analysis remains the same, the out-of-plane points remain unstable. In our future work, we plan to treat these cases with general relativity.

In discussing the out-of-plane equilibrium points, we observed that equation (12) has a solution $y \neq 0$, so, we solve the remaining two equations (11) and (13) numerically (Tables 1–3), using power series, and taking $y_0 \in (0,\ 0.3]$. The numerical computations of the coordinates, when $y_0 = 0$ do not suffice, because the solution on the z-axis results in infinity, while for $y_0 < 0.1$, these points do not appear realistic. For $y_0 = 0.01$, we get $x = -1.13547 \times 10^7$ and z is imaginary. The effects of the various parameters involved are shown graphically in Figures 2–4.

The stability of the out-of-plane equilibrium points are completely determined by the roots of the characteristic equation (23). The stability of the system has been studied using numerical approach in order to express the partial derivatives clearly. Our numerical exploration in the computations of these roots reveals the existence of at least a positive root and/or a positive real part. Consequently, the motion is unbounded, and thus unstable.

References

AbdulRaheem, A., Singh, J. 2006, *Astron. J.*, **131**, 1880–1885.

Alain Chenciner 2007, *Scholarpedia*, **2(10)**, 2111.

Anand, S. P. S. 1965, *Proceedings of the National Academy of Sciences of the United States of America*, **54(1)**, 23–26.

Arutyunyan, G. G., Sedrakyan, D. M., Chubaryan, É. V. 1971, *Astrophysics*, **7(3)**, 274–280.

Boshkayev *et al.* 2013, *The Astrophys. J.*, **762(2)**, article id. 117, 14 pp.

Bruno, A. D. 1994, de Gruyter (1994).

Chernikov, Yu. A. 1970, *Sov. Astron. A. J*, **14**, 175–176.

Damiani, C. *et al.* 2011, *J. Atmos. Solar-Terrestrial Phys.*, **73**, 241.

Das, M. K. *et al.* 2009, *JAA*, **30**, 177–185.

Douskos, C. N., Markellos, V. V. 2006, *A&A*, **446**, 357–360.

Ershkov, S. V. 2012, *IJPAM*, **78(3)**.

Fienga, A. *et al.* 2014, arXiv:1409.4932 [astro-ph.EP].

Fokker, A. D. 1921, *Proceedings*, **23**, 729–738.

Gutzwiller, M. 1998, *Moon–Earth–Sun Reviews of Modern Physics*, **70(2)**.

Hamilton, D. P., Burns, J. A. 1992, *Icarus*, **96**, 43–64.

Hartle, James, B. 1967, *Astrophys. J.*, **150**, 1005.

Iorio, L. 2007a, *AP&SS*, **310(1-2)**, 73–76.

Iorio, L. 2007b, *AP&SS*, **312(3-4)**, 337–341.

Iorio, L. 2011, *Physical Review D*, **84(12)**, 124001.

Iorio, L. 2013a, *Classical and Quantum Gravity*, **30(19)**, article id 195011.

Iorio, L. *et al.* 2013b, *Acta Astronautica*, **91**, 141–148.

Iorio, L. 2014, *Frontiers in Astronomy and Space Sciences*, **1(3)**.

Jonker, P. G. *et al.* 2007, *The Astrophysical Journal*, **665**, 147.

Kushvah, B. S. 2008, *Ap&SS*, **315**, 231–241.

Kushvah, B.S. 2009, *Res. Astron. Astrophys.*, **9**, 1049–1060.

Kushvah, B. S, Sharma, J. P, Ishwar, B. 2007a, *Bulletin of the Astronomical Society of India*, **35**, 319–338.

Kushvah, B. S, Sharma, J. P., Ishwar, B 2007c, *Earth Moon and Planets*, **101**, 55–64.

Langer, N. *et al.* 2002, in: ASP Conf. Ser (2002) 252–261.

Murray, C. D. 1994, *Icarus (1994)*, **122**, 465–484.

Narayan, A. *et al.* 2015, *IJAA*, **3(1)**, 8–16.

Nordvedt, K. 1968, *Phys. Rev.*, **170(5)**, 1186.

Oberti, P., Vienne, A. 2003, *Astron. Astrophys.*, **397**, 353–359.

Pinto, R. F. *et al.* 2011, *Astrophys. J.*, **737**, 72.

Pitjeva, E. V., Pitjev, N. P. 2012, *Solar Syst. Res.*, **46**, 78.

Pitjeva, E. V., Pitjev, N. P. 2014, *Celest. Mech. Dyn. Astr.*, **119**, 237.

Poynting, J. H. 1903, *PTSLA*, **202**, 525–552.

Radzievsky, V. V. 1950, *Astron Zh*, **27**, 250.

Ragos, O. *et al.* 2001, *Nonlinear Analysis*, **47**, 3413–3418.

Ragos, O., Zafiropoulos, F. A. 1995, *A&A*, **300**, 568.

Renzetti, G. 2012, *Earth, Moon, and Planets*, **109(1–4)**, 55–59.

Renzetti, G. 2013, *JOAA*, **34(4)**, 341–348.

Renzetti, G. 2014, *AP&SS*, **352(2)**, 493–496.
Robertson, H. P. 1937, *MNRAS*, **97**, 423–438.
Rozelot, J. P., Fazel, Z. 2013, *Solar Physics*, **287**, 161–170.
Schuerman, D. W. 1980, *Astrophys J*, **238**, 337–342.
Shahbaz, T. *et al.* 2014, *MNRAS*.
Shankaran *et al.* 2011, *Ap&SS*, **332**, 115.
Sharma, R. K. 1987, *Ap&SS*, **135**, 271–281.
Sharma, R. K., SubbaRao, P. V. 1967, *Celest. Mech.*, **13**, 137–149.
Singh Jagadish 2009, *The Astronomical Journal*, **137**, 3286–3292.
Singh Jagadish 2012, *Ap&SS*, **342**, 303.
Singh, J., Mohammed, H. L. 2012, *E.M.P*, **109**, 1–11.
Singh, J., Umar, A. 2012, *The Astronomical Journal*, **143**, 109.
Singh, J., Umar, A. 2013, *Ap&SS*, **341**, 349.
Singh, J., Amuda, T. O. 2014, *Ap&SS*, **350**, 119–126.
Singh, J., Leke, O. 2014, *Ap&SS*, **350**, 109–117.
Soffel, M. *et al.* 1988, *Celestial Mechanics*, **42(1–4)**, 81–89.
Valtonen, M., Karttunen, H. 2006, Cambridge University Press.
Xuetang, Z., Lizhong, Y. 1993, *Chinese Phys. Lett.*, **10(1)**, 61–64.