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Canonical Ensemble Model for Black Hole Radiation

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Abstract. In this paper, a canonical ensemble model for the black hole quantum tunnelling radiation is introduced. In this model the probability distribution function corresponding to the emission shell is calculated to second order. The formula of pressure and internal energy of the thermal system is modified, and the fundamental equation of thermodynamics is also discussed.

Key words. Canonical ensemble model—black hole—quantum tunnelling—fundamental equation of thermodynamics.

The quantum tunnelling method, proposed by Parikh & Wilczek, is an effective method of investigating the black hole radiation (Parikh & Wilczek 2000; Parikh 2004). Zhang (2013) investigated the black hole quantum tunnelling radiation and presented a statistical model based on the Parikh–Wilczek tunnelling framework, which is different from the method proposed in Zhao *et al.* (2006). It showed that if the emission shell is treated as a thermodynamical system which reached a thermal equilibrium with the black hole, in the second-order accuracy, the probability distribution function will be equal to the tunnelling rate of this shell. In Zhang (2013), the probability distribution function is calculated to second order, which is more accurate than the calculation in statistical physics. In this paper, we will discuss the formula of pressure and internal energy, as well as the fundamental equation of thermodynamics. In Zhang (2013), the emission shell is treated as a thermodynamical system. The statistical operator of this system is

$$\hat{\rho} = \sum_i |\psi_i\rangle P_i \langle\psi_i|, \tag{1}$$

where $|\psi_i\rangle$ denotes the quantum states of the emission shell, P_i is the probability of the state $|\psi_i\rangle$.

For a canonical ensemble system, the probability distribution function is

$$P_i = \frac{\Omega_{\rm BH}(E - E_i)}{\Omega(E)}, \tag{2}$$

where E and E_i denote the total energy of the isolated system and energy of the emission shell, respectively. $\Omega_{\mathrm{BH}}(E-E_i)$ is the number of microscopic states of the black hole, and $\Omega(E)$ is that of the isolated system. In Zhang (2013), $\ln\Omega_{\mathrm{BH}}(E-E_i)$ is expanded into the form of Taylor series,

$$\ln\Omega_{\mathrm{BH}}(E-E_i) = \ln\Omega_{\mathrm{BH}}(E) + \left(\frac{\partial\ln\Omega_{\mathrm{BH}}}{\partial E_{\mathrm{BH}}}\right)_{E_{\mathrm{BH}}=E}(-E_i) + \frac{1}{2}\left(\frac{\partial^2\ln\Omega_{\mathrm{BH}}}{\partial E_{\mathrm{BH}}^2}\right)_{E_{\mathrm{BH}}=E}(-E_i)^2+\cdots. \quad (3)$$

If we ignore the terms higher than third order, then we have

$$P_i = \frac{\Omega_{\mathrm{BH}}(E-E_i)}{\Omega(E)} = \frac{1}{Z}\mathrm{e}^{-\beta E_i - \frac{K_B}{2C_{\mathrm{BH}}}\beta^2 E_i^2}, \quad (4)$$

where

$$Z = \sum_i \mathrm{e}^{-\beta E_i - \frac{K_B}{2C_{\mathrm{BH}}}\beta^2 E_i^2}. \quad (5)$$

Therefore, we get

$$\hat{\rho} = \frac{1}{Z}\mathrm{e}^{-\beta\hat{H} - \frac{K_B}{2C_{\mathrm{BH}}}\beta^2\hat{H}^2}, \quad (6)$$

where $\hat{H}$ is the Hamiltonian operator of the thermodynamic system. For a Schwarzschild black hole, we have

$$E = M,\quad T = \frac{1}{8\pi M K_B},\quad E_i = \omega,\quad C_{\mathrm{BH}} = -\frac{K_B\beta^2}{8\pi}. \quad (7)$$

Substituting equation (7) into equation (4) we get

$$P_i \propto \mathrm{e}^{-8\pi M\omega(1-\frac{\omega}{2M})} = \mathrm{e}^{\Delta S_{\mathrm{BH}}} = \Gamma, \quad (8)$$

where Γ is the tunnelling rate of the emission particle. It means that with the canonical ensemble model, the probability distribution function of the emission shell system is the same as the emission rate of a spherical shell. If we calculate the probability distribution function to second order, then the fundamental equation of thermodynamics and the formula about the thermodynamical quantity, such as pressure and internal energy, will be very different.

The internal energy of the thermodynamical system is

$$U \equiv \langle\hat{H}\rangle = \mathrm{tr}(\hat{\rho}\hat{H}) = \frac{1}{Z}\sum_i E_i \mathrm{e}^{-\beta E_i - \frac{K_B}{2C_{\mathrm{BH}}}\beta^2 E_i^2}. \quad (9)$$

The generalized force

$$Y \equiv \left\langle\frac{\partial\hat{H}}{\partial y}\right\rangle = \mathrm{tr}\left(\hat{\rho}\frac{\partial\hat{H}}{\partial y}\right) = \frac{1}{Z}\sum_i\frac{\partial E_i}{\partial y}\mathrm{e}^{-\beta E_i - \frac{K_B}{2C_{\mathrm{BH}}}\beta^2 E_i^2}. \quad (10)$$

For entropy, there is no corresponding thermodynamical quantity, without loss of generalization. Let us define an entropy operator

$$\hat{S} = -K_B \ln \hat{\rho}. \tag{11}$$

Then, the mean value of entropy is

$$S \equiv \langle \hat{S} \rangle = \mathrm{tr}(\hat{\rho}\hat{S}) = -K_B \mathrm{tr}(\hat{\rho} \ln \hat{\rho}). \tag{12}$$

For ideal gases, let $y = V$, then the pressure of the system is

$$P = \frac{1}{Z} \sum_i \frac{\partial E_i}{\partial V} \mathrm{e}^{-\beta E_i - \frac{K_B}{2C_{\mathrm{BH}}} \beta^2 E_i^2}, \tag{13}$$

which cannot be formulated as the simple form of the function of Z. Moreover, if we calculate the differential $\mathrm{d}U - Y\mathrm{d}y$, we find that

$$\mathrm{d}U - Y\mathrm{d}y = M\mathrm{d}\beta + N\mathrm{d}y, \tag{14}$$

where

$$M = \frac{1}{Z} \sum_i E_i \frac{\partial \rho_i}{\partial \beta} - U \frac{\partial \ln Z}{\partial \beta}, \tag{15}$$

$$N = \frac{1}{Z} \sum_i E_i \frac{\partial \rho_i}{\partial y} - U \frac{\partial \ln Z}{\partial y} \tag{16}$$

and

$$\rho_i = \mathrm{e}^{-\beta E_i - \frac{K_B}{2C_{\mathrm{BH}}} \beta^2 E_i^2}. \tag{17}$$

Obviously, in the case of calculating the probability distribution function to second order, we cannot find any integral factor A, which satisfy

$$A(\mathrm{d}U - Y\mathrm{d}y) = \mathrm{d}S. \tag{18}$$

It means that in this case the fundamental equation of thermodynamics does not exist.

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References

Parikh, M. K., Wilczek, F. 2000, *Phys. Rev. Lett.*, **85**, 5042.
Parikh, M. K. 2004, *Int. J. Mod. Phys. D*, **13**, 2355.
Parikh, M. K. 2004, arXiv:0402166.
Zhang, J. 2013, *Chin. Phys. Lett.*, **30**, 070401.
Zhao, R., Zhang, L. C., Hu, S. Q. 2006, *Acta Phys. Sin. (in chinese)*, **55**, 3898.