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Effects of Triaxiality, Oblateness and Gravitational Potential from a Belt on the Linear Stability of $L_{4,5}$ in the Restricted Three-Body Problem

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Abstract. In this paper we have considered the restricted three body problem (R3BP) in which the more massive primary is triaxial, the less massive primary and infinitesimal body are oblate spheroids, and are encompassed by a belt of homogenous material points. Analytically and numerically, we have studied the effects of triaxiality of the more massive primary, oblateness of both the less massive primary and infinitesimal body and the gravitational potential generated by the belt on the location of the triangular libration points $L_{4,5}$ and their linear stability. $L_{4,5}$ do not form equilateral triangles with the primaries in the presence of all or any of the aforementioned perturbations. Due to triaxiality of the more massive primary and oblateness of the infinitesimal body the triangular libration points are seen to move away from the line linking the primaries, whereas they shift closer to the line owing to the oblateness of the less massive primary and the potential from the belt. The range $0 < \mu < \mu_c$ of stability of the triangular points is reduced in the presence of any of the perturbations, except when considering the potential from the belt the range increases, where μ_c is the critical mass ratio. The oblateness of a test particle (of infinitesimal mass) shifts the location of its libration positions away from the primaries and reduces its range of stability.

Key words. Restricted three-body problem (R3BP)—potential from the belt—triaxiality effect—oblateness effect.

1. Introduction

The general problem of obtaining a closed form solution for the motion of $N > 2$ bodies in space under their mutual gravitational interaction has remained intractable. This is rather famously known as the N-body problem. More precisely, the three-body problem ($N = 3$) has been explored extensively for more than three centuries. Unlike the case with two bodies, the three-body problem has no simple *general* solution that can be utilized to describe the motion for any arbitrary initial conditions at all time (Szebehely 1967; Szebehely & Mark 2004). As recognized by Leonhard Euler (1772), the assumption that one body is of infinitesimal mass, and therefore does not influence the motion of the remaining two primary bodies, yields a more manageable and useful problem, called the restricted three-body problem (R3BP) (Szebehely 1967; Bruno 1994; Gutzwiller 1998; Valtonen & Karttunen 2006; Chenciner 2007). Three-body problem also exist in the General Relativity (Renzetti 2012a; Iorio 2014). In circular (C) R3BP, the primary bodies are fixed in a coordinate system rotating with the origin (axis of rotation) at the centre of their masses. Lagrange showed that in this rotating (synodic) frame there are five libration points at which the infinitesimal mass would remain fixed if there exists zero velocity. Three of the points L_1, L_2, L_3 are collinear with the primaries while, the other two L_4, L_5 are in equilateral triangular configurations with the primaries. The collinear points L_1, L_2, L_3 are linearly unstable, while the triangular points L_4, L_5 are linearly stable for the mass ratio of the primaries less than the Routhian value 0.03852..., (Szebehely 1967).

In the classical R3BP the sphericity of the bodies is presumed, whereas in reality numerous celestial bodies are either oblate or triaxial in shape. For instance, Jupiter and Saturn are oblate while Pluto and its moon Charon are triaxial. The asphericity of either the Sun or the major bodies are seen to play a role in several orbital problems at classical and relativistic level (Rozelot & Damiani 2011; Damiani *et al.* 2011; Iorio 2006, 2009, 2010, 2011, 2013; Smith *et al.* 2012; Rozelot & Fazel 2013; Yan *et al.* 2013; Iorio *et al.* 2011, 2013; Renzetti 2012b, 2013, 2014). An important example of perturbations arising due to oblateness in the solar system is the orbit of the satellite of Jupiter, called Amalthea. This planet is very oblate and the satellite's orbit is too small that its line of apsides advances about 900° in one year (Moulton 1914). This prompted Khanna & Bhatnagar (1999), Sharma *et al.* (2001a, b, c), Jain *et al.* (2006) and Singh (2013) to involve at least one of the primaries in their studies of the CR3BP as triaxial rigid bodies. While Ishwar (1997, 1998), Elshaboury & Mostafa (2011), Singh & Umar (2013), Abouelmagd *et al.* (2013) and Singh & Haruna (2014) have considered at least one of the primaries and the infinitesimal body as oblate spheroids in their investigations of the CR3BP.

There are ring-type belts of dust particles in the stellar systems which are regarded as the young analogues of the Kuiper belt (Greaves *et al*. 1998). Trilling *et al.* (2007) detected debris rings in many main-sequence stellar binary systems using the *Spitzer Space Telescope*. Among the observed 69 A3-F8 main sequence binary star systems, nearly 60% showed dust rings surrounding the binary stars. This inspired many scientists to study the CR3BP by taking into consideration the additional gravitational potential from the belt. Jiang & Yeh (2003, 2004a, b) studied the CR3BP by considering the influence from a belt for planetary systems and found that the chance to have libration points around the inner part of the belt is higher than that near the outer part. Jiang & Yeh (2003) and Yeh & Jiang (2006) have incorporated additional gravitational potential from the belt in their studies of the CR3BP. They found that the gravitational potential from the belt makes the structure of the CR3BP quite different such that new libration points exist only under certain conditions. The orbital motion of a test particle around a primary is greatly affected in the presence of a massive belt (Iorio 2007, 2012). Of late, a bi-dimensional ring model for the minor asteroids of the solar system has been implemented and modeled in the latest EPM2013 planetary ephemerides (Pitjeva & Pitjev 2014). Kushvah (2008) studied analytically and numerically the effects of radiation pressure of the more massive primary, oblateness of less massive primary and gravitational potential from the belt on the linear stability of libration points in the R3BP. Singh & Taura (2013) examined the combined effect of radiation and oblateness of both primaries, together with additional gravitational potential from the belt on the motion of an infinitesimal body in the CR3BP. Singh & Taura (2014a) studied the effects of oblateness up to J_4 of the less massive primary and gravitational potential from a belt on the linear stability of triangular equilibrium points in the photogravitational CR3BP. Singh & Taura (2014b) examined the effect of triaxiality of the more massive primary, oblateness up to the zonal harmonic J_4 of the smaller primary and gravitational potential from a belt on the linear stability of the triangular libration points in the CR3B.

In this study, we aim to examine the effect of triaxiality of the more massive primary, oblateness of both the less massive primary and the infinitesimal body and the gravitational potential from the belt on the locations of $L_{4,5}$ and their linear stability in the CR3BP. This study can be useful in the investigation of motion of an oblate test particle near triaxial and oblate binaries surrounded by a cluster of material points.

This paper is arranged as follows Section 2 presents equations of motion of the infinitesimal body, locations of $L_{4(5)}$ and their linear stability are established in sections 3 and 4, while section 5 contains conclusion.

2. Equations of motion

Let m_3 be the mass of the infinitesimal body moving in the orbital plane of a more massive triaxial primary of mass m_1 and a less massive oblate primary of mass m_2. We take a coordinate system *oxyz* with origin at the centre of mass of the primaries and the x-axis is the line joining the primaries; while y-axis is perpendicular to it in the orbital plane, the z-axis is perpendicular to the orbital plane of the primaries. The circumbinary belt is centered at the origin of the coordinate system *oxyz* (Fig. 1). The distances of m_3 from m_1, m_2 are r_1, r_2, respectively, and the distance between the primaries is R. The units for the mass and length are chosen such that the sum of the masses of the primaries and their separation distance are unity. The unit of time is chosen such that the gravitational constant is also unity. Let $\mu = \frac{m_2}{m_1+m_2}$ be the mass parameter, then we have the masses $m_2 = \mu$ and $m_1 = 1 - \mu$; let *oxyz* rotate about the z-axis, then the coordinates of m_1, m_2 and m_3 are $(x_1, 0, 0) = (-\mu, 0, 0)$, $(x_2, 0, 0) = (1 - \mu, 0, 0)$ and $(x, y, 0)$, respectively.

We assume that the smaller primary and infinitesimal body have their equatorial planes coinciding with the plane of motion; and let us represent the oblateness coefficients of the smaller primary and infinitesimal body as A_i, $0 < A_i = \frac{\mathrm{R}_{ei}^2 - R_{pi}^2}{5R^2} << 1$, where R_{ei} $i = 2, 3$ are equatorial radii and, R_{pi} $i = 2, 3$ are the polar radii of m_2 and m_3, respectively.

Then, the equations of motion of m_3 under the influence of triaxiality of m_1, oblateness of m_2, m_3 and gravitational potential from a circumbinary belt centred at

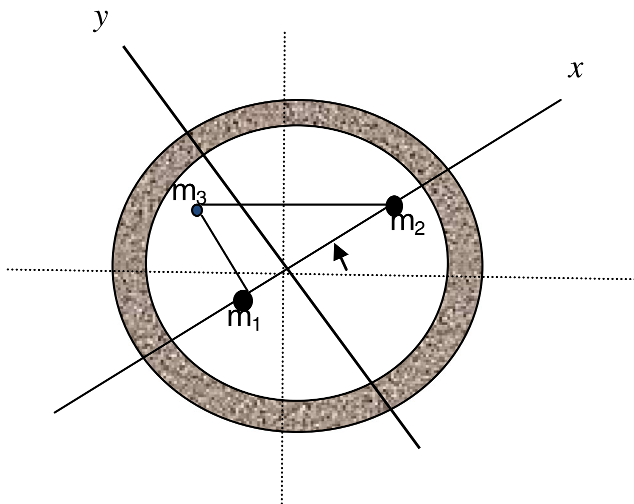

Figure 1. The planar configuration of the problem.

the origin of the coordinate system *oxyz* (Sharma *et al.* 2001a, Singh & Taura 2013) can be expressed as

$$\ddot{x} - 2n\dot{y} = \Omega_x, \quad \ddot{y} + 2n\dot{x} = \Omega_y, \tag{1}$$

where

$$\Omega = \frac{n^2}{2}[(1-\mu)r_1^2+\mu r_2^2]+\left(\frac{1}{r_1}+\frac{(2\sigma_1-\sigma_2)}{2r_1^3}-\frac{3(\sigma_1-\sigma_2)y^2}{2r_1^5}+\frac{A_3}{2r_1^3}\right)(1-\mu)$$
$$+\left(\frac{1}{r_2}+\frac{A_2}{2r_2^3}+\frac{A_3}{2r_2^3}\right)\mu+\frac{M_b}{(r^2+T^2)^{1/2}}, \tag{2}$$

$$r_1^2 = (x+\mu)^2 + y^2, \quad r_2^2 = (x+\mu-1)^2 + y^2, \tag{3}$$

$$\sigma_1 = \frac{a'^2 - c'^2}{5R^2}, \quad \sigma_2 = \frac{b'^2 - c'^2}{5R^2},$$

The over dot denotes differentiation with respect to time t, $\sigma_1, \sigma_2 << 1$ characterize the triaxiality of the more massive primary with a', b', c' as lengths of its semi-axes, $M_b << 1$ is the total mass of the belt, r is the radial distance of the infinitesimal mass given by $r^2 = x^2 + y^2$, $T = a + b$, a and b are parameters which determine the density profile of the belt (Miyamoto & Nagai 1975, Jiang & Yeh 2004b, Yeh & Jiang 2006; Kushvah 2008). The parameter a known as the flatness parameter controls the flatness of the profile. The parameter b controls the size of the core of density profile and is called the *core parameter*. When $a = b = 0$, the potential is equal to one by a point mass. n is the mean motion of the primaries, given by

$$n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2 + A_2) + \frac{2M_b r_c}{(r_c^2 + T^2)^{3/2}}, \tag{4}$$

where $r_c^2 = 1 - \mu + \mu^2$ is the radial distance of the infinitesimal body through the triangular points in the classical R3BP.

3. Location of triangular libration points

The libration points represent stationary solutions of the CR3BP. All the derivatives of coordinates with respect to time are zero at these points (Szebehely 1967). Thus the libration points of the perturbed CR3BP if they exist, are determined by setting all velocities and accelerations of the equations of motion (1) to zero, i.e.

$$\begin{aligned}\Omega_x &= n^2x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)(x+\mu)(2\sigma_1-\sigma_2)}{2r_1^5}\\ &\quad + \frac{15(1-\mu)(x+\mu)(\sigma_1-\sigma_2)y^2}{2r_1^7} - \frac{3(1-\mu)(x+\mu)A_3}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2}{2r_2^5}\\ &\quad - \frac{\mu(x+\mu-1)}{r_2^3} - \frac{3\mu(x+\mu-1)A_3}{2r_2^5} - \frac{M_b x}{(r^2+T^2)^{3/2}} = 0,\end{aligned}$$

$$\begin{aligned}\Omega_y &= n^2y - \frac{(1-\mu)y}{r_1^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)y}{2r_1^5} + \frac{15(1-\mu)(\sigma_1-\sigma_2)y^3}{2r_1^7}\\ &\quad - \frac{3(1-\mu)yA_3}{2r_1^5} - \frac{\mu y}{r_2^3} - \frac{3\mu yA_2}{2r_2^5} - \frac{3\mu yA_3}{2r_2^5} - \frac{M_b y}{(r^2+T^2)^{3/2}} = 0.\end{aligned}$$

Now, from $\Omega_x = 0$, $\Omega_y = 0$ with $y \neq 0$, we have

$$\left.\begin{aligned}&n^2x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)(x+\mu)(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{15(1-\mu)(x+\mu)(\sigma_1-\sigma_2)y^2}{2r_1^7}\\ &\quad - \frac{3(1-\mu)(x+\mu)A_3}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2}{2r_2^5} - \frac{\mu(x+\mu-1)}{r_2^3}\\ &\quad - \frac{3\mu(x+\mu-1)A_3}{2r_2^5} - \frac{M_b x}{\left(r^2+T^2\right)^{3/2}} = 0,\\ &n^2 - \frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2r_1^5} + \frac{15(1-\mu)(\sigma_1-\sigma_2)y^2}{2r_1^7} - \frac{3(1-\mu)A_3}{2r_1^5}\\ &\quad - \frac{\mu}{r_2^3} - \frac{3\mu A_2}{2r_2^5} - \frac{3\mu A_3}{2r_2^5} - \frac{M_b}{(r^2+T^2)^{3/2}} = 0.\end{aligned}\right\} \tag{5}$$

If the effect of triaxiality of the more massive primary, oblateness of the less massive primary and the infinitesimal mass and the potential from the belt are neglected, i.e., $\sigma_1 = \sigma_2 = A_2 = A_3 = M_b = 0$, the equations of (5) reduce to

$$\left.\begin{aligned}&n^2x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} = 0,\\ &n^2 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} = 0\end{aligned}\right\}$$

with the solutions $r_1 = r_2 = 1$ (Szebehely 1967). Therefore, due to the perturbations, the values of r_1 and r_2 may change slightly, say by ε_1 and ε_2 so that

$$r_1 = 1+\varepsilon_1, \quad r_2 = 1+\varepsilon_2, \quad \text{where } \varepsilon_i << 1 \ \ (i=1,2). \tag{6}$$

By considering only linear terms in ε_1 and ε_2, using equation (6) in equation (3) we obtain

$$\left.\begin{aligned} x &= \frac{1}{2} - \mu + \varepsilon_1 - \varepsilon_2, \\ y &= \pm\frac{\sqrt{3}}{2}\left(1 + \frac{2}{3}(\varepsilon_1 + \varepsilon_2)\right). \end{aligned}\right\} \tag{7}$$

Substituting equations (4), (6) and (7) in equation (5), and considering only linear terms in small quantities, we have

$$\left.\begin{aligned} \varepsilon_1 &= \frac{A_3}{2} - \frac{A_2}{2} - \frac{11}{8}\sigma_1 + \frac{11}{8}\sigma_2 - \frac{M_{\text{b}}(2r_{\text{c}} - 1)}{3(r_{\text{c}}^2 + T^2)^{3/2}}, \\ \varepsilon_2 &= \frac{A_3}{2} + \left[-\frac{3}{2} + \frac{1}{2\mu}\right]\sigma_1 + \left[1 - \frac{1}{2\mu}\right]\sigma_2 - \frac{M_{\text{b}}(2r_{\text{c}} - 1)}{3(r_{\text{c}}^2 + T^2)^{3/2}}. \end{aligned}\right\} \tag{8}$$

Hence, from equations (7) and (8) we obtain the first approximation in terms of the small parameters, the coordinates of the triangular libration points $L_4(x, y)$ and $L_5(x, -y)$ as

$$\left.\begin{aligned} x &= \frac{1}{2}\left[1 - 2\mu + \left(\frac{1}{4} - \frac{1}{\mu}\right)\sigma_1 + \left(\frac{3}{4} + \frac{1}{\mu}\right)\sigma_2 - A_2\right], \\ y &= \pm\frac{\sqrt{3}}{2}\left(1 + \frac{1}{3}\left\langle\left(\frac{1}{\mu} - \frac{23}{4}\right)\sigma_1 + \left(\frac{19}{4} - \frac{1}{\mu}\right)\sigma_2 - A_2 + 2A_3 - \frac{4M_{\text{b}}(2r_{\text{c}} - 1)}{3(r_{\text{c}}^2 + T^2)^{3/2}}\right\rangle\right). \end{aligned}\right\} \tag{9}$$

The effects of various parameters on the location of triangular points $L_{4(5)}$ are shown in Figs 2–6, when $\mu = 0.003$ and $T = 0.01$.

The coordinates in equation (11) are functions of $\sigma_1,\ \sigma_2,\ A_2,\ A_3$ and M_{b}. It is significant that the y-coordinates of the triangular points depend on the mass parameter μ, such that at $\mu = 0$ the coordinates in equation (11) do not exist, which is contrary to those of Szebehely (1967) and Singh & Ishwar (1999), but it is in agreement with Sharma *et al.* (2001a) and Singh & Begha (2011). However, when $\sigma_1 = \sigma_2 = A_2 = A_3 = M_{\text{b}} = 0$, the coordinates reduce to $x = \frac{1}{2}(1 - 2\mu)$, $y = \pm\frac{\sqrt{3}}{2}$ which correspond to the classical case of Szebehely (1967). In the presence of the only triaxiality of the more massive primary (i.e., $A_2 = A_3 = M_b = 0$), these coordinates completely agree with those of Sharma *et al.* (2001a) when the more massive primary is non-luminous. On considering only the triaxiality of the more massive primary and oblateness of the less massive primary (i.e., $A_3 = M_{\text{b}} = 0$), the coordinates (7) confirm those of Sharma *et al.* (2001b) when the less massive primary is oblate spheroid instead of triaxial, Singh & Begha (2011) in the absence of perturbations in the Coriolis and centrifugal forces, and Singh (2013) in the absence of radiation of the primaries, perturbations in the Coriolis and centrifugal forces and when less massive primary is oblate spheroid instead of triaxial. It is noted in this problem that, the triangular points no longer form equilateral triangles with the primaries as they do in the classical case; rather they form scalene triangles (Figs 2, 3, 6) and isosceles triangles (Figs 4, 5) with the primaries. The triangular points move away from

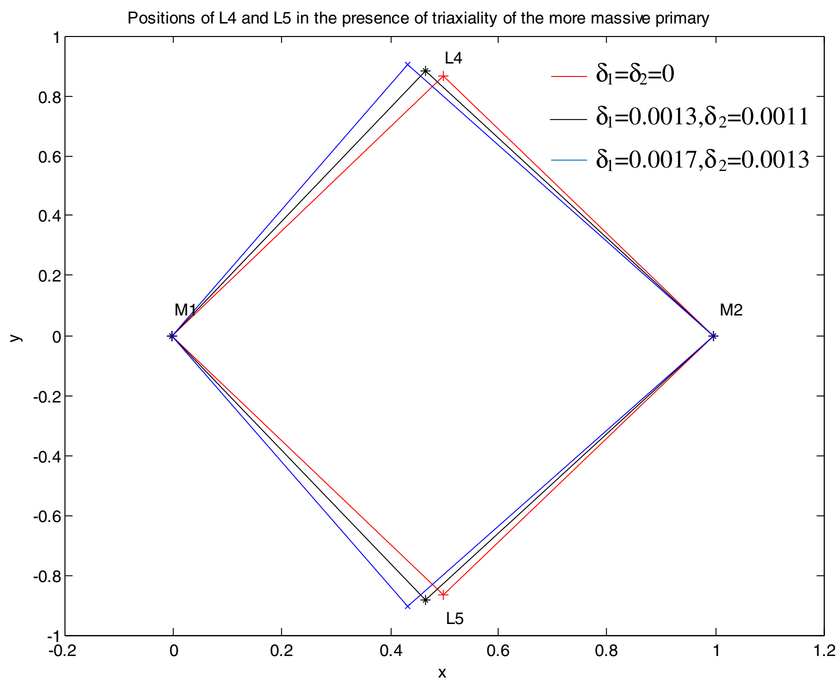

Figure 2. Effects of triaxiality of the more massive primary on the location of $L_{4,5}$ ($A_2 = A_3 = M_b = 0$).

the line joining the primaries in the presence of the triaxiality of the more massive primary (Fig. 2) and oblateness of the infinitesimal body (Fig. 4), whereas the points shift toward the line due to the oblateness of the less massive primary (Fig. 3) and the potential from the belt (Fig. 5). In the presence of all of these perturbations, the points move away from the x-axis (Fig. 6).

4. Linear stability of the triangular libration points

Let (x^*, y^*) be the coordinates of any of the triangular libration points. Let η and ξ be small displacements in x^* and y^* respectively, then $(x^* + \eta, y^* + \xi)$ is a point in the vicinity of (x^*, y^*). Thus the linear variational equations of motion of the infinitesimal body are

$$\ddot{\eta} - 2n\dot{\xi} = (\Omega^0_{xx})\eta + (\Omega^0_{xy})\xi,$$
$$\ddot{\xi} + 2n\dot{\eta} = (\Omega^0_{yx})\eta + (\Omega^0_{yy})\xi, \tag{10}$$

where

$$\Omega^0_{xx} = \frac{\partial^2\Omega}{\partial x^2}(x^*, y^*), \quad \Omega^0_{xy} = \frac{\partial^2\Omega}{\partial xy}(x^*, y^*),$$
$$\Omega^0_{yx} = \frac{\partial^2\Omega}{\partial yx}(x^*, y^*), \quad \Omega^0_{yy} = \frac{\partial^2\Omega}{\partial y^2}(x^*, y^*).$$

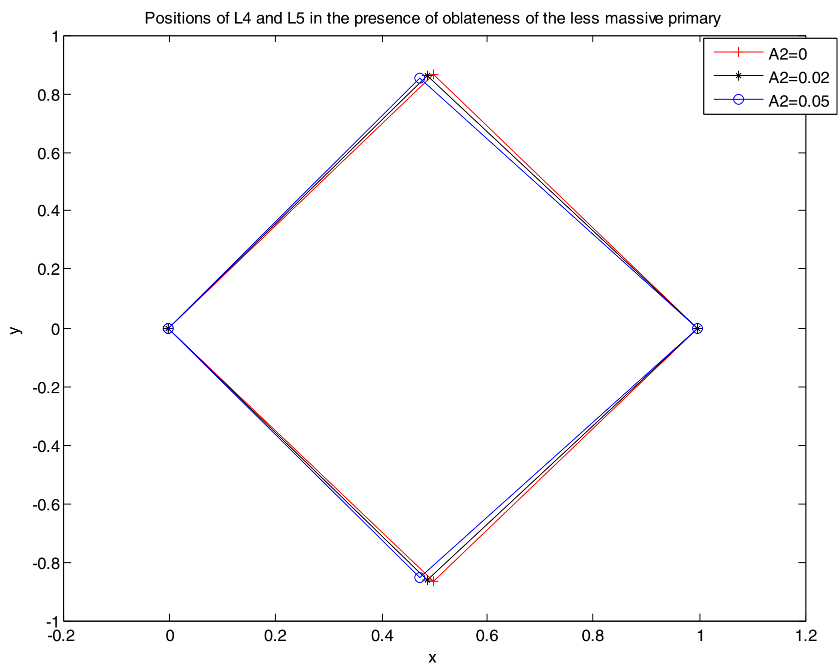

Figure 3. Effects of oblateness of the less massive primary on the location of $L_{4,5}$ ($\delta_1 = \delta_1 = A_3 = M_b = 0$).

Let $\eta = A e^{\lambda t}$, $\xi = B e^{\lambda t}$ (A, B, λ are constants) be the solutions of the equations of (10), then the characteristic equation for the triangular libration point is

$$\lambda^4 + (4n^2 - \Omega^0_{xx} - \Omega^0_{yy})\lambda^2 + (\Omega^0_{xx}\Omega^0_{yy} - \Omega^{0,2}_{xy}) = 0. \tag{11}$$

Now, confining only to the linear terms in $A_2, A_3, \sigma_1, \sigma_2, M_b$, using equation (9) we obtain

$$\begin{aligned}
\Omega^0_{xx} &= \frac{3}{4} + \frac{57\sigma_1}{16} - \frac{3\sigma_2}{16} + \frac{3A_2}{8} + \frac{3\mu}{2}\left(\frac{15\sigma_1}{8} - \frac{31\sigma_2}{8} + \frac{\sigma_2}{\mu^2} - \frac{\sigma_1}{\mu^2} + 2A_2\right) \\
&\quad + \frac{5M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{1}{4} - \mu + \mu^2)}{(r_c^2 + T^2)^{5/2}}, \\
\Omega^0_{yy} &= \frac{9}{4} + \frac{87\sigma_1}{16} - \frac{21\sigma_2}{16} + \frac{33A_2}{8} + 3A_3 + \frac{3\mu}{2}\left(-\frac{15\sigma_1}{8} + \frac{15\sigma_2}{8} + \frac{\sigma_1}{\mu^2} - \frac{\sigma_2}{\mu^2}\right) \\
&\quad + \frac{7M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{9M_b}{4(r_c^2 + T^2)^{5/2}},
\end{aligned}$$

$$\Omega^0_{xy} = \sqrt{3}\left\{\begin{aligned}
&\frac{3}{4} + \frac{47\sigma_1}{16} - \frac{9\sigma_2}{16} + \frac{7A_2}{8} + \frac{A_3}{2} + \mu\left(-\frac{89\sigma_1}{16} + \frac{37\sigma_2}{16}\right. \\
&\left. - \frac{1}{2\mu^2}\langle 3\mu^2 + \sigma_1 - \sigma_2\rangle - \frac{13A_2}{4} - A_3 - \frac{11M_b(2r_c - 1)}{6(r_c^2 + T^2)^{3/2}}\right) \\
&+ \frac{11M_b(2r_c - 1)}{12(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{1}{2} - \mu)}{2(r_c^2 + T^2)^{5/2}}
\end{aligned}\right\}.$$

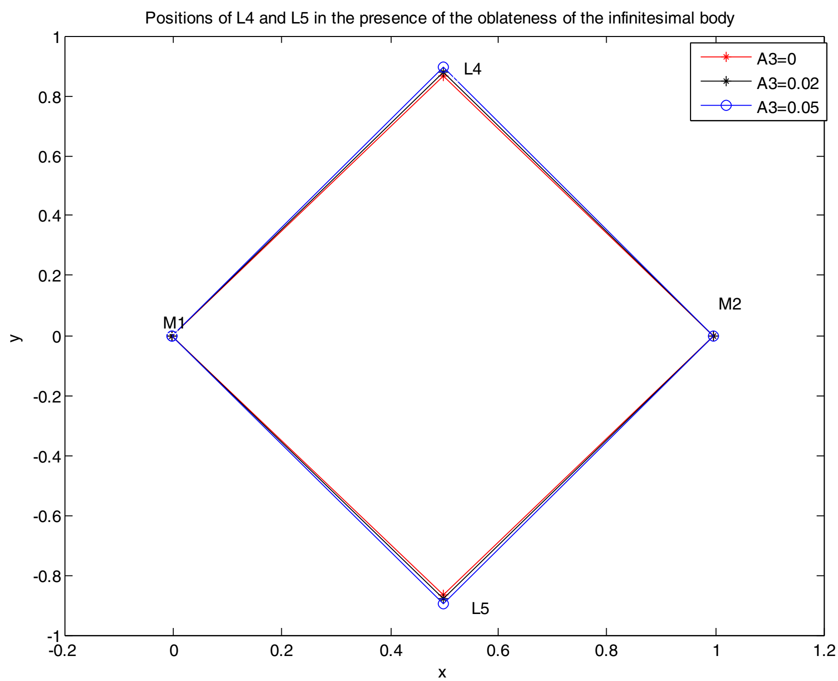

Figure 4. Effects of oblateness of the infinitesimal body on the location of $L_{4,5}$ ($\delta_1 = \delta_2 = A_2 = M_b = 0$).

Utilizing the values of Ω^0_{xx}, Ω^0_{yy}, Ω^0_{xy} in equation (11) yield

$$\Lambda^2 + Q\Lambda + N = 0, \tag{12}$$

where

$$\begin{aligned}
\Lambda &= \lambda^2, \\
Q &= 1 + 3\sigma_1 + 3\left(\mu - \frac{3}{2}\right)\sigma_2 + 3\left(\frac{1}{2} - \mu\right)A_2 - 3A_3 + \frac{M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} \\
&\quad - \frac{3M_b r_c^2}{(r_c^2 + T^2)^{5/2}} > 0, \\
N &= -\frac{9\sigma_1}{8} + \frac{9\sigma_2}{8} + \left(\frac{27}{4} + \frac{891\sigma_1}{16} - \frac{423\sigma_2}{16} + \frac{117A_2}{4} + 9A_3 + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}}\right. \\
&\quad \left. + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}}\right)\mu - \left(\frac{27}{4} + \frac{801\sigma_1}{16} - \frac{333\sigma_2}{16} - \frac{117A_2}{4} - 9A_3\right. \\
&\quad \left. + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}}\right)\mu^2.
\end{aligned}$$

The roots of equation (12) are

$$\Lambda = \frac{1}{2}[-Q \pm \sqrt{\Delta}], \tag{13}$$

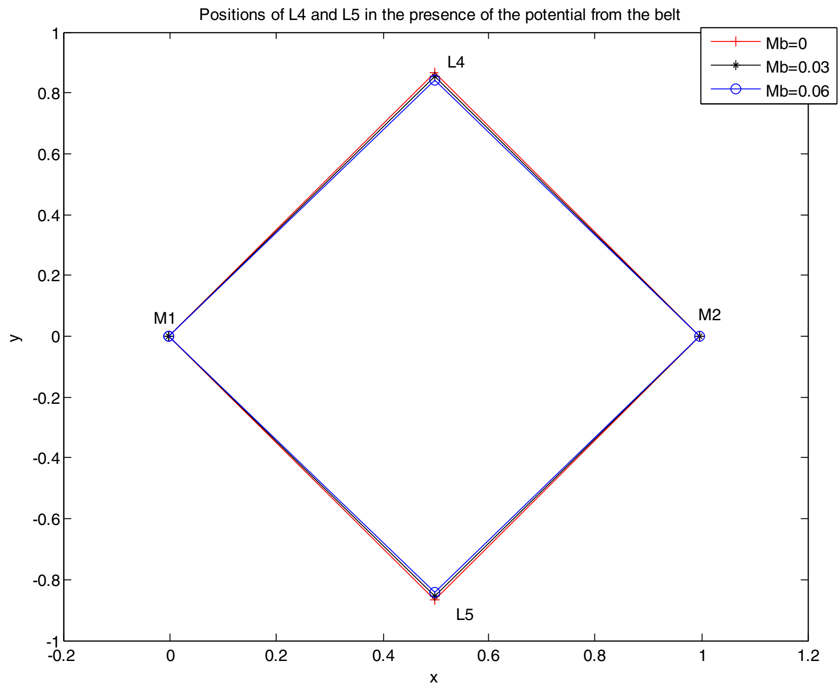

Figure 5. Effects of potential from the belt on the location of $L_{4,5}$ ($\delta_1 = \delta_2 = A_2 = A_3 = 0$).

where $\Delta = Q^2 - 4N$ is called the discriminant, which is given by

$$\begin{aligned}\Delta \;=\; & 1 + \frac{21\sigma_1}{2} - \frac{27\sigma_2}{2} + 3A_2 - 6A_3 + \frac{2M_{\mathrm{b}}(2r_{\mathrm{c}}+3)}{(r_{\mathrm{c}}^2+T^2)^{3/2}} - \frac{6M_{\mathrm{b}}r_{\mathrm{c}}^2}{(r_{\mathrm{c}}^2+T^2)^{5/2}} \\ & - \left(27 + \frac{891\sigma_1}{4} - \frac{447\sigma_2}{4} + 123A_2 + 36A_3 + \frac{66M_{\mathrm{b}}(2r_{\mathrm{c}}-1)}{(r_{\mathrm{c}}^2+T^2)^{3/2}} + \frac{27M_{\mathrm{b}}}{(r_{\mathrm{c}}^2+T^2)^{5/2}}\right)\mu \\ & + \left(27 + \frac{801\sigma_1}{4} - \frac{333\sigma_2}{4} + 117A_2 + 36A_3 + \frac{66M_{\mathrm{b}}(2r_{\mathrm{c}}-1)}{(r_{\mathrm{c}}^2+T^2)^{3/2}} + \frac{27M_{\mathrm{b}}}{(r_{\mathrm{c}}^2+T^2)^{5/2}}\right)\mu^2. \end{aligned} \tag{14}$$

Thus, the nature of the roots

$$\lambda_{1,2} = \pm\sqrt{\Lambda_1}, \quad \lambda_{3,4} = \pm\sqrt{\Lambda_2}, \tag{15}$$

depend on the discriminant Δ which is a quadratic function of the mass parameter μ. Since $0 < \mu < \frac{1}{2}$ we are interested in the behavior of Δ in the interval $(0, \frac{1}{2})$.

The domain of Δ is $(-\infty, \infty)$ at μ very close to 0,

$$\begin{aligned}\Delta \;=\; & 1 + \frac{21\sigma_1}{2} - \frac{27\sigma_2}{2} + 3A_2 - 6A_3 + \frac{2M_{\mathrm{b}}(2r_{\mathrm{c}}+3)}{(r_{\mathrm{c}}^2+T^2)^{3/2}} - \frac{6M_{\mathrm{b}}r_{\mathrm{c}}^2}{(r_{\mathrm{c}}^2+T^2)^{5/2}}, \\ \Delta \;=\; & 1 + \left\langle \frac{21\sigma_1}{2} - \frac{27\sigma_2}{2} + 3A_2 - 6A_3 + \frac{2M_{\mathrm{b}}(2r_{\mathrm{c}}+3)}{(r_{\mathrm{c}}^2+T^2)^{3/2}} - \frac{6M_{\mathrm{b}}r_{\mathrm{c}}^2}{(r_{\mathrm{c}}^2+T^2)^{5/2}} \right\rangle > 0. \end{aligned} \tag{16}$$

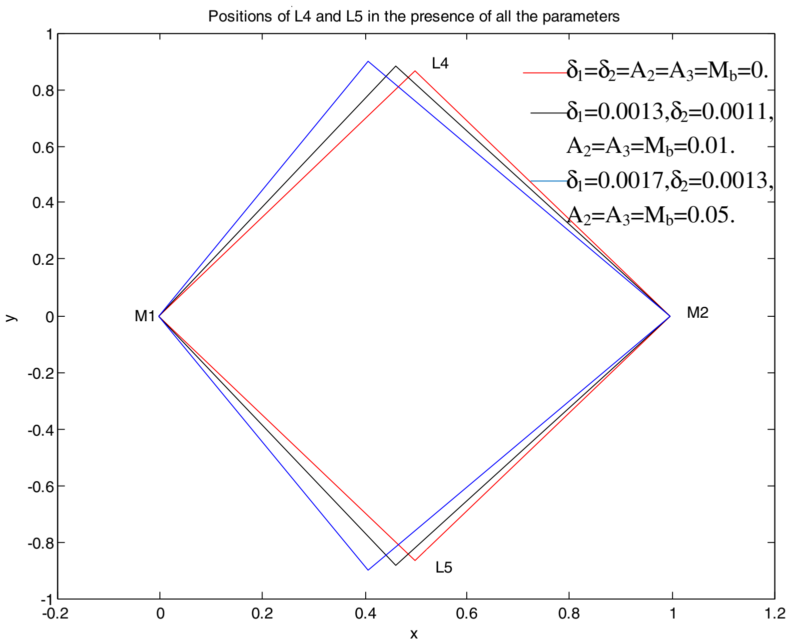

Figure 6. The combined effects of the parameters on the location of $L_{4,5}$.

When $\mu = \frac{1}{2}$,

$$\begin{aligned}
\Delta &= 1 + \frac{21\sigma_1}{2} - \frac{27\sigma_2}{2} + 3A_2 - 6A_3 + \frac{2M_\mathrm{b}(2r_\mathrm{c}+3)}{(r_\mathrm{c}^2+T^2)^{3/2}} - \frac{6M_\mathrm{b}r_\mathrm{c}^2}{(r_\mathrm{c}^2+T^2)^{5/2}} \\
&\quad - \frac{27}{2} - \left(\frac{891\sigma_1}{4} - \frac{447\sigma_2}{4} + 123A_2 + 36A_3 + \frac{66M_\mathrm{b}(2r_\mathrm{c}-1)}{(r_\mathrm{c}^2+T^2)^{3/2}} + \frac{27M_\mathrm{b}}{(r_\mathrm{c}^2+T^2)^{5/2}}\right)\frac{1}{2} \\
&\quad \frac{27}{4} + \left(\frac{801\sigma_1}{4} - \frac{333\sigma_2}{4} + 117A_2 + 36A_3 + \frac{66M_\mathrm{b}(2r_\mathrm{c}-1)}{(r_\mathrm{c}^2+T^2)^{3/2}} + \frac{27M_\mathrm{b}}{(r_\mathrm{c}^2+T^2)^{5/2}}\right)\frac{1}{4}. \\
\Delta &= \left\langle \frac{21\sigma_1}{2} - \frac{27\sigma_2}{2} + 3A_2 - 6A_3 + \frac{2M_\mathrm{b}(2r_\mathrm{c}+3)}{(r_\mathrm{c}^2+T^2)^{3/2}} - \frac{6M_\mathrm{b}r_\mathrm{c}^2}{(r_\mathrm{c}^2+T^2)^{5/2}} \right. \\
&\quad - \frac{891\sigma_1}{8} + \frac{447\sigma_2}{8} - \frac{123}{2}A_2 - 18A_3 - \frac{33M_\mathrm{b}(2r_\mathrm{c}-1)}{(r_\mathrm{c}^2+T^2)^{3/2}} - \frac{27M_b}{2(r_\mathrm{c}^2+T^2)^{5/2}} \\
&\quad \left. + \frac{801\sigma_1}{16} - \frac{333\sigma_2}{16} + \frac{117}{4}A_2 + 9A_3 + \frac{33M_\mathrm{b}(2r_\mathrm{c}-1)}{2(r_\mathrm{c}^2+T^2)^{3/2}} + \frac{27M_\mathrm{b}}{4(r_\mathrm{c}^2+T^2)^{5/2}} \right\rangle - \frac{23}{4}. \\
\Delta &= \left\langle -\frac{813\sigma_1}{16} + \frac{345\sigma_2}{16} - \frac{117A_2}{4} - 15A_3 - \frac{M_\mathrm{b}(58r_\mathrm{c}-45)}{2(r_\mathrm{c}^2+T^2)^{3/2}} - \frac{3M_\mathrm{b}(8r_\mathrm{c}^2+9)}{4(r_\mathrm{c}^2+T^2)^{5/2}} \right\rangle - \frac{23}{4} < 0.
\end{aligned} \tag{17}$$

The derivative of the discriminant Δ with respect to the mass parameter μ is

$$\frac{\mathrm{d}\Delta}{\mathrm{d}\mu} = \left(27+\frac{801\sigma_1}{4}-\frac{333\sigma_2}{4}+117A_2+36A_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)2\mu$$
$$-\left(27+\frac{891\sigma_1}{4}-\frac{447\sigma_2}{4}+123A_2+36A_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right). \quad (18)$$

Considering

$$\sigma_1,\ \sigma_2,\ A_2,\ A_3,\ M_{\mathrm{b}} << 1,\ \frac{\mathrm{d}\Delta}{\mathrm{d}\mu}<0,\quad \text{for all } \mu\in\left(0,\frac{1}{2}\right). \quad (19)$$

Since Δ is a polynomial it is continuous and differentiable in (0, $\frac{1}{2}$), thus in equation (18), Δ is a constantly decreasing function of μ in the interval (0, $\frac{1}{2}$); and we conclude with respect to equations (16) and (17) that μ exists in the interval (0, $\frac{1}{2}$) for which $\Delta = 0$. The value of μ for which the discriminant vanishes is termed the critical value of mass parameter represented as μ_{c}. Thus, three regions of μ values can be considered:

(1) When $0 < \mu < \mu_{\mathrm{c}}$ implies $\Delta = Q^2 - 4N > 0$ and Δ invarially decreasing in the interval (0, $\frac{1}{2}$) implies $N > 0$. Thus $Q^2 - 4N < Q^2 \Rightarrow \sqrt{Q^2-4N} < Q$ (since $Q > 0$ equation (12)) or $\Lambda = \frac{1}{2}\left[-Q \pm \sqrt{\Delta}\right] < 0$ and consequently the four values of λ in equation (15) are distinct pure imaginary numbers. This indicates stability of the triangular points.
(2) If $\mu_{\mathrm{c}} < \mu < \frac{1}{2}$ $(\Delta < 0)$, the real parts of two of the values of λ in equation (15) are positive. Thus, the triangular points are unstable.
(3) When $\Delta = 0(\mu = \mu_{\mathrm{c}})$, we have double roots of (15) and consequently give secular terms, showing that the triangular points are unstable.

Solving the equation $\Delta = 0$, gives the value of the critical mass parameter μ_{c}

$$\mu_{\mathrm{c}} = \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)+\frac{1}{6}\left(\frac{5}{2}-\frac{49}{3\sqrt{69}}\right)\sigma_1-\frac{1}{18}\left(\frac{19}{2}-\frac{23}{\sqrt{69}}\right)\sigma_2+\frac{1}{9}\left(1-\frac{13}{\sqrt{69}}\right)A_2$$
$$-\frac{22}{9\sqrt{69}}A_3+\left[\frac{(76-8r_{\mathrm{c}})\left(r_{\mathrm{c}}^2+T^2\right)^{5/2}-9(1+6r_{\mathrm{c}}^2)\left(r_{\mathrm{c}}^2+T^2\right)^{3/2}}{27\sqrt{69}\left(r_{\mathrm{c}}^2+T^2\right)^4}\right]M_{\mathrm{b}}. \quad (20)$$

In Table 1, we have utilized equation (20) to examine the effect of triaxiality of the bigger primary, oblateness of both the smaller primary and infinitesimal mass; and the potential from the belt on the critical mass value, by taking arbitrary values for $\sigma_1,\ \sigma_2,\ A_2,\ A_3$ and M_{b}.

Equation (20) provides the critical mass parameter μ_{c}, which is used to establish the range of stability of the triangular points. It reflects the influence of triaxiality of the more massive primary, oblateness of both the less massive primary and

Table 1. The effects of the various perturbations on the critical mass value, $T = 0.01$, $r_c = 0.997$.

Case	σ_1	σ_2	A_2	A_3	M_b	μ_c
1	0	0	0	0	0	0.03852
2	0.0013	0.0011	0	0	0	0.03823
	0.0015	0.0012	0	0	0	0.03821
	0.0017	0.0014	0	0	0	0.03815
3	0	0	0.02	0	0	0.03727
	0	0	0.03	0	0	0.03664
	0	0	0.04	0	0	0.03601
4	0	0	0	0.02	0	0.03264
	0	0	0	0.03	0	0.02969
	0	0	0	0.04	0	0.02675
5	0	0	0	0	0.01	0.03854
	0	0	0	0	0.02	0.03897
	0	0	0	0	0.03	0.03919
6	0.0013	0.0011	0.02	0.02	0.01	0.03131
	0.0015	0.0012	0.03	0.03	0.02	0.02794
	0.0017	0.0013	0.04	0.04	0.03	0.02458

infinitesimal body, and gravitational potential from the belt. On ignoring the effect of these perturbations (i.e., $\sigma_1 = \sigma_2 = A_2 = A_3 = M_b = 0$), μ_c reduces to

$$\frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) = 0.03852\ldots,$$

which corresponds to the classical case of Szebehely (1967). If the triaxiality of the more massive primary, the oblateness of the infinitesimal body and the potential from the belt are ignored (i.e., $\sigma_1 = \sigma_2 = A_3 = M_b = 0$), the value of μ_c coincides with the result of Sharma (1987) when the more massive primary is non radiating. If the more massive primary is oblate instead of triaxial and in the absence of oblateness of the infinitesimal body and potential from the belt (i.e., $\sigma_1 = \sigma_2, \quad A_3 = M_b = 0$), the value of μ_c reduces to that of Singh & Ishwar (1999) when the primaries are non-luminous. It can be deduced from Table 1 that the value of μ_c reduces in the presence of triaxiality of more massive primary or oblateness of less massive primary or oblateness of infinitesimal body, while it increases due to the potential from the belt. The combined effect of these perturbations also reduces the value of μ_c. Since the triangular points are linearly stable in the range $0 < \mu < \mu_c$ then, triaxiality of more massive primary, oblateness of less massive primary and oblateness of the infinitesimal body reduce the range of stability; whereas the potential from the belt increases the range of stability.

5. Conclusion

The problem considered in this paper is a perturbation of the R3BP when the gravitational potential from a belt, and the oblateness and triaxiality of the bodies are assumed. The positions of triangular libration points and their linear stability are affected by the triaxiality of more massive primary, oblateness of less massive primary, oblateness of infinitesimal body and gravitational potential from a belt. Triangular points no longer form equilateral triangles with the primaries in presence of all or any of the aforementioned perturbations. Due to triaxiality of the more massive primary and oblateness of the infinitesimal body the points move away from the x-axis, while owing to the oblateness of less massive primary and potential from the belt the points draw closer to the x-axis. The range $0 < \mu < \mu_c$ of stability of the triangular points is reduced by the triaxiality of more massive primary, oblateness of less massive primary and oblateness of infinitesimal body. However, the potential from the belt increases the range. The oblateness of a test particle (of infinitesimal mass) shifts the location of its equilibrium positions away from the primaries and limits its range of stability.

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