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Parametric Models of Periodogram

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Abstract. The maximum likelihood estimator is used to determine fit parameters for various parametric models of the Fourier periodogram followed by the selection of the best-fit model amongst competing models using the Akaike information criteria. This analysis, when applied to light curves of active galactic nuclei can be used to infer the presence of quasi-periodicity and break or knee frequencies. The extracted information can be used to place constraints on the mass, spin and other properties of the putative central black hole and the region surrounding it through theoretical models involving disk and jet physics.

Key words. Methods: statistical—galaxies: active—galaxies: Seyfert.

1. Introduction and periodogram fit models

Broadband flux variability arises from disk- and jet-based phenomena. The periodogram (e.g., Van der Klis 1989) of Light Curves (LCs) of active galactic nuclei implicitly reflects this. Parametric models of the binned periodogram approximate the true expected shape, the Power Spectral Density (PSD) which is ideally normally distributed making it useful to work with. A statistically appropriate model aids in theoretical modelling of variability.

Commonly used parametric models include: power law model (e.g., Papadakis & Lawrence 1993), $I(f) = Nf^{-\alpha}$, where α is the red-noise slope in the range of -1 and -2.5; broken power law model (e.g., Uttley *et al.* 2002), $I(f) = N(f/f_{\mathrm{Brk}})^{-\alpha_{\mathrm{hi}}}$, $f > f_{\mathrm{Brk}}$ and $I(f) = N(f/f_{\mathrm{Brk}})^{-\alpha_{\mathrm{low}}}$, $f < f_{\mathrm{Brk}}$ where f_{Brk} is the break frequency, α_{hi} is the slope of the high frequency region and α_{low} is the slope of the low frequency region; bending power law model (e.g., Uttley *et al.* 2002), $I(f) = N(1 + (f/f_{\mathrm{Knee}})^2)^{-\alpha/2}$ where f_{Knee} is the knee frequency, α is the slope in the high frequency region above the knee frequency; power law with a Lorentzian QPO model (e.g., Nowak 2000), $I(f) = Nf^{-\alpha} + \frac{R^2 Q f_0^2/\pi}{f_0^2 + Q^2(f - f_0)^2}$, where α is the red-noise slope and the second term is the Lorentzian function with an amplitude R, central frequency f_0 and quality factor $Q = f_0/\Delta f$, where Δf is the frequency spread in the bin hosting the central frequency.

2. Fit procedure, model selection and significance testing

The likelihood and log-likelihood functions are:

$$L(\theta_k) = \prod_{j=1}^{(n-1)} \frac{1}{I(f_j, \theta_k)} e^{-P(f_j)/I(f_j, \theta_k)},$$

$$S(\theta_k) = 2 \sum_{j=1}^{n-1} (\ln(I(f_j, \theta_k)) + P(f_j)/I(f_j, \theta_k)), \tag{1}$$

where $L(\theta_k)$ is the likelihood function, θ_k are the parameters of the parametric models, $I(f_j, \theta_k)$, to be estimated and $P(f_j)$ is the data periodogram. Determining θ_k which minimize S yields the maximum likelihood values. Confidence levels are determined in a similar manner as that described in Press *et al.* (2007) for the $\Delta \chi^2$ method. Model selection is carried out using the Akaike Information Criteria (AIC) and the Relative Likelihood (RL). AIC measures the information lost when a model is fit to the data. The model with least information loss (least AIC) is assumed to be the null model. The likelihood of other models describing the data as well as the null is determined using the relative likelihood (e.g., Burnham & Anderson 2004). AIC and RL are defined as

$$\text{AIC} = S(\theta_k) + 2p_k, \tag{2}$$

$$\Delta_i = \text{AIC}_{\min(\text{model } i)} - \text{AIC}_{\min(\text{null})},$$

$$L(\text{model } i|\text{data}) = e^{-\Delta_i/2},$$

$$\text{RL} = 1/L(\text{model } i|\text{data}),$$

where p_k is the penalty term = number of parameters θ_k used in the model, $L(\text{model } i|\text{data})$ is the likelihood of model i given in the data, RL is the likelihood ratio of model i relative to that of the null. Models with $\Delta_i \leq 2$ are close to the best fit, those with $4 \leq \Delta_i \leq 7$ are considerably less supported and those with $\Delta_i > 10$ and $RL > 150$ cannot be supported (Burnham & Anderson) in which case the null remains the best fit. On subtraction of PSD from it, the periodogram ideally comprises of χ_2^2 distributed white noise (e.g., Chatfield 2009). The residue from the best fit to the data is tested against the χ_2^2 distribution using a goodness of fit test. A 99% significance level (based on its cumulative distribution function) then reveals the outlying QPO.

3. Application, results and conclusions

XMM Newton X-ray light curve (0.3–10 KeV) of REJ 1034 + 396 which revealed a QPO of $\sim$3733 s (Gierliński *et al.* 2008) is analysed using the periodogram (296 bins) with the above fit models (results in Table 1). The power law with QPO model (Fig. 1) is the best fit model with an AIC = 149.8 and a significance of QPO > 99.94%. Parameter estimation with MLE and model selection with AIC is computationally efficient when compared to Monte-Carlo simulations based procedures (Uttley *et al.* 2002). Any other parametric model can be easily incorporated into this

Table 1. Results of the periodogram analysis with fit models.

Fit model	Properties	Results
Power law fit parameters	(AIC, RL)	$(170.1, 25.3 \times 10^3)$
	Power law slope (α)	(-1.33 ± 0.013)
	Significance (%)	>99.99
Broken power law fit parameters	(AIC, RL)	$(172.1, 68.1 \times 10^3)$
	Break frequency f_{Brk} (Hz)	0.000320 ± 0.000074
	High frequency region slope α_{hi}	-1.6 ± 0.42
	Low frequency region slope α_{low}	-1 ± 0.33
	Significance (%)	>99.99
Bending power law fit parameters	(AIC, RL)	$(198.4, 3.5 \times 10^{10})$
	Knee frequency f_{Knee} (Hz)	0.00084 ± 0.00018
	High frequency region slope α	-3.9 ± 0.74
	Significance (%)	>99.99
Power law and QPO fit parameters	(AIC, RL)	$(149.8, 1)$
	Central frequency f_0 (Hz)	0.000269
	Amplitude R	0.050 ± 0.014
	Quality factor Q	32 ± 6.5
	Significance (%)	>99.94
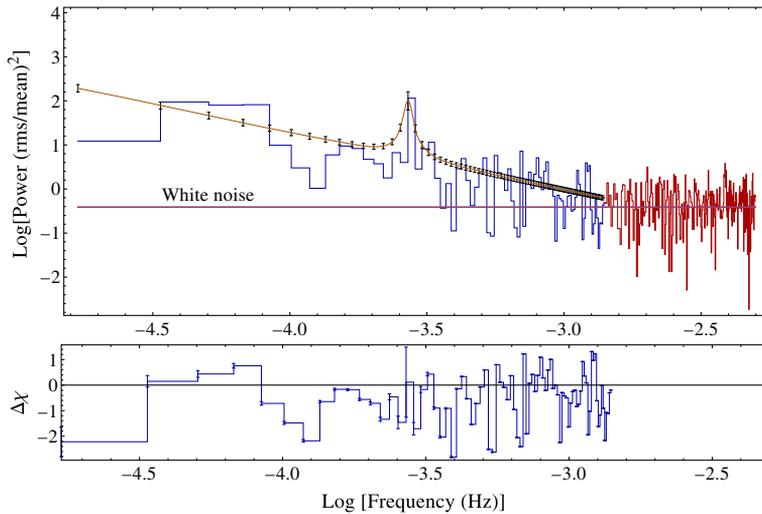
Figure 1. Binned periodogram: fit portion is in blue and white noise region is in red. The best fit is the power law with the Lorentzian QPO model and the residue $\Delta\chi = $ (data model)$/\sigma$ is shown below.

procedure. If a statistically significant QPO is detected, a lower limit to the black hole mass and constraints on its spin can be placed assuming that QPO is from an orbital signature. Theoretical models considering general relativistic effects and emission region structure can be used to simulate PSDs which can then be compared with observations to yield constraints on physical parameters.

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